Polynomial and analytic monads, revisited

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Plan of the talk

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1. Motivation (paradigmatic example).
2. (Co)completeness of and in 2-categories.
3. Main theme: extensions of representations.
4. Kleisli and Eilenberg-Moore objects in 2-category of monoidal objects and lax monoidal functors.
5. Kleisli and Eilenberg-Moore objects in 2-category of actions of monoidal objects.
6. Examples.
7. A construction of opetopic sets via Burroni fibration.
8. More examples.

Motivating example

The category $Set_{/\omega}$ of *untyped algebraic signatures* with the substitution tensor

$$\{A_n\}_n \otimes \{B_n\}_n = \{ \coprod_{k,m_1,\ldots,m_k\in\omega;\sum_{i=1}^{k} m_i=n} A_k \times B_{m_1} \times \ldots B_{m_k}\}_n$$

is a monoidal category. It acts on Set

$$* : Set_{/\omega} \times Set \longrightarrow Set$$

$$\langle \{A_n\}_n, X\rangle \mapsto \coprod_n A_n \times X^n$$

... and by exponential adjunction we get a strong monoidal functor

$$\mathbf{r} : Set_{/\omega} \longrightarrow End(Set)$$

that has a (lax monoidal) right adjoint ($U(H) = \{H(\underline{n})\}_{n\in\omega}$), for $H : Set \to Set$.

Motivation

Fact

1. The closure under isomorphism in $End(Set)$ of the 'image' of

$$\mathbf{r} : Set_{/\omega} \longrightarrow End(Set)$$

   is the category **Poly** of (finitary) polynomial endo-functors and cartesian natural transformations.

2. The closure under reflexive coequalizers **Poly** in $End(Set)$ is the category **An** of (finitary) analytic endo-functors and weakly cartesian natural transformations.

Motivation

NB. Both classes of functors have abstract characterizations:

1. polynomial endofunctor on *Set* are finitary wide pullback preserving functors;
2. analytic endofunctor on *Set* are finitary weak wide pullback preserving functors.

We can describe the analytic functors on *Set* as follows.

A symmetric signature (A, α) is a graded set $\{A_n : n \in \omega\}$ equipped with (right) actions of symmetric groups $\alpha_n : A_n \times S_n \to A_n$, for $n \in \omega$.

A morphism of symmetric sets $f : (A, \alpha) \to (B, \beta)$ is a family of morphisms of actions $f_n : (A_n, \alpha_n) \to (B_n, \beta_n)$, for $n \in \omega$.

On the category of symmetric signatures we can also define a substitution tensor but it is more complicated since we need to take actions of symmetric groups into account.

Notation. We write $a \in A$ to mean that $a \in \coprod_n A_n$ and if $a \in A$, then we write $|a| = n$ to mean that $a \in A_n$. Thus, for $a \in A$, we have $a \in A_{|a|}$.

The substitution tensor product on symmetric signatures

Let (A,α), (B,β) be two symmetric signatures. The *tensor product*

$$(A,\alpha)\otimes_O(B,\beta)=(A\otimes B,\alpha\otimes\beta)$$

is defined as follows

$(A\otimes_O B)_n =$

$$=\{\langle a,\langle b_i\rangle_{i\in(|a|]},\sigma\rangle : \sum_i |b_i| = n,\ b_i\in B,\ a\in A,\ \text{for } i\in(|a|],\ \sigma\in S_n\}_{/\sim}$$

where the equivalence relation $\sim$ is defined as follows:

$$\langle a\cdot\tau;\langle b_{\tau(i)}\cdot\sigma_{\tau(i)}\rangle_i;\sigma\rangle \sim \langle a,\langle b_i\rangle_i,\tau*(\sigma_{\tau(1)},\ldots,\sigma_{\tau(|a|)})\circ\sigma\rangle$$

where $\tau\in S_{|a|}$, $\sigma_i\in S_{|b_i|}$, $\sigma\in S_{\sum_i|b_i|}$, $*$ is the composition in the operad of symmetries, and $\circ$ is the usual composition of permutations.

The equivalence class of the element $\langle a, \langle b_i \rangle_{i \in (|a|]}, \sigma \rangle$ with be denoted by $[\langle a, \langle b_i \rangle_{i \in (|a|]}, \sigma \rangle]_\sim$.

The action $(A \otimes B, \alpha \otimes \beta)$ is defined in the obvious way

$$[\langle a, \langle b_i \rangle_{i \in (|a|]}, \sigma \rangle]_\sim \cdot \tau = [\langle a, \langle b_i \rangle_{i \in (|a|]}, \sigma \circ \tau \rangle]_\sim .$$

The category of symmetric signatures $Sig^{\mathcal{S}}$ acts also on the category *Set*

$$\ddot{*} : Sig^{\mathcal{S}} \times Set \longrightarrow Set$$

$$\langle \{A_n\}_n, X \rangle \mapsto \coprod_n (A_n \times X^n)_{\sim_n}$$

where $\sim_n$ is an equivalence relation such that for $n \in \omega$, $a \in A_n$ and $\vec{x} : \{1, \ldots, n\} \to X$, and $\sigma \in S_n$ we have

$$\langle a, \vec{x} \rangle \sim \langle a \cdot \sigma, \vec{x} \circ \sigma \rangle$$

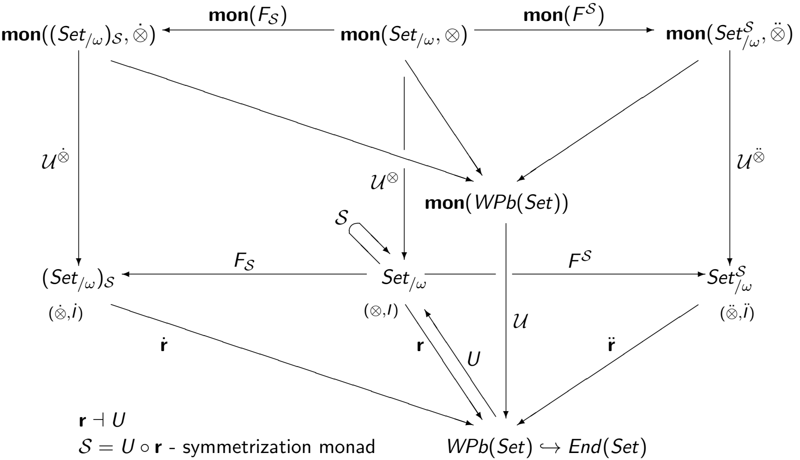
... and by exponential adjunction we get again a strong monoidal functor

$$\ddot{\mathbf{r}} : Sig^{\mathcal{S}} \longrightarrow End(Set).$$

In fact the substitution tensor product on $Sig^{\mathcal{S}}$ was so defined to make the fact true.

Motivation

Monoidal categories and categories of monoids

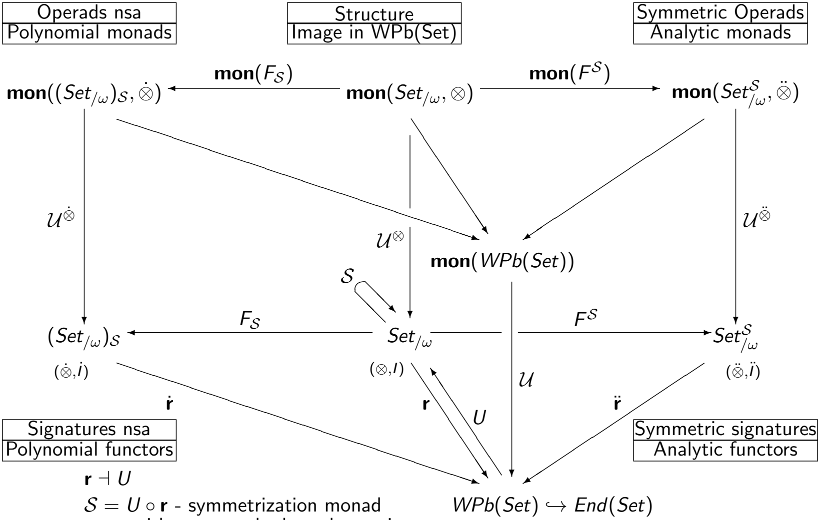


$\mathbf{r} \dashv U$

$\mathcal{S} = U \circ \mathbf{r}$ - symmetrization monad

Motivation

Monoidal categories and categories of monoids

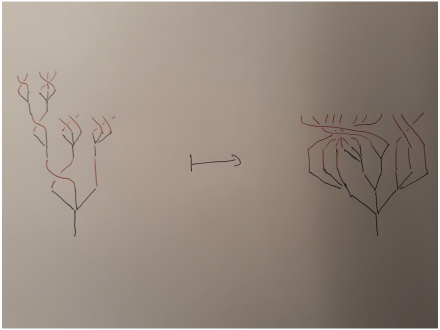


$\mathbf{r} \dashv U$

$\mathcal{S} = U \circ \mathbf{r}$ - symmetrization monad

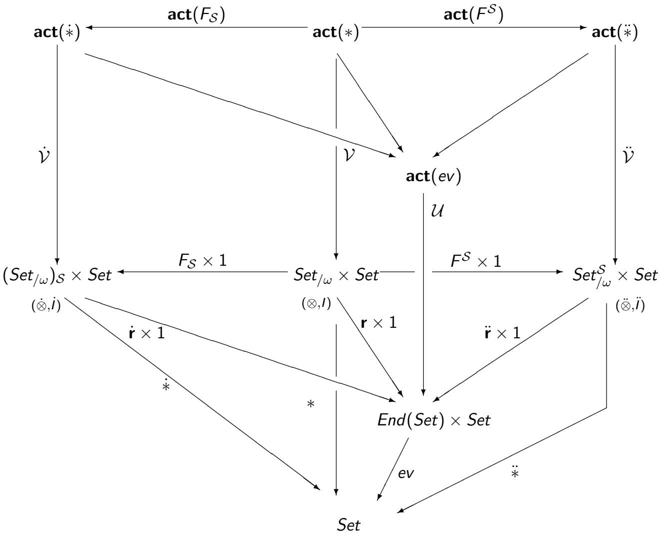
nsa - with non-standard amalgamation

The functor $\mathcal{U}^{\otimes} : \mathbf{mon}((Set_{/\omega}) \longrightarrow Set_{\omega}$ has a left adjoint $\mathcal{F}^{\otimes} \vdash \mathcal{U}^{\otimes}$ and the resulting monad $T^{\otimes}$ of free monoids distributes over symmetrization monad $\mathcal{S}$. The distributive law is, what Baez and Dolan call *combing trees*. This I will explain on a picture:



NB. The functors $\mathcal{U}^{\dot{\otimes}}$, $\mathcal{U}^{\ddot{\otimes}}$ are also monadic.

Motivation
Actions of monoidal categories and categories of actions of monoids



Some limits and colimits in 2-categories

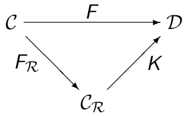
Let $\mathcal{A}$ be a 2-category with finite products. We will consider the following weighted limits and colimits in $\mathcal{A}$

1. Kleisli and Eilenberg-Moore objects (for monads in $\mathcal{A}$);
2. objects of monoids (for monoidal objects in $\mathcal{A}$);
3. objects of actions along actions (for actions of monoidal objects of $\mathcal{A}$ on 0-cells of $\mathcal{A}$);

2-categories: global (co)completeness and exactness

...and moreover we ask for exactness properties:

1. Kleisli objects commute with finite products;
2. Comparison morphisms from Kleisli objects are full and faithful



$F \dashv U$

$\mathcal{R} = UF$

We do not assume $\mathcal{A}$ has all of these (co)limits but just in some interesting cases.
Concerning the exactness properties, we expect them to hold whenever the constructions are available.

2-categories: colimits in 0-cells

Reflexive coequalizers in 0-cells A 0-cell $\mathcal{C}$ in a 2-category $\mathcal{A}$ has *reflexive coequalizer* (or *is rc*) if for any 0-cell $\mathcal{X}$ of $\mathcal{A}$ the category $\mathcal{A}(\mathcal{X},\mathcal{C})$ has coequalizers of reflexive pairs of morphisms and for any 1-cell $H : \mathcal{Y} \to \mathcal{X}$ the functor

$$\mathcal{A}(H,\mathcal{C}) : \mathcal{A}(\mathcal{X},\mathcal{C}) \longrightarrow \mathcal{A}(\mathcal{Y},\mathcal{C})$$

preserves them.
A 1-cell $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{A}$ *preserves reflexive coequalizers* (or *is rc*) if for any $\mathcal{X}$ in $\mathcal{A}$ the functor

$$\mathcal{A}(\mathcal{X}, F) : \mathcal{A}(\mathcal{X},\mathcal{C}) \longrightarrow \mathcal{A}(\mathcal{X},\mathcal{D})$$

preserves coequalizers of reflexive pairs of morphisms.

Extensions of representations

Extensions of representations

In a 2-category $\mathcal{A}$:

r representation 1-cell (faithful, conservative), i.e.,

$\mathcal{C}$ - ‘abstract’, $\mathcal{M}$ - ‘concrete’ and rc.

$$\begin{array}{c} \mathcal{C} \\ \mathbf{r} \downarrow \\ \mathcal{M} \end{array}$$

1.
2.

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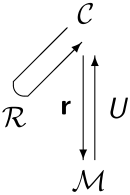

$\mathbf{r} \dashv U$

$\mathcal{R} = U\mathbf{r}$

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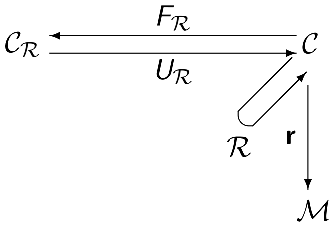

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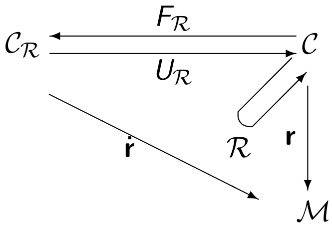

1. $\dot{\mathbf{r}}$ is full and faithful;
2.

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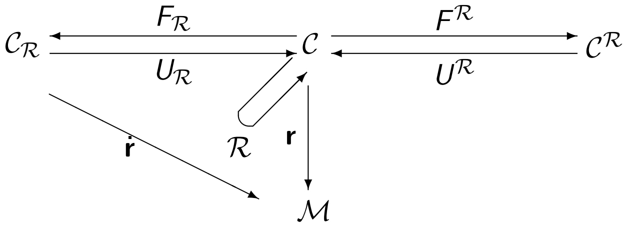

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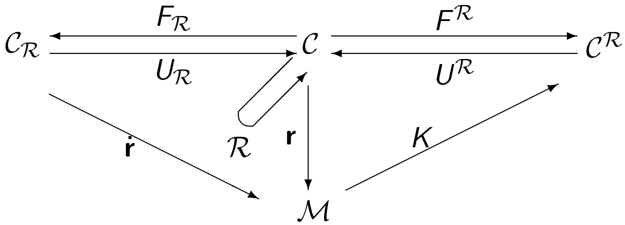

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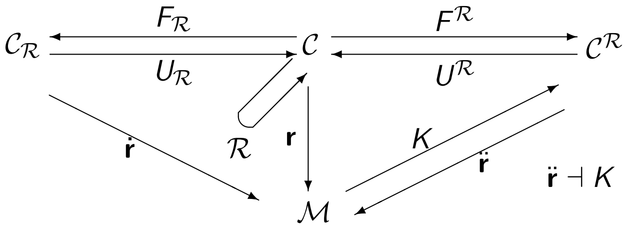


1. $\dot{\mathbf{r}}$ is full and faithful;
2. $\ddot{\mathbf{r}}$ is also expected to be full and faithful with $\mathcal{C}^\mathcal{R}$ keeping some nice properties of $\mathcal{C}$.

Kleisli and Eilenberg-Moore monoidal objects
$\mathcal{A}$ - 2-category with finite products.

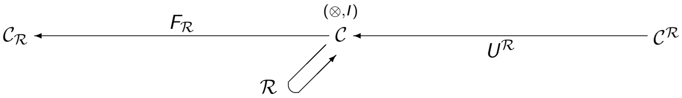
$\mathbf{Mon}_l(\mathcal{A})$ - 2-category:

- 0-cells: monoidal objects in $\mathcal{A}$;
- 1-cells: lax monoidal 1-cells in $\mathcal{A}$;
- 2-cells: monoidal 2-cells in $\mathcal{A}$.

Kleisli and Eilenberg-Moore monoidal objects

Theorem

Let $\mathcal{R}$ be an rc-lax monoidal monad on an rc-monoidal category $(C, \otimes, I, \alpha, \lambda, \varrho)$ in $\mathcal{A}$ ($\mathcal{C}$ and $\otimes$ are rc). If $\mathcal{R}$ admits both Kleisli and Eilenberg-Moore objects for monads in $\mathcal{A}$, then $\mathcal{R}$ admits both Kleisli and Eilenberg-Moore objects in $\mathbf{Mon}_I(\mathcal{A})$ and they are both standard. The tensor in $\mathcal{C}^{\mathcal{R}}$ is given by Linton's formula.



Contributors to this result: F. Linton (1969), R. Guitar, I. Moerdijk, P. McCrudden, S. Szawiel, MZ, G. Seal.

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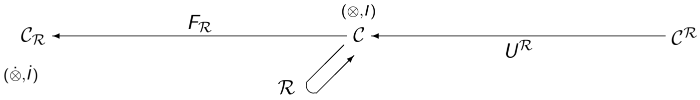

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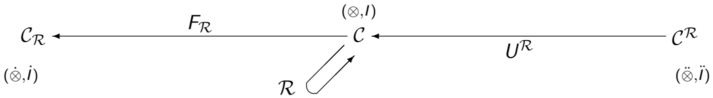

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Kleisli and Eilenberg-Moore monoidal objects

$(\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho)$ rc-monoidal object in $\mathcal{A}$ ($\mathcal{C}$ and $\otimes$ are rc),

$\mathcal{X}$ an exponentiable rc-0-cell in $\mathcal{A}$, and

$$\mathcal{C} \times \mathcal{X} \xrightarrow{(\star, \psi)} \mathcal{X},$$

(strong) action of $\mathcal{C}$ on $\mathcal{X}$. By exponential adjunction we get a strong monoidal representation

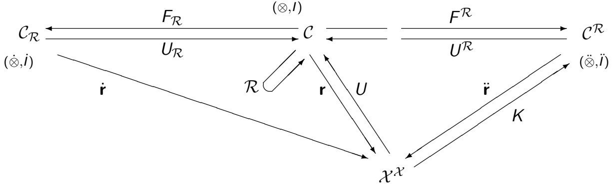
$$\mathcal{C} \xrightarrow{(\mathbf{r}, \phi)} \mathcal{X}^{\mathcal{X}},$$

which can have a (lax) monoidal right adjoint ($\mathbf{r} \dashv U$)

$$\mathcal{C} \underset{\mathbf{r}}{\overset{U}{\leftrightarrows}} \mathcal{X}^{\mathcal{X}},$$

inducing a lax monoidal monad $\mathcal{R} = U\mathbf{r}$ on $\mathcal{C}$.

Thus we have a situation



If free $\otimes$-monoids exist, i.e., $\mathcal{U}^{\otimes}$ has a left adjoint, then the induced monad $\mathcal{T}^{\otimes}$ on $\mathcal{C}$ distributes over $\mathcal{R}$, i.e., we have a distributive law:

$$\kappa : \mathcal{T}^{\otimes}\mathcal{R} \longrightarrow \mathcal{R}\mathcal{T}^{\otimes}$$

Kleisli and Eilenberg-Moore monoidal objects

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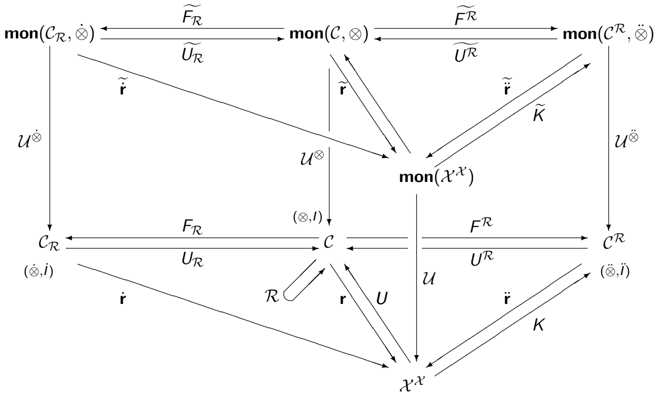

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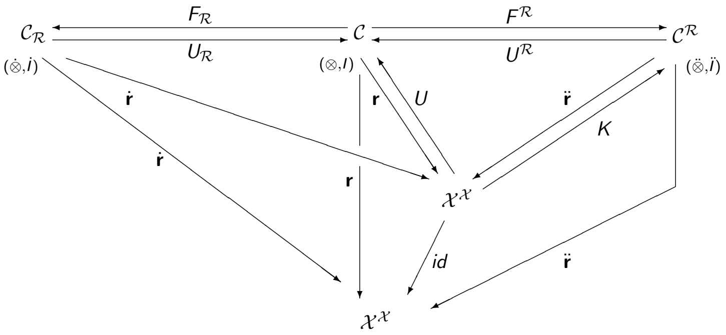
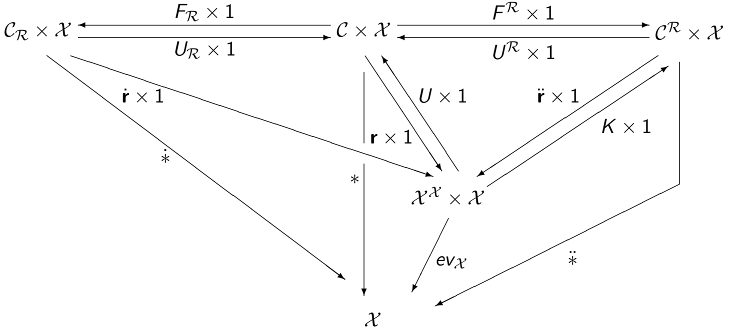
Kleisli and Eilenberg-Moore actions of monoidal objects

All the above lifts to actions of monoidal objects. First we lift the Kleisli and Eilenberg-Moore objects to the lax slice $\mathbf{Mon}_l(\mathcal{A})_{/_l \mathcal{X}^{\mathcal{X}}}$

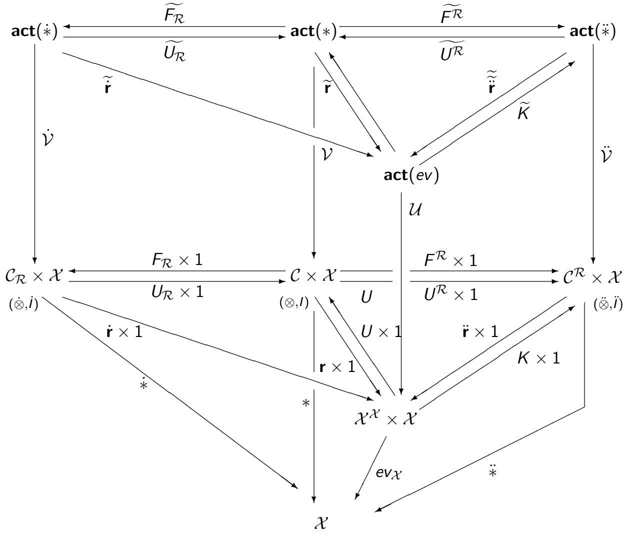


Kleisli and Eilenberg-Moore actions of monoidal objects

Then we move this diagram to the isomorphic 2-category of actions $\mathbf{Act}_l\mathbf{Mon}_l(\mathcal{A},\mathcal{X})$ of monoidal objects in $\mathcal{A}$ on $\mathcal{X}$



Finally, we can take the objects of actions along actions



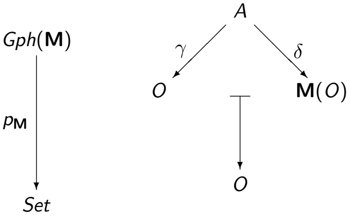
Examples: Joyal

Analytic and polynomial endofunctors on slices of *Set*

2-category $\mathbf{Fib}_{/Set}$:

- 0-cells: fibrations over *Set*,
- 1-cells: functors commuting over the base (Fact of life: substitution tensor is NOT cartesian!),
- 2-cells: vertical natural transformations.

Burroni fibration of signatures



M - is the free monoid monad on *Set*.

Examples: Joyal

The tautologous action of Burroni fibration of signatures

$$
\begin{array}{ccc}
Gph(\mathbf{M}) \times_{Set} Set^{\rightarrow} & \xrightarrow{\ \star\ } & Set^{\rightarrow} \\
& \searrow \quad \swarrow_{cod} & \\
& Set &
\end{array}
$$

is defined on objects by

$$
\begin{array}{ccc}
& A & \\
{}^{\gamma}\swarrow & & \searrow^{\delta} \\
O & & \mathbf{M}(O)
\end{array}
\qquad
\begin{array}{c}
X \\
\downarrow d \\
O
\end{array}
\quad \longmapsto \quad
\begin{array}{c}
A \star X \\
\downarrow \\
O
\end{array}
$$

where the right vertical arrow in the above diagram is the composite of the upper horizontal arrows in the following diagram

$$
\begin{array}{ccccc}
O & \xleftarrow{\ \gamma\ } & A & \longleftarrow & A \star X \\
& & {\scriptstyle \delta}\downarrow & & \downarrow \\
& & \mathbf{M}(O) & \xleftarrow[\mathbf{M}(d)]{} & \mathbf{M}(X)
\end{array}
$$

in which the square is a pullback.

Examples: Joyal

Images of the extensions

By the exponential adjunction, we get a strong monoidal morphism of (lax) monoidal fibrations

$$\begin{array}{ccc} Gph(\mathbf{M}) & \xrightarrow{\mathbf{r}} & Exp(Set^{\to}) \\ & {\scriptstyle p_{\mathbf{M}}} \searrow \quad \swarrow {\scriptstyle p_{exp}} & \\ & Set & \end{array}$$

with $\mathbf{r}$ conservative but not full even on isomorphisms. $\mathbf{r}$ has a right adjoint U (in $\mathbf{Fib}_{/Set}$) and the induced monad $\mathcal{F}$ is

$$\mathcal{F}(A,\partial)_n = \coprod_{m\in\omega} \mathbb{F}(\underline{m},\underline{n}) \times A_m$$

for a signature (A,∂) in $Gph(\mathbf{M})_O$, $n \in \omega$, $\underline{n} = \{1,\ldots,n\}$.

Examples: Joyal

The monad $\mathcal{F}$ has various submonads including symmetrization submonad $\mathcal{S}$ related to subcategory $\mathbb{B}$ (of finite sets and bijections) of $\mathbb{F}$.

$$\mathcal{S}(A, \partial)_n = \coprod_{m \in \omega} \mathbb{B}(\underline{m}, \underline{n}) \times A_m = S_n \times A_n$$

S_n -symmetric group.
This monad gives a finer extension of the representation on the category of signatures. It gives rise to polynomial (finitary) functors with cartesian natural transformations as Kleisli extension and analytic (finitary) functor with weakly cartesian natural transformations as Eilenberg-Moore fibration.

Relative Burroni fibrations and relative T-categories

The construction of a lax monoidal fibration of T-graphs can be performed even on a fibred monad on a fibration. Suppose $p : \mathcal{E} \to \mathbf{B}$ is a fibration such that the fibres of p have pullbacks. Moreover (T, η, μ) is a monad on the category $\mathcal{E}$ so that T commutes over the base

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;T\;} & \mathcal{E} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathbf{B} & \end{array}$$

and η, μ are fibred natural transformations (i.e., their components lie in the fibres of p).

Opetopic sets through Burroni fibrations

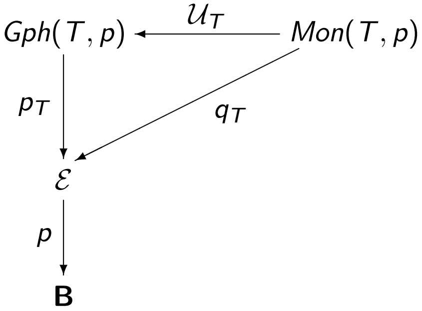
Having such data we can repeat the construction of the category of T-graphs but restricting the objects to such spans

$$\begin{array}{ccc} & A & \\ {}^{\gamma}\swarrow & & \searrow^{\delta} \\ O & & T(O) \end{array}$$

that are in fibres of p (i.e., $p(\gamma) = p(\delta) = 1_{p(O)}$). The morphisms are defined as before. In this way, we get a relative Burroni fibration $p_T : Gph(T, p) \to \mathcal{E}$ of T-graphs over p. Clearly, p_T is a lax monoidal fibration with the tensor structure defined as before.

Opetopic sets through Burroni fibrations

Thus we have a fibration of monoids with a forgetful to $Gph(T, p)$ as in the diagram



of functors and categories. As for any category $\mathcal{C}$, the functor $! : \mathcal{C} \longrightarrow \mathbf{1}$ into the terminal category $\mathbf{1}$ is a fibration, this construction is a generalization of the previous one.

Remark We can also define a basic fibration $cod : \mathcal{E}^{\to,p} \to \mathcal{E}$ relative to a fibration $p : \mathcal{E} \to \mathbf{B}$, so that the objects of $\mathcal{E}^{\to,p}$ are morphisms of $\mathcal{E}$ in fibres of p and morphisms are commuting squares. Then, as previously for the Burroni fibrations, we have a tautologous action the lax monoidal fibration $p_T : Gph(T,p) \to \mathcal{E}$ on a fibration $cod : \mathcal{E}^{\to,p} \to \mathcal{E}$

$$\begin{array}{ccc} Gph(T,p) \times_{\mathcal{E}} \mathcal{E}^{\to,p} & \xrightarrow{\ \star^{(T,p)}\ } & \mathcal{E}^{\to,p} \\ & \searrow \quad \swarrow_{cod} & \\ & \mathcal{E} & \end{array}$$

If we take the exponential adjoint of this morphism, as before, we obtain a (relative) representation of relative T-graphs and relative T-categories.

Free relative T-categories

In his thesis T. Leinster has given a reasonable sufficient conditions for a cartesian monads T so that the forgetful functor $\mathcal{U}_T$ defined above has a left adjoint and permitting to iterate the construction in the terminal fiber.

NB. In fact this is a joint effort of at least J. Adamek, A. Burroni, H.J. Baues, M. Jibladze, A.Tonks, and G. M. Kelly.

Below we give a characterization of those fibrations p and fibred monads T on them for which one can iterate the process of taking T-graphs over a fibration p. In the exposition we use ideas from all the mentioned papers.

Opetopic sets through Burroni fibrations

The notions of a suitable fibrations and a fibrewise suitable monad are very much inspired by the notions of a suitable category and a suitable monad, respectively, proposed by T. Leinster in his book.

The main difference of our approach with respect to that of T. Leinster is that we iterate whole fibrations over fibrations and get as a final result the category of opetopic sets, whereas T. Leinster's the construction is done fibre by fibre and gives the set of opetopes as a result. From the perspective of our construction this set of opetopes is the set of cells in the terminal opetopic set.

Opetopic sets through Burroni fibrations

We say that a fibration $p : \mathcal{E} \to \mathbf{B}$ is *suitable* if and only if

1. p has fibred pullbacks, finite coproducts, and filtered colimits,
2. finite coproducts and filtered colimits are universal in fibres of p,
3. filtered colimits commutes with pullbacks in fibres of p.

Let $p : \mathcal{E} \to \mathbf{B}$ be a fibration with fibred pullbacks. A monad (T, η, μ) on $\mathcal{E}$ is *cartesian relative to p* if and only if (T, η, μ) is a fibred monad over p (i.e., $p \circ T = p$, $p(\eta) = 1_p = p(\mu)$) and the restriction of the monad (T, η, μ) to every fibre of p is a cartesian monad on this fibre.

Let $p : \mathcal{E} \to \mathbf{B}$ be a suitable fibration. We say that a monad (T, η, μ) on $\mathcal{E}$ is *suitable relative to p* if and only if (T, η, μ) is cartesian relative to p and T preserves filtered colimits in the fibres of p.

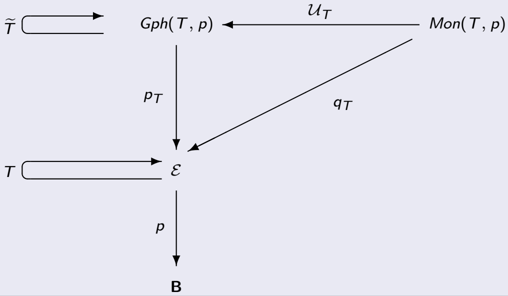
The following theorem is the key to the definition of the tower of fibrations that defines the category of opetopic sets.

Opetopic sets through Burroni fibrations

Theorem

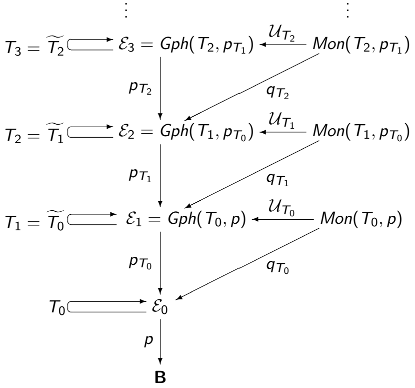
Let (T, η, μ) be a suitable monad relative to a suitable fibration $p : \mathcal{E} \to \mathbf{B}$. Then

1. *the fibration p_T over p is again suitable;*
2. *the forgetful functor $\mathcal{U}_T$ is monadic;*
3. *the monad $(\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$ induced by the adjunction $\mathcal{F}_T \dashv \mathcal{U}_T$ is suitable relative to p_T.*



Opetopic sets through Burroni fibrations

Using the above Theorem 1, and starting with any fibrewise suitable monad T_0 on a fibrewise suitable fibration $p : \mathcal{E}_0 \to \mathbf{B}$, we can build a tower of (fibrewise suitable) lax monoidal fibrations and fibrewise suitable monads as in the diagram below:



by interation of the construction.

Opetopic sets through Burroni fibrations

The identity monad 1_{Set} on Set is of course a fibrewise suitable on the fibrewise suitable fibration $! : Set \to \mathbf{1}$, where $\mathbf{1}$ is the terminal category. Thus we can build a tower of fibrations, as above, starting form this fibration. We obtain

$$
\begin{array}{ccccc}
 & & \vdots & & \vdots \\
T_3 = \widetilde{T_2} & \hookrightarrow & \mathcal{O}_3 = Gph(T_2, p_{T_1}) & \xleftarrow{\mathcal{U}_{T_2}} & Mon(T_2, p_{T_1}) \\
 & & \downarrow p_{T_2} & \swarrow q_{T_2} & \\
T_2 = \widetilde{T_1} & \hookrightarrow & \mathcal{O}_2 = Gph(T_1, p_{T_0}) & \xleftarrow{\mathcal{U}_{T_1}} & Mon(T_1, p_{T_0}) \\
 & & \downarrow p_{T_1} & \swarrow q_{T_1} & \\
T_1 = \widetilde{T_0} & \hookrightarrow & \mathcal{O}_1 = Gph(T_0, !) & \xleftarrow{\mathcal{U}_{T_0}} & Mon(T_0, !) \\
 & & \downarrow p_{T_0} & \swarrow q_{T_0} & \\
T_0 = 1_{Set} & \hookrightarrow & \mathcal{O}_0 = Set & & \\
 & & \downarrow ! & & \\
 & & \mathbf{1} & &
\end{array}
$$

Opetopic sets through Burroni fibrations

An *opetopic set* is an infinite sequence of objects $\{A_n\}_{n\in\omega}$ such that

1. A_n is an object in $\mathcal{O}_n$,
2. A_{n+1} lies in the fibre over A_n, i.e., $p_{T_n}(A_{n+1}) = A_n$,

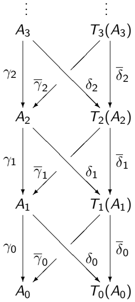
for $n \in \omega$. A *morphism of opetopic sets* $\{f_n\}_{n\in\omega} : \{A_n\}_{n\in\omega} \longrightarrow \{B_n\}_{n\in\omega}$ is a family of morphisms such that

1. $f_n : A_n \longrightarrow B_n$ is a morphism in $\mathcal{O}_n$
2. f_{n+1} lies in the fibre over f_n, i.e., $p_{T_n}(f_{n+1}) = f_n$,

for $n \in \omega$.

Opetopic sets through Burroni fibrations

Unraveling this definition, we see that an opetopic set (in the above sense) is an ∞-span as the diagram below:



with

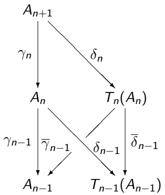
$$\gamma_n \circ \gamma_{n+1} = \overline{\gamma}_n \circ \delta_{n+1}, \qquad \delta_n \circ \gamma_{n+1} = \overline{\delta}_n \circ \delta_{n+1}$$

$$\gamma_n \circ \overline{\gamma}_{n+1} = \overline{\gamma}_n \circ \overline{\delta}_{n+1}, \qquad \delta_n \circ \overline{\gamma}_{n+1} = \overline{\delta}_n \circ \overline{\delta}_{n+1}$$

for $n \in \omega$.

Opetopic sets through Burroni fibrations

To describe the terminal opetopic set A, we need to start with $A_0 = 1$ the terminal object in Set. Then choose A_{n+1} as the terminal object in the fibre of p_{T_n} over A_n. Thus A_1 is 1 and A_{n+1} for $n > 0$ can be taken as the limit in the following diagram:



The disjoint union of the sets $\{A_n\}_{n\in\omega}$ is the set of opetopes in the sense of T. Leinster.

Opetopic sets through Burroni fibrations

Theorem

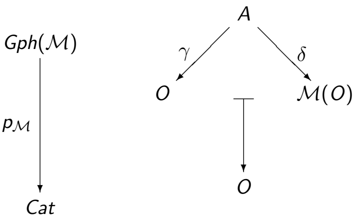
The category of opetopic sets so defined is equivalent to the category of many-to-one polygraphs.

NB. This category of opetopic sets can defined as the limit of the diagram of categories $\mathcal{O}_n$.

Examples: Fiore-Gambino-Hyland-Winskel

Analytic endofunctors on presheaf categories

Let $\mathcal{M}$ be the monad on *Cat* for strict monoidal categories.
(Modified) Burroni fibration of signatures over ***Cat***



two-sided discrete bifibration
$(A : O^{op} \times \mathcal{M}(O) \to Set)$

NB. $\mathcal{M}$ - preserves (op)fibrations and two-sided discrete fibration.

Examples: Fiore-Gambino-Hyland-Winskel

The tautologous action of modified Burroni fibration of signatures

$$
\begin{array}{ccc}
Gph(\mathcal{M}) \times_{Cat} DFib & \xrightarrow{\;\star\;} & DFib \\
& \searrow \quad \swarrow_{cod} & \\
& Cat &
\end{array}
$$

is defined on objects (as before) by

$$
\begin{array}{ccc}
& A & \\
{}^{\gamma}\swarrow & & \searrow^{\delta} \\
O & & \mathbf{M}(O)
\end{array}
\qquad
\begin{array}{c}
X \\
\downarrow d \\
O
\end{array}
\qquad \longmapsto \qquad
\begin{array}{c}
A \star X \\
\downarrow \\
O
\end{array}
$$

from the diagram

$$
\begin{array}{ccccc}
O & \xleftarrow{\;\gamma\;} & A & \longleftarrow & A \star X \\
& & \downarrow \delta & & \downarrow \\
& & \mathbf{M}(O) & \xleftarrow[\mathbf{M}(d)]{} & \mathbf{M}(X)
\end{array}
$$

NB. The fibres of $cod : DFib \to Cat$ presheaf categories.

Examples: Fiore-Gambino-Hyland-Winskel

Images of the extensions

By the exponential adjunction, we get a strong monoidal morphism of (lax) monoidal fibrations

$$
\begin{array}{ccc}
Gph(\mathcal{M}) & \xrightarrow{\mathbf{r}} & Exp(DFib) \\
 & {\scriptstyle p_{\mathcal{M}}} \searrow \quad \swarrow {\scriptstyle p_{exp}} & \\
 & Cat &
\end{array}
$$

with $\mathbf{r}$ conservative but not full even on isomorphisms. $\mathbf{r}$ has a right adjoint U (in $\mathbf{Fib}_{/Cat}$).

Examples: Fiore-Gambino-Hyland-Winskel

For $H : \widehat{O} \to \widehat{O}$, $U(H)$ is the two-sided discrete fibration

$$O \xleftarrow{\ \gamma\ } U(H) \xrightarrow{\ \delta\ } \mathbf{M}(O)$$

corresponding to

$$\overline{H} : O^{op} \times \mathcal{M}(O) \longrightarrow Set$$

which is an adjoint to

$$\mathcal{M}(O) \xrightarrow{\iota_O} \widehat{O} \xrightarrow{H} \widehat{O}$$

Examples: Fiore-Gambino-Hyland-Winskel

The induced monad $\mathcal{F} = U\mathbf{r}$ is

$$\mathcal{F}(A,\gamma,\delta)(p,\vec{p}) = \coprod_{m\in\omega,\ \sigma:\underline{m}\to\underline{n},\ q_1,\dots,q_m\in O} A(p,\vec{q}) \times \prod_{i\in\underline{m}} O(q_i, p_{\sigma(i)})$$

for a signature (A,γ,δ) in $Gph(\mathcal{M})_O$, where $\vec{p} = \langle p_1,\dots,p_n\rangle$, $p, p_i \in O$.

Examples: Fiore-Gambino-Hyland-Winskel

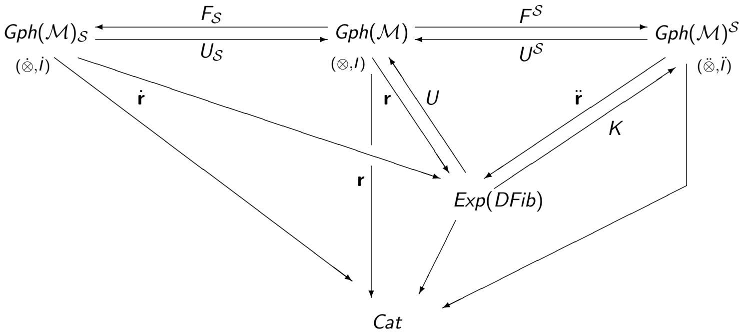
As before, the monad $\mathcal{F}$ has various submonads including symmetrization submonad $\mathcal{S}$ related to subcategory $\mathbb{B}$ (of finite sets and bijections) of $\mathbb{F}$.
The induced monad $\mathcal{F} = U\mathbf{r}$ is

$$\mathcal{S}(A,\gamma,\delta)(p,\vec{p}) = \coprod_{\sigma:\underline{n}\to\underline{n}\in S_n,\ q_1,\ldots,q_n\in O} A(p,\vec{q}) \times \prod_{i\in\underline{m}} O_{iso}(q_i, p_{\sigma(i)})$$

for a signature (A,γ,δ) in $Gph(\mathcal{M})_O$, where $\vec{p} = \langle p_1,\ldots,p_n\rangle$, $p, p_i \in O$.

Examples: Fiore-Gambino-Hyland-Winskel

The monad $\mathcal{S}$ gives, again, a finer extension of the representation on the category of signatures. The image of the extended representation $\ddot{\mathbf{r}}$ consists of the analytic (endo)functors on presheaf categories of Fiore-Gambino-Hyland-Winskel



More Examples

1. Batanin-like context: take the Burroni fibration for the strict ω-category monad over the category of ω-graphs.
   NB. Without additional modifications the notion of analytic functor does not add anything new as the representation is already full on isomorphisms.
2. Kock-Gambino: diagrams (defining polynomial functors) in a lcc category $\mathcal{C}$ form a fibration over $\mathcal{C}$ that acts on basic fibration over $\mathcal{C}$. We get a representation by an exponential adjoint...
3. Joyal-Gambino...

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Thank You for Your Attention!