On phase semantics and denotational semantics: the exponentials

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January 25, 2001

Abstract

We extend to the exponential connectives of linear logic the study initiated in [BE00]. We define an indexed version of propositional linear logic and provide a sequent calculus for this system. To a formula A of indexed linear logic, we associate an underlying formula $\underline{A}$ of linear logic, and a family $\langle A \rangle$ of elements of $|\underline{A}|$, the interpretation of $\underline{A}$ in the category of sets and relations. Then A is provable in indexed linear logic iff the family $\langle A \rangle$ is contained in the interpretation of some proof of $\underline{A}$. We extend to this setting the product phase semantics of indexed multiplicative additive linear logic introduced in [BE00], defining the symmetric product phase spaces. We prove a soundness result for this truth-value semantics and show how a denotational model of linear logic can be associated to any symmetric product phase space.

Considering a particular symmetric product phase space, we obtain a new coherence space model of linear logic, which is *non-uniform* in the sense that the interpretation of a proof of $!A \multimap B$ contains informations about the behavior of this proof when applied to "chimeric" arguments of type A (for instance: booleans whose value can change during the computation). In this coherence semantics, an element of a web can be strictly coherent with itself, or two distinct elements can be "neutral" (that is, neither strictly coherent, nor strictly incoherent).
Keywords: linear logic, denotational semantics, phase semantics, coherence spaces.

This is a preliminary version of a paper which will appear in *Annals of Pure and Applied Logic*, North-Holland.

Introduction

Starting from a study of logical relations in a monoid-enriched coherence space model of linear logic, we arrived in [BE00] to the observation that, when logical relations satisfy certain conditions (closure under restriction), they can be faithfully described in terms of phase semantics, the truth-value semantics of linear logic. We observed also that, given a formula S of multiplicative additive linear logic, a J-indexed family of elements of $|S|$, the set interpreting S in the category of sets and relations[1], can itself be seen as a formula A of an *indexed* system of multiplicative additive linear logic. In this system, a formula has a *domain*: here, the domain of A is J, the set of indices of the corresponding family of points of $|S|$. The formula S itself can be retrieved from A, by forgetting all domain informations. This system is designed in such a way that it has a natural truth-value semantics in the *product phase spaces*[2] introduced for describing logical relations[3]. The key property of this indexed system of multiplicative additive linear logic is that the provability of A is equivalent to the existence of a proof of S (in multiplicative additive linear logic) whose denotation in the model of sets and relations contains all the elements of the family of points of $|S|$ corresponding to A.

In the category of sets and relations, the formulae $!S$ and $?S$ are interpreted as the set $|!S| = |?S|$ of all finite multisets[4] of elements of $|S|$. So a reasonable idea is to extend the indexed system of multiplicative additive linear logic by adding exponentials, in such a way that the "key property" mentioned above remain true for this extended system, with respect to the denotational semantics of first order propositional linear logic in the category of sets and relations.

This is precisely what we do in the sections 1 and 2 of the present paper, assuming once and for all given a global set of indices I which is infinite (and denumerable). Our definition of this indexed logic $\mathsf{LL}(I)$ is based on the following observation. Given a set $K \subseteq I$ and a formula S of linear logic, a K-indexed family $\xi = (\xi_k)_{k\in K}$ of elements of $|!S| = |?S|$ can be described as follows: for each $k \in K$, it suffices to specify a finite subset J_k of I and an enumeration $\alpha^k \in |S|^{J_k}$ of the elements of ξ_k (taking repetitions into account). This can be done in such a way that the sets J_k be pairwise disjoint. If we call J the disjoint union of the sets J_k, and if we denote by α the element of $|S|^J$ obtained by "gluing together" the families α^k and by u the function $J \to K$ which to each $j \in J$ associates the unique $k \in K$ such that $j \in J_k$, we see that the family $(J_k, \alpha^k)_{k\in K}$ which describes the family ξ can also be presented as follows: it suffices to specify a set $J \subseteq I$, an element α of $|S|^J$ and a function $u : J \to K$ in such a way that, for each $k \in K$, the restriction of α to $u^{-1}(k)$ be an enumeration of the multiset ξ_k. Due to the finiteness of the multisets ξ_k, the function u satisfies the following property: for all $k \in K$, the set $u^{-1}(k)$ is finite. Such a function u will be called an *almost injective* function in the sequel and such a pair (α, u) will be called a *representative* of ξ. Of course, ξ admits in general an infinity of different such representatives (α, u). If we admit that we have been able to represent the J-indexed family α of elements of $|S|$ by a formula A of domain J in the system $\mathsf{LL}(I)$ that we aim at defining, then it is natural to accept in this system both formulae $!_u A$ and $?_u A$ of domain K, representing the K-indexed family ξ of elements of $|!S| = |?S|$. So the system $\mathsf{LL}(I)$ is defined exactly like the system $\mathsf{MALL}(I)$, with two additional ways of building indexed formulae: if $J, K \subseteq I$, if A is a formula of domain J and u is an almost injective function from J to K, then $!_u A$ and $?_u A$ are formulae of domain K. To a formula A of domain J, we can associate easily an underlying formula $\underline{A}$ of linear logic, as well as a J-indexed family $\langle A\rangle$ of elements of $|\underline{A}|$. For any formula S of *multiplicative additive* linear logic and any set J, there is a bijection between the J-indexed families α of elements of $|S|$ and

[1]This category is an extremely simple model of linear logic, where the orthogonal of an object is this object itself.

[2]A product phase space is a phase space of the shape $(P_0^I, \bot)$ where P_0 is a commutative monoid which has an absorbing element 0, P_0^I is the I-product of P_0 equipped with its product monoid structure induced by the monoid structure of P_0, and $\bot$ is a non-empty subset of P_0^I subject to the following condition: if $p \in \bot$, any element of P_0^I obtained by replacing some components of p by 0 must belong to $\bot$.

[3]We also proved completeness for an extension of this notion of phase space.

[4]It is an interesting piece of folklore that in this pure relational setting, one cannot replace multisets by sets like in coherence semantics, where the exponentials admit two natural interpretations: the usual one, where the web of $!E$ is the set of all finite cliques of E, and the "co-free" one, where the web of $!E$ is the set of all finite multicliques of E.

the formulae A of $\mathsf{MALL}(I)$ whose common domain is J and which satisfy $\underline{A} = S$. It is no more the case in $\mathsf{LL}(I)$, but one checks easily that any J-indexed family of $|S|$ can be represented by at least one formula A of domain J with $\underline{A} = S$. We extend the sequent calculus $\mathsf{MALL}(I)$ to $\mathsf{LL}(I)$ in such a way that the "key property" still holds for this extension: given a formula S of linear logic and a J-indexed family α of elements of $|S|$, there exists a proof of S in linear logic whose denotation contains the range of α if and only if there exists a formula A of $\mathsf{LL}(I)$ with $\underline{A} = S$ and $\langle A \rangle = \alpha$ and which is provable in this new sequent calculus (and then all such formulae A are provable). The rules of $\mathsf{LL}(I)$ are indexed versions of the usual rules of the linear sequent calculus. The "key property" expresses the fact that each rule of $\mathsf{LL}(I)$ describes in a proof-theoretic way the denotational interpretation of the corresponding rule of the linear sequent calculus in its purely relational model.

Then, in section 3, we consider the interpretation of the formulae of $\mathsf{LL}(I)$ in product phase spaces. Given a product phase space $M = (P_0^I, \perp)$, one defines for each $J \subseteq I$ a local space $M(J) = (P_0^J, \perp(J))$ where $\perp(J)$ is obtained by projecting $\perp$ on P_0^J. Then a formula of domain J is interpreted as a fact of the local space $M(J)$. For interpreting the exponentials, one observes that, given an almost injective function $u : J \to K$, one can define a monoid morphism $u_* : P_0^J \to P_0^K$ by setting, for $p \in P_0^J$, $(u_*(p))_k = \prod_{j \in u^{-1}(k)} p_j$, this definition making sense precisely because u is almost injective. Then one interprets the formula $!_u A$ as $(u_* F)^{\perp\perp}$, where F is the fact interpreting the formula A in $M(J)$. We prove the soundness theorem which states that, if the formula A of domain J is provable in $\mathsf{LL}(I)$, then the fact interpreting A in the local space $M(J)$ contains the unit of the monoid P_0^J. For proving this result, it is crucial to impose to the product phase space $M = (P_0^I, \perp)$ an additional *symmetry* condition which, roughly speaking, states that $\perp$ has to be invariant under all permutations of I.

Next, in section 4, given a symmetric product phase space $M = (P_0^I, \perp)$, we construct a category of *M-spaces* and endow this category with the structures required for being a denotational model of linear logic. An M-space is a pair $X = (|X|, \widehat{X})$ where $|X|$ is a set (the *web*), and $\widehat{X} = (\widehat{X}_J)_{J \subseteq I}$ is a family of mappings $\widehat{X}_J$ from the J-indexed families of elements of $|X|$ to the facts of the local phase space $M(J)$. We require this family to be natural in J, with respect to *injective* reindexing[5]. Then it is possible to define, for each logical connective of linear logic, a corresponding operation on M-spaces. On the webs, this operation is simply the corresponding operation in the plain relational semantics of linear logic: disjoint sums for the additives, cartesian product for the multiplicatives and set of all finite multisets for the exponentials. As to the natural transformations, the idea is to perform simply the corresponding operation in the phase space M, the only interesting case being the exponential case. Given a family $\xi = (\xi_k)_{k \in K}$ of finite multisets of elements of $|X|$, we take an arbitrary representative (α, u) of ξ (with $\alpha \in |X|^J$ and $u : J \to K$ almost injective, for some $J \subseteq I$). Then, thanks to the naturality of $\widehat{X}$, the fact $!_u(\widehat{X}_J(\alpha))$ *does not depend* on the arbitrary choices of J, u and α, but only on ξ, so that it makes sense to set $\widehat{!X}_K(\xi) = !_u(\widehat{X}_J(\alpha))$. We obtain a category of M-spaces by defining a notion of *clique* of an M-space X: it is a subset x of $|X|$ such that, for any family $\alpha \in |X|^J$ of elements of x, the fact $\widehat{X}_J(\alpha)$ contains the unit of the local monoid P_0^J. Then the morphisms from X to Y are defined in the usual way as cliques of $X \multimap Y$.

We describe last a concrete example of this general construction of a categorical model of linear logic from a symmetric product phase space, focusing our attention on the monoid $P_0 = \{0, 1, \tau\}$ with τ satisfying $\tau\tau = \tau$. We define $\perp$ as the set of all elements p of P_0^I for which, if $p_j = \tau$, then, $p_i = 0$ for all $i \neq j$. In this special setting, we show that the facts of $M(J)$ can be described

[5]Indeed, reindexing a fact of $M(J)$ by a function $u : K \to J$ gives rise to a subset of P_0^K which is generally not a fact, unless u is injective.

as three-valued symmetric and anti-reflexive graphs with J as set of vertices. Then M-spaces can be described as some new kind of coherence spaces where two elements of the web can have three different kinds of relation between them: coherence, neutrality and incoherence. The only requirement on such a *non-uniform coherence space* is symmetry, and no kind of reflexivity is necessary: a point of the web can be neutral, coherent or incoherent with itself. The cliques in these spaces are the obvious generalization to this setting of the usual notion of clique in a coherence space (in particular, a singleton $\{a\}$ is not a clique if a is incoherent with itself). We describe, for each connective of linear logic, the corresponding operation on non-uniform coherence spaces, and observe that they are completely similar to the usual ones, but for the exponentials, which are of a different nature.

As already quoted in [BE00], the present work bears some similarities with a previous work by Lamarche [Lam95], who had the idea of generalizing coherence spaces and hypercoherences to a setting where coherence is not simply a boolean valued predicate, but a predicate taking its values in a quantale (a generalization of phase spaces). The main difference between the present approach and Lamarche's constructions is that in our work, instead of associating truth values to *sets* of points of the webs, we associate truth values to *families* thereof. Moreover, a symmetric product phase space has an additional "horizontal" structure: its underlying monoid is the I-product of a "1-dimensional" monoid P_0. This allows facts to be located at different places (sets of indexes) in the phase space, a crucial feature for our interpretation of the additives, where the facts to be combined must be located at disjoint places. Winskel in [Win94] considered a variation on the theme of hypercoherences where coherent sets were replaced by coherent families. He obtained in that way a model of intuitionistic linear logic (with a non-involutive negation). The notion of M-space introduced in the present work can probably be seen as a phase-parameterized and logically symmetrized version of Winskel's hypercoherences. The importance of localization in the present setting is reminiscent of analogous phenomena in Girard's *ludics* [Gir99, Gir00b, Gir00a]. For instance, in both settings, the "with" connective of linear logic can be considered as an intersection or as a cartesian product. However, the precise connection between ludics and indexed linear logic is yet to be explored. Also, M-spaces, which are webs (sets) endowed with relations of varying arity subject to a naturality condition, present formal similarities with the setting of Kripke logical relations of [JT93, OR95], although the precise connection, if any, is not clear yet.

The present paper requires from the reader a general knowledge of the phase semantics and of the denotational semantics of linear logic, basic references for these topics being [Gir87, GLT89, Gir95, AC98]. For a better understanding of the underlying intuitions, we advise the reader to have a look at [BE00].

1 Indexed linear logic

For us, a function is a triple (J, K, u) (notation $u : J \to K$) where J and K are sets (the domain and codomain of u) and u is a total functional relation on $J \times K$. Observe that u is not necessarily surjective onto its codomain K.

Let J and K be two sets. A function $u : J \to K$ is *almost injective* if, for any $k \in K$, the set $u^{-1}(k)$ is finite.

Let I be an infinite denumerable set.

If E and F are sets and $\alpha \in E^J$ and $\beta \in F^J$ (for some $J \subseteq I$), we denote by (α, β) the element of $(E \times F)^J$ given by $(\alpha, \beta)_j = (\alpha_j, \beta_j)$ for each $j \in J$. If L and R are disjoint sets, we denote by $L + R$ their union. If $L, R \subseteq I$ are disjoint and if $\alpha \in E^L$ and $\beta \in E^R$, we denote by $\alpha + \beta$ the element of E^{L+R} defined by case: $(\alpha + \beta)(l) = \alpha(l)$ if $l \in L$ and $(\alpha + \beta)(r) = \beta(r)$ if $r \in R$.

The logical system $\mathsf{LL}(I)$ is defined as follows. Each formula A has a domain $d(A)$, which is a subset of I.

- The constants $\top$ and 0 are formulae of empty domain.
- If $J \subseteq I$, the constants $\bot_J$ and 1_J are formulae of domain J.
- If A and B are formulae of domain $J \subseteq I$, then $A \otimes B$ and $A \parr B$ are formulae of domain J.
- If A is a formula of domain J and B is a formula of domain K, with $J \cap K = \emptyset$, then $A \oplus B$ and $A \mathbin{\&} B$ are formulae of domain $J + K$ (the disjoint union of J and K).
- If A is a formula of domain J and $u : J \to K$ is an almost injective function, then $!_u A$ and $?_u A$ are formulae of domain K.

The *orthogonal* $A^\bot$ of a formula A of domain J is the formula of domain J obtained by applying recursively the usual De Morgan laws between dual connectives, for instance $(!_u A)^\bot = ?_u(A^\bot)$.

A *sequent* of $\mathsf{LL}(I)$ is an expression of the shape $\vdash_J \Gamma$ where J is a subset of I and Γ is a (possibly empty) sequence $(A_1, \ldots, A_n)$ of formulae of $\mathsf{LL}(I)$ such that each A_i has domain J (a sequence Γ of formulae satisfying this condition will sometimes be called *homogeneous*, and we shall denote by $d(\Gamma)$ the common domain of the elements of Γ, when Γ is not empty).

If A is a formula of $\mathsf{LL}(I)$ with $d(A) = J$, and if $K \subseteq I$, we define the *restriction* of A by K, denoted by $A|_K$, which is a formula of $\mathsf{LL}(I)$ with domain $J \cap K$, as follows:

- $\top|_K = \top$ and $0|_K = 0$.
- $\bot_J|_K = \bot_{J\cap K}$ and $1_J|_K = 1_{J\cap K}$.
- $(A \otimes B)|_K = A|_K \otimes B|_K$, $(A \parr B)|_K = A|_K \parr B|_K$, $(A \oplus B)|_K = A|_K \oplus B|_K$ and $(A \mathbin{\&} B)|_K = A|_K \mathbin{\&} B|_K$.
- $(!_u A)|_K = !_v(A|_{u^{-1}(K\cap J)})$ where $v : u^{-1}(K \cap J) \to K \cap J$ is obtained by co-restricting u. The definition of $(?_u A)|_K$ is similar.

If $\Gamma = (A_1, \ldots, A_n)$ is an homogeneous sequence of formulae, one defines $\Gamma|_K = (A_1|_K, \ldots, A_n|_K)$ so that again, $d(\Gamma|_K) = d(\Gamma) \cap K$. Last, observe that trivially $A^\bot|_K = (A|_K)^\bot$.

Lemma 1 *Let A be a formula of $\mathsf{LL}(I)$ and let K and L be subsets of I. Then $(A|_K)|_L = A|_{K\cap L}$.*

The proof is a straightforward induction.

When $u : J \to K$ is a *bijection*, one can define, for each formula A of domain J, a formula $u_* A$ of domain K, as follows:

- $u_* \top = \top$ and $u_* 0 = 0$.
- $u_* \bot_J = \bot_K$ and $u_* 1_J = 1_K$.
- $u_*(A \otimes B) = u_* A \otimes u_* B$ and $u_*(A \parr B) = u_* A \parr u_* B$.
- $u_*(A \oplus B) = v_* A \oplus w_* B$ where $v : d(A) \to u(d(A))$ and $w : d(B) \to u(d(B))$ are obtained by restricting u (observe that $K = u(d(A)) + u(d(B))$ as u is bijective). The formula $u_*(A \mathbin{\&} B)$ is defined in a similar way.

- $u_*(!_v A) = !_{u \circ v} A$ and $u_*(?_v A) = ?_{u \circ v} A$.

We describe a sequent calculus for these sequents.

We have the following axioms:

$$\frac{}{\vdash_J 1_J}$$

and

$$\frac{}{\vdash_\emptyset \Gamma, \top}$$

this latter making sense only under the assumption that Γ is empty, or has empty domain.

The multiplicative rules are without surprises.

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, \bot_J}$$

$$\frac{\vdash_J \Gamma, A \quad \vdash_J \Delta, B}{\vdash_J \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash_J \Gamma, A, B}{\vdash_J \Gamma, A \parr B}$$

In the introduction rules for $\oplus$, observe that B must have an empty domain.

$$\frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, A \oplus B} \qquad \frac{\vdash_J \Gamma, A}{\vdash_J \Gamma, B \oplus A}$$

Next we give the introduction rule for &. Assume that $d(A) = L$, $d(B) = R$ with $L \cap R = \emptyset$, and that $d(\Gamma) = L + R$.

$$\frac{\vdash_L \Gamma|_L, A \quad \vdash_R \Gamma|_R, B}{\vdash_{L+R} \Gamma, A \mathbin{\&} B}$$

We give now the exponential rules. For A a formula of empty domain, 0_J denoting the empty function from $\emptyset$ to J, the weakening rule is the following:

$$\frac{\vdash_J \Gamma}{\vdash_J \Gamma, ?_{0_J} A}$$

For A a formula of domain J, u an almost injective function from J to K, J_1 and J_2 two subsets of J such that $J = J_1 + J_2$, u_i (for $i = 1, 2$) the almost injective function $J_i \to K$ obtained by restricting u to J_i, the contraction rule is the following:

$$\frac{\vdash_K \Gamma, ?_{u_1}(A|_{J_1}), ?_{u_2}(A|_{J_2})}{\vdash_K \Gamma, ?_u A}$$

Let $u : J \to K$ be a *bijection*. The dereliction rule is the following:

$$\frac{\vdash_K \Gamma, u_* A}{\vdash_K \Gamma, ?_u A}$$

Let $(A_i)_{i=1,\dots,n}$ be a family of formulae and let J_i be the domain of A_i. Let K be a subset of I and, for each $i = 1,\dots,n$, let u_i be an almost injective function from J_i to K. Let A be a formula of domain K and let $v : K \to L$ be an almost injective function. The promotion rule is the following:

$$\frac{\vdash_K ?_{u_1}A_1,\dots,?_{u_n}A_n, A}{\vdash_L ?_{v\circ u_1}A_1,\dots,?_{v\circ u_n}A_n, !_v A}$$

which makes sense, because the composite of two almost injective functions is almost injective.

The only structural rule is the exchange rule, which is

$$\frac{\vdash_J A_1,\dots,A_n}{\vdash_J A_{\sigma(1)},\dots,A_{\sigma(n)}}$$

where σ is any permutation of $\{1,\dots,n\}$.

Last, the cut rule is standard.

$$\frac{\vdash_J \Gamma, A \quad \vdash_J \Delta, A^\perp}{\vdash_J \Gamma, \Delta}$$

To any formula A of $\mathsf{LL}(I)$, we can associate in an obvious way a formula of linear logic, simply by forgetting all the indexing sets and functions. We denote by $\underline{A}$ this formula of linear logic.

2 The relational denotational model of linear logic

The category of sets and relations is a (compact) model of linear logic, the various connectives corresponding to the following operations on sets. Let X and Y be sets.

- $0 = \top = \emptyset$.
- $\perp = 1 = \{*\}$ where $*$ is an arbitrary distinguished element.
- $X^\perp = X$.
- $X \mathbin{\&} Y = X \oplus Y = (\{1\} \times X) \cup (\{2\} \times Y)$ is the disjoint union of X and Y.
- $X \otimes Y = X \mathbin{⅋} Y = X \times Y$.
- $!X = ?X$ is the set of all finite multisets of elements of X.

In that way, one associates to each formula S of linear logic a set $|S|$. If $\Phi = (S_1,\dots,S_n)$ is a sequence of formulae of linear logic, one defines $|\Phi| = |S_1| \times \cdots \times |S_n|$.

To each proof π of a sequent $\vdash \Phi$ of linear logic, one associates a subset π^* of $|\Phi|$. This is done exactly like in the coherence semantics of linear logic (see [Gir87, AC98]), except that here, when interpreting the contraction and promotion rules, there is no coherence restriction in building multisets. We recall this interpretation of proofs in section 6.

If J is a finite set and α is a J-indexed family of elements of a set X, we denote by $\mathsf{m}(\alpha)$ the multiset of elements of X which maps each element of X to its number of occurrences in α:

$$\mathsf{m}(\alpha)(a) = \#\{j \in J \mid \alpha_j = a\}\,,$$

that is, the multiset obtained by forgetting the indexes in the enumeration α. If α is a J-indexed family of elements of a set X, and if $K \subseteq J$, we denote by $\alpha|_K$ the K-indexed family obtained by restricting α to K.

To any formula A of $\mathsf{LL}(I)$, one associates an element $\langle A\rangle$ of $|\underline{A}|^{d(A)}$ as follows.

- $\langle 0 \rangle = \langle \top \rangle$ is the empty family.
- $\langle \bot_J \rangle = \langle 1_J \rangle$ is the J-indexed family which is constantly equal to $*$.
- $\langle A \,\&\, B \rangle = \langle A \oplus B \rangle = \langle A \rangle + \langle B \rangle = \gamma$ where we recall that

$$\gamma_j = \begin{cases} (1, \langle A \rangle_j) & \text{if } j \in d(A) \\ (2, \langle B \rangle_j) & \text{if } j \in d(B) \end{cases}$$

- $\langle A \otimes B \rangle = \langle A \parr B \rangle = (\langle A \rangle, \langle B \rangle) = \gamma$ where we recall that $\gamma_j = (\langle A \rangle_j, \langle B \rangle_j)$ for all $j \in d(A) = d(B)$.
- $\langle !_u A \rangle_j = \langle ?_u A \rangle_j = \mathsf{m}(\langle A \rangle|_{u^{-1}(j)})$ which is well defined as u is almost injective.

When $\Gamma = (A_1, \ldots, A_n)$ is a vector of formulae of $\mathsf{LL}(I)$ of domain J, one defines $\gamma = \langle \Gamma \rangle \in |\underline{\Gamma}|^J$ as

$$\gamma_j = (\langle A_1 \rangle_j, \ldots, \langle A_n \rangle_j) \ .$$

Lemma 2 *Let S be a formula of linear logic, J be a subset of I, and let $\alpha \in |S|^J$. Then there exists a formula A of domain J of $\mathsf{LL}(I)$ such that $\underline{A} = S$ and $\langle A \rangle = \alpha$.*

The proof is a straightforward induction, using the obvious fact that, for any finite multiset m of a set X, one can find a finite subset K of I and an element β of X^K such that $\mathsf{m}(\beta) = m$.

Lemma 3 *Let Γ be an homogeneous vector of formulae of $\mathsf{LL}(I)$ of domain J.*

- *Let $K \subseteq J$. Then $\langle \Gamma|_K \rangle = \langle \Gamma \rangle|_K$.*
- *Let $u : J \to L$ be a bijection. Then $\langle u_* \Gamma \rangle_l = \langle \Gamma \rangle_{u^{-1}(l)}$ for each $l \in L$.*

Proof: Straightforward induction. We deal here only with the first part of the lemma, in the exponential case. So let A be a formula of domain L and let $u : L \to J$ be a function. Let $L' = u^{-1}(K)$ and let $v : L' \to K$ be the restriction of u to L'. By definition,

$$(!_u A)|_K = !_v (A|_{L'}) \ .$$

Let $\xi = \langle (!_u A)|_K \rangle$ and let $k \in K$. We have

$$\begin{aligned} \xi_k &= \mathsf{m}(\langle A|_{L'} \rangle|_{v^{-1}(k)}) \\ &= \mathsf{m}((\langle A \rangle|_{L'})|_{v^{-1}(k)}) \quad \text{by inductive hypothesis} \\ &= \mathsf{m}(\langle A \rangle|_{v^{-1}(k)}) \quad \text{since } v^{-1}(k) \subseteq L' \\ &= \mathsf{m}(\langle A \rangle|_{u^{-1}(k)}) \\ &= \langle !_u A \rangle_k \end{aligned}$$

and we are done. ■

Proposition 4 *Let Φ be a vector of formulae of linear logic. Let π be a proof (resp. a cut-free proof) of $\vdash \Phi$, let J be a subset of I and let $\varphi \in (\pi^*)^J$. Let Γ be any vector of formulae of $\mathsf{LL}(I)$ of domain J such that $\underline{\Gamma} = \Phi$ and $\langle \Gamma \rangle = \varphi$. Then $\vdash_J \Gamma$ is provable (resp. cut-free provable) in $\mathsf{LL}(I)$.*

Proof: By induction on π. The induction steps associated to the rules and axioms dealing with constants are straightforward. The multiplicative rules are easy too. We consider only the additive and exponential rules, as well as the cut rule.

Assume that the proof ends with

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Psi, T\end{array}}{\vdash \Psi, S \,\&\, T}$$

Then φ can be written $\varphi = (\psi, \gamma)$ where $\psi \in |\Psi|^J$ and $\gamma \in |S \,\&\, T|^J = (|S| + |T|)^J$. We can write, in a unique way, $\gamma = \alpha + \beta$ with $\alpha \in |S|^L$ and $\beta = |T|^R$, for two subsets L and R of J such that $L + R = J$ and $(\psi|_L, \alpha) \in ({\pi_1}^*)^K$ and $(\psi|_R, \beta) \in ({\pi_2}^*)^L$. Since $\underline{\Gamma} = (\Psi, S \,\&\, T)$, we have $\Gamma = (\Delta, A \,\&\, B)$, with $\underline{\Delta} = \Psi$, $\underline{A} = S$ and $\underline{B} = T$. By hypothesis, $\langle \Gamma \rangle = (\psi, \gamma)$, so by lemma 3 we get

$$\langle \Delta|_L, A \rangle = (\psi|_L, \alpha) \quad \text{and} \quad \langle \Delta|_R, B \rangle = (\psi|_R, \beta)\ .$$

By inductive hypothesis, the sequents $\vdash_L \Delta|_L, A$ and $\vdash_R \Delta|_R, B$ are provable and so, applying a &-rule, the sequent $\vdash_J \Gamma$ is provable.

Assume that the proof ends with a left plus rule:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array}}{\vdash \Psi, S \oplus T}$$

Then φ can be written $\varphi = (\psi, \gamma)$ where $\varphi \in |\Gamma|^J$ and $\gamma \in |S \oplus T|^J$ is such that there is $\alpha \in |S|^J$ with $\gamma_j = (1, \alpha_j)$ for each $j \in J$ and $(\gamma, \alpha) \in ({\pi_1}^*)^J$. As above, one has $\Gamma = (\Delta, A \oplus B)$ with $\underline{\Delta} = \Psi$, $\langle \Delta \rangle = \psi$, $\underline{A} = S$, $\langle A \rangle = \alpha$, $\underline{B} = T$ and $\langle B \rangle = \emptyset$. By inductive hypothesis, the sequent $\vdash_J \Gamma, A$ is provable, and so, applying a left $\oplus$-rule, the sequent $\vdash_J \Gamma, A \oplus B$ is provable.

Assume that the proof ends with a weakening rule:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi\end{array}}{\vdash \Psi, ?S}$$

Then φ can be written $\varphi = (\psi, \zeta)$ where $\psi \in |\Psi|^J$ and $\zeta \in |?S|^J$ is defined by $\zeta_j = []$ (the empty multiset) for each $j \in J$. As above, one has $\Gamma = (\Delta, ?_u A)$ with

$$\underline{\Delta} = \Psi \quad \text{and} \quad \langle \Delta \rangle = \psi$$

and

$$\underline{A} = S \quad \text{and} \quad \langle ?_u A \rangle = \zeta$$

for some almost injective function $u : d(A) \to J$. For each $j \in J$, we must have $\mathsf{m}(\langle A \rangle|_{u^{-1}(j)}) = []$, that is, $u^{-1}(j)$ must be the empty set. So $d(A) = \emptyset$ and $u = 0_J$ (the unique function from $\emptyset$ to J, whose graph is empty). We know moreover that $\psi \in ({\pi_1}^*)^J$ so that by inductive hypothesis, the sequent $\vdash_J \Delta$ is provable. We conclude by applying a weakening rule in $\mathsf{LL}(I)$.

Assume that the proof ends with a contraction rule:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, ?S, ?S\end{array}}{\vdash \Psi, ?S}$$

Then φ can be written $\varphi = (\psi, \zeta)$ where $\psi \in |\Psi|^J$, and $\zeta \in |?S|^J$ is such that each ζ_j can be written $\zeta_j = \xi_j + \xi'_j$ (sum of multisets) in such a way that the J-indexed family (ψ, ξ, ξ') (whose j-th element is (ψ_j, ξ_j, ξ'_j)) belongs to $(\pi_1{}^*)^J$. As above, one has $\Gamma = (\Delta, ?_u A)$ with

$$\underline{\Delta} = \Psi \quad \text{and} \quad \langle \Delta \rangle = \psi$$

and

$$\underline{A} = S \quad \text{and} \quad \langle ?_u A \rangle = \zeta$$

for some almost injective function $u : d(A) \to J$. For each $j \in J$, we must have $\mathsf{m}(\langle A \rangle|_{u^{-1}(j)}) = \xi_j + \xi'_j$. So there exist two disjoint subsets L_j and R_j of $u^{-1}(j)$ such that $L_j + R_j = u^{-1}(j)$ and

$$\mathsf{m}(\langle A \rangle|_{L_j}) = \xi_j \quad \text{and} \quad \mathsf{m}(\langle A \rangle|_{R_j}) = \xi'_j \ ,$$

that is (using lemma 3)

$$\mathsf{m}(\langle A|_{L_j} \rangle) = \xi_j \quad \text{and} \quad \mathsf{m}(\langle A|_{R_j} \rangle) = \xi'_j \ .$$

We set

$$L = \bigcup_{j \in J} L_j \quad \text{and} \quad R = \bigcup_{j \in J} R_j$$

(observe that these unions are disjoint and that $L \cap u^{-1}(j) = L_j$ for all $j \in J$, and similarly for R). Let $v : L \to J$ be obtained by restricting u to L and $w : R \to J$ be obtained by restricting u to R. Then we clearly have

$$\langle ?_v(A|_L) \rangle = \xi \quad \text{and} \quad \langle ?_w(A|_R) \rangle = \xi' \ .$$

By inductive hypothesis, the sequent $\vdash_J \Delta, ?_v(A|_L), ?_w(A|_R)$ is provable and we conclude that the sequent $\vdash_J \Gamma$ is provable by applying a contraction rule in $\mathsf{LL}(I)$.

Assume that the proof ends with a dereliction rule:

$$\dfrac{\begin{array}{c} \vdots\ \pi_1 \\ \vdash \Psi, S \end{array}}{\vdash \Psi, ?S}$$

Then φ can be written $\varphi = (\psi, \zeta)$ where $\psi \in |\Psi|^J$ and $\zeta \in |?S|^J$ is such that each ζ_j is a singleton multiset, that is $\zeta_j = [\alpha_j]$ for some $\alpha \in |S|^J$. Moreover, the J-indexed family (φ, α) belongs to $(\pi_1{}^*)^J$. As above, one has $\Gamma = (\Delta, ?_u A)$ with

$$\underline{\Delta} = \Psi \quad \text{and} \quad \langle \Delta \rangle = \psi$$

and

$$\underline{A} = S \quad \text{and} \quad \langle ?_u A \rangle = \zeta$$

for some almost injective function $u : d(A) \to J$. For each $j \in J$, we must have $\mathsf{m}(\langle A \rangle|_{u^{-1}(j)}) = [\alpha_j]$. So $u^{-1}(j)$ must be a singleton for each $j \in J$, and this means that u is a bijection[6] from $d(A)$ to J. So $u_* A$ is a formula of domain J, and one has $\langle u_* A \rangle_j = \alpha_j$ for each $j \in J$, by lemma 3. Hence, by inductive hypothesis, the sequent $\vdash_J \Delta, u_* A$ is provable, and we conclude, applying a dereliction rule in $\mathsf{LL}(I)$.

[6]This explains our choice for the indexed version of the promotion rule: the most obvious solution would have been to introduce $?_{\mathrm{Id}} A$ instead of $?_u A$ (for an arbitrary bijection u), but then this step of the proof of proposition 4 would have been problematic.

Assume that the proof ends with a promotion rule:

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash ?S^1,\ldots,?S^n,S\end{array}}{\vdash ?S^1,\ldots,?S^n,!S}$$

Then φ can be written $\varphi = (\xi^1,\ldots,\xi^n,\xi)$ where, for $l = 1,\ldots,n$, $\xi^l = (\xi^l_j)_{j\in J} \in |?S^l|^J$ and $\xi = (\xi_j)_{j\in J} \in |!S|^J$. Since $\underline{\Gamma} = (?S^1,\ldots,?S^n,!S)$, the sequence of formulae Γ is of the shape

$$\Gamma = (?_{u_1}A^1,\ldots,?_{u_n}A^n,!_v A)\,.$$

Let us denote by L^l the domain of A^l (for $l = 1,\ldots,n$) and by K the domain of A, so that $u_l : L^l \to J$ and $v : K \to J$. For each $j \in J$, let us set

$$L^l_j = u_l^{-1}(j) \quad \text{for } l = 1,\ldots,n, \text{ and} \quad K_j = v^{-1}(j)\,.$$

We know that

$$\mathsf{m}(\langle A^l\rangle|_{L^l_j}) = \xi^l_j \quad \text{for } l = 1,\ldots,n, \text{ and} \quad \mathsf{m}(\langle A\rangle|_{K_j}) = \xi_j\,.$$

Let $\alpha = \langle A\rangle \in |S|^K$. By definition of the denotation of proofs, for each $j \in J$ and each $l = 1,\ldots,n$, we can write ξ^l_j as a sum of multisets indexed by K_j, $\xi^l_j = \sum_{k\in K_j}\xi^l_{j,k}$, in such a way that, for each $k \in K_j$, $(\xi^1_{j,k},\ldots,\xi^n_{j,k},\alpha_k) \in {\pi_1}^*$. Each set L^l_j can thus be written as a disjoint union $L^l_j = \sum_{k\in K_j} L^l_{j,k}$ in such a way that $\mathsf{m}(\langle A^l\rangle|_{L^l_{j,k}}) = \xi^l_{j,k}$ for each $k \in K_j$. For $l = 1,\ldots,n$, let $w_l : L^l \to K$ be the function which to $r \in L^l$ associates the unique $k \in K$ such that $r \in L^l_{v(k),k}$. Then we clearly have $v \circ w_l = u_l$. To conclude observe that

$$\langle(?_{w_1}A^1,\ldots,?_{w_n}A^n,A)\rangle = (\xi^1_{v(k),k},\ldots,\xi^n_{v(k),k},\alpha_k)_{k\in K} \in ({\pi_1}^*)^K$$

so that by inductive hypothesis the sequent $\vdash_K\ ?_{w_1}A^1,\ldots,?_{w_n}A^n,A$ is provable. Applying a promotion rule in $\mathsf{LL}(I)$, we conclude that the sequent $\vdash_J\ ?_{u_1}A^1,\ldots,?_{u_n}A^n,!_v A$ is provable.

The case where the proof ends with an exchange rule is trivial.

Last, assume that the proof ends with a cut rule

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Phi,S\end{array}\qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Psi,S^\perp\end{array}}{\vdash \Phi,\Psi}$$

then $\varphi = (\chi,\psi)$ where $\chi \in |\Phi|^J$ and $\psi \in |\Psi|^J$, and we know that there exists a family $\alpha \in |S|^J$ such that $(\chi,\alpha) \in ({\pi_1}^*)^J$ and $(\psi,\alpha) \in ({\pi_2}^*)^J$. Then $\Gamma = (\Delta,\Lambda)$ with $\underline{\Delta} = \Phi$, $\underline{\Lambda} = \Psi$, $\langle\Delta\rangle = \chi$ and $\langle\Lambda\rangle = \psi$. By lemma 2, there exists a formula A of $\mathsf{LL}(I)$ such that $\underline{A} = S$ and $\langle A\rangle = \alpha$. By inductive hypothesis, the sequents $\vdash_J \Delta, A$ and $\vdash_J \Lambda, A^\perp$ are provable, so, applying a cut rule in $\mathsf{LL}(I)$, the sequent $\vdash_J \Delta,\Lambda$ is provable. ∎

We now establish a converse to proposition 4.

Proposition 5 *Let Γ be a vector of formulae of $\mathsf{LL}(I)$ of domain J. To any proof (resp. cut-free proof) σ of the sequent $\vdash_J \Gamma$ in $\mathsf{LL}(I)$, one can associate a proof (resp. a cut-free proof) $\underline{\sigma}$ of the sequent $\vdash \underline{\Gamma}$ in LL such that $\langle\Gamma\rangle \in (\underline{\sigma}^*)^J$.*

Proof: Of course, $\underline{\sigma}$ is obtained by simply removing all domain informations (indexing sets and functions associated to the exponentials) occurring in the proof σ of $\vdash_J \Gamma$; one clearly obtains in that way a proof of $\vdash \underline{\Gamma}$. The proof of the proposition is just an essentially straightforward verification.

Let us just check the promotion case. So assume that

$$\Gamma = (?_{v \circ u_1} A^1, \ldots, ?_{v \circ u_n} A^n, !_v A)$$

with $d(A^l) = L^l$ (for $l = 1, \ldots, n$), $d(A) = K$ and $v : K \to J$, $u_l : L^l \to K$ almost injective functions. Let $S^l = \underline{A^l}$ and $S = \underline{A}$. Assume also that the proof σ of Γ ends with a promotion rule:

$$\frac{\begin{array}{c}\vdots\, \sigma_1 \\ \vdash_K ?_{u_1} A^1, \ldots, ?_{u_n} A^n, A\end{array}}{\vdash_J ?_{v \circ u_1} A^1, \ldots, ?_{v \circ u_n} A^n, !_v A}$$

Let $\varphi = \langle (?_{u_1} A^1, \ldots, ?_{u_n} A^n, A) \rangle \in |(?S^1, \ldots, ?S^n, S)|^K$. For each $k \in K$, φ_k is a vector $\varphi_k = (\varphi_k^1, \ldots, \varphi_k^n, \varphi_k')$ where $\varphi_k^l \in |?S^l|$ (for $l = 1, \ldots, n$) and $\varphi_k' \in |S|$.

Then, by inductive hypothesis, $\varphi \in (\underline{\sigma_1}^*)^K$. The proof $\underline{\sigma}$ in LL reads:

$$\frac{\begin{array}{c}\vdots\, \underline{\sigma_1} \\ \vdash ?S^1, \ldots, ?S^n, S\end{array}}{\vdash ?S^1, \ldots, ?S^n, !S}$$

Since $\varphi \in (\underline{\sigma_1}^*)^K$, we have $\psi \in (\underline{\sigma}^*)^J$ where ψ is given by $\psi_j = (\psi_j^1, \ldots, \psi_j^n, \psi_j')$ with, for each $j \in J$, $\psi_j' = \mathsf{m}((\varphi_k')_{k \in v^{-1}(j)})$ and $\psi_j^l = \sum_{k \in v^{-1}(j)} \varphi_k^l$ for each $l = 1, \ldots, n$. But it is easily checked that $\psi = \langle \Gamma \rangle$ and we are done. ∎

From these propositions, and from the cut elimination theorem of linear logic (see [Gir87, Gir95]), we derive a cheap proof of cut elimination for $\mathsf{LL}(I)$.

Proposition 6 *Let Γ be a sequence of formulae of $\mathsf{LL}(I)$ of domain J. If the sequent $\vdash_J \Gamma$ is provable in $\mathsf{LL}(I)$, it is cut-free provable in $\mathsf{LL}(I)$.*

Proof: Assume that $\vdash_J \Gamma$ is provable in $\mathsf{LL}(I)$, with possibly some uses of the cut rule. By proposition 5, there exists a proof π of $\vdash \underline{\Gamma}$ in linear logic such that $\langle \Gamma \rangle \in (\pi^*)^J$. Let π' be obtained from π by applying a cut-elimination procedure to π, so that $\pi'^* = \pi^*$. Then π' is a cut-free proof of $\vdash \underline{\Gamma}$ in linear logic, and $\langle \Gamma \rangle \in (\pi'^*)^J$, so by proposition 4, the sequent $\vdash_J \Gamma$ is cut-free provable in $\mathsf{LL}(I)$. ∎

3 Product phase spaces for $\mathsf{LL}(I)$

We recall first the following notations: if U, V are subsets of a monoid Q (with multiplicative notation), then UV denotes the set $\{qq' \mid q \in U \text{ and } q' \in V\}$, and if $q \in Q$, qV denotes the set $\{q\}V$.

Let $\mathcal{I}(I)$ be the category whose objects are the subsets of I and whose morphisms are the *injective* functions between them.

Given a set E, we denote by Fam_E the contravariant functor from **Set** to **Set** which to J associates the set of all J-indexed families of elements of E: $\mathrm{Fam}_E(J) = E^J$ and if $u : J \to K$,

$\mathrm{Fam}_E(u)(\alpha) = \alpha \circ u$, for any $\alpha \in E^K$. In the sequel, we shall only consider the restriction of this functor to $\mathcal{I}(I)$ (which is a small subcategory of **Set**), and we shall denote $\mathrm{Fam}_E(u)$ by u^*, according to a well established tradition. In particular, if J is a subset of K and u is the corresponding injection, the associated function u^* is just the obvious projection function $\pi_J : E^K \to E^J$. If $u : J \to K$ is an injection and if V is a subset of E^K, we set $u^*V = \{u^*(\alpha) \mid \alpha \in V\}$.

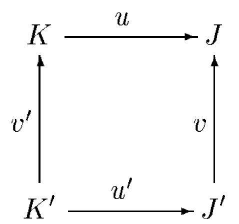
If Q is a monoid, and if each set Q^J is equipped with the product structure of monoid, then the functor Fam_Q becomes a contravariant functor from $\mathcal{I}(I)$ to the category of monoids and monoid homomorphisms.

Let J and K be two subsets of I and let $u : J \to K$ be an almost injective function from J to K. If Q is a *commutative* monoid, then one defines, in a covariant functorial way, a monoid morphism $u_* : Q^J \to Q^K$ as follows:

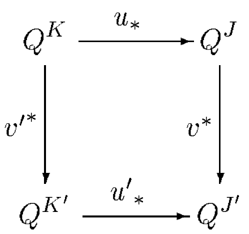
$$(u_*(p))_k = \prod_{j \in u^{-1}(k)} p_j$$

for $p \in Q^J$ and $k \in K$. In particular, when u is a bijection from J to K, one has $u_* = (u^{-1})^*$. When $U \subseteq Q^J$, we set $u_*U = \{u_*(p) \mid p \in U\}$. These two functorial actions are related by the following easy property.

Lemma 7 *If the following diagram in* **Set** *is a pull-back*



where $K, J, K', J' \subseteq I$ and where u is almost injective, then u' is almost injective, and the following diagram is commutative.



We shall show that any (symmetric) I-product phase space gives naturally rise to a model of $\mathsf{LL}(I)$, and we shall explore the connection between these phase spaces and the web-based denotational semantics of LL.

We recall that a product phase space M is given by a commutative monoid P_0 with an absorbing element 0, together with a subset $\perp$ of P_0^I. For $J \subseteq I$, we denote by ε_J the characteristic function of J, that is, the element of P_0^I given by

$$(\varepsilon_J)_i = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{otherwise} \end{cases}$$

We assume moreover that $\varepsilon_J \perp \subseteq \perp$ for each $J \subseteq I$ (this condition will be called "closure under restrictions") and that $\perp$ is not empty (that is, $0 \in \perp$). We recall then that any fact F (we recall below what a fact is) of the phase space $(P_0^I, \bot)$ satisfies the same closure property, namely that F is not empty and that $\varepsilon_J F \subseteq F$ for each $J \subseteq I$.

From now on, we assume given a product phase space $M = (P_0^I, \bot)$.

Let $J \subseteq I$. We denote by $\bot(J)$ the projection of $\bot$ on P_0^J, $\bot(J) = \pi_J(\bot)$. Then $(P_0^J, \bot(J))$ can in turn be considered as a product phase space, that we call the *local product phase space at J* associated to M, and denote by $M(J)$. We denote by 1^J the unit of the monoid P_0^J, that is $1_j^J = 1$ for each $j \in J$. When we use the notation $U^\bot$ for a subset U of P_0^J, we always mean that the orthogonal is taken in the local space $M(J)$, with respect to $\bot(J)$, that is $U^\bot = \{p' \in P_0^J \mid \forall p \in U,\ pp' \in \bot(J)\}$. We recall that a subset F of P_0^J is a fact if $F^{\bot\bot} = F$, and we recall also the following properties, which hold in any phase space, and that we shall use tacitly in the sequel. Let $U, V \subseteq P_0^J$.

- If $U \subseteq V$ then $V^\bot \subseteq U^\bot$.
- $U^{\bot\bot\bot} = U^\bot$.
- $(U \cup V)^\bot = U^\bot \cap V^\bot$.
- $(UV^{\bot\bot})^\bot = (UV)^\bot$.

In particular, $U^\bot$ is always a fact, and for showing that $U^{\bot\bot} \subseteq F$ (when F is a fact), it suffices to show that $U \subseteq F$.

Lemma 8 *Let $K \subseteq J \subseteq I$. If F is a fact of $M(J)$, then $\pi_K(F)$ is a fact of $M(K)$. Moreover, $(\pi_K(F))^\bot = \pi_K(F^\bot)$.*

Proof: It results easily from the closure under restriction condition fulfilled by $\bot$; see [BE00]. ■

We now show how to interpret a formula of $\mathsf{LL}(I)$ in M. More precisely, we interpret a formula A of domain J as a fact of $M(J)$. Rather than defining the interpretation of formulae by induction, we shall directly define the logical operations on facts, which obviously amounts to the same thing.

- If $J \subseteq I$, one defines $1(J) = \bot(J)^\bot$.
- $\top$ and 0 are defined as the only non-empty subset of $P_0^\emptyset$ (which is a singleton).
- If F and G are two facts of $M(J)$, then $F \otimes G = (FG)^{\bot\bot}$ and $F \parr G = (F^\bot G^\bot)^\bot$.
- Let L and R be two disjoint subsets of I. Let F be a fact of $M(L)$ and let G be a fact of $M(R)$. One defines

$$\begin{aligned} F \mathbin{\&} G &= \{p \in P_0^{L+R} \mid \pi_L(p) \in F \text{ and } \pi_R(p) \in G\} \\ &= \pi_L^{-1}(F) \cap \pi_R^{-1}(G) \end{aligned}$$

  where π_L and π_R are the projections from P_0^{L+R} to P_0^L and P_0^R respectively. This subset of P_0^{L+R} is indeed a fact of $M(L+R)$. Identifying P_0^{L+R} with $P_0^L \times P_0^R$, one has $F \mathbin{\&} G = F \times G$.

  Then one sets

$$F \oplus G = (F^\bot \mathbin{\&} G^\bot)^\bot \ .$$

  One defines $\zeta_L : P_0^L \to P_0^{L+R}$ by

$$\zeta_L(p)_i = \begin{cases} p_i & \text{if } i \in L \\ 0 & \text{otherwise} \end{cases}$$

Observe that this map *is not* a monoid morphism (it does not preserve the unit). And one defines similarly $\zeta_R : P_0^R \to P_0^{L+R}$. Then one easily checks that

$$F \oplus G = (\zeta_L(F) \cup \zeta_R(G))^{\perp\perp} \ .$$

Indeed, $(\zeta_L F)^\perp = \pi_L^{-1}(F^\perp)$.

- Let $J, K \subseteq I$ and let $u : J \to K$ be an almost injective function. Let F be a fact of $M(J)$. Then one sets

$$!_u F = (u_* F)^{\perp\perp} \quad \text{and} \quad ?_u F = (u_* F^\perp)^\perp \ .$$

In that way, we associate to any formula A of $\mathsf{LL}(I)$ of domain $J \subseteq I$ a fact $A^\bullet$ of $M(J)$. If $\Gamma = (A^1, \ldots, A^n)$ is a sequence of formulae having all the same domain $J \subseteq I$, one defines as usual its semantics as a fact of $M(J)$ by $\Gamma^\bullet = (A^1)^\bullet \parr \ldots \parr (A^n)^\bullet$.

Lemma 9 *Let L, R and J be subsets of I such that L and R are disjoint. Let $l : L \to J$ and $r : R \to J$ be almost injective functions. Let F be a fact of $M(L)$ and G be a fact of $M(R)$. Then*

$$!_l F \otimes !_r G = !_{l+r}(F \,\&\, G)$$

where $l+r : L+R \to J$ is the almost injective function defined by cases using l and r in the obvious way (the "co-pairing" of l and r).

Proof: It is sufficient to show that

$$(l_* F)(r_* G) = (l+r)_*(F \,\&\, G) \, .$$

Let $p \in F$ and $q \in G$. Let $s = (p, q) \in P_0^{L+R} \simeq P_0^L \times P_0^R$, so that $s \in F \,\&\, G$. One has

$$l_*(p) r_*(q) = (l+r)_*(s)$$

as easily checked, and the result follows. ∎

3.1 Projecting and reindexing facts

We first study the behavior of facts under projection.

Let $K \subseteq J \subseteq I$. From lemma 8, one derives easily the following properties.

- $\pi_K(\perp(J)) = \perp(K)$ and $\pi_K(1(J)) = 1(K)$.
- If F and G are facts of $M(K)$, then $\pi_K(F \otimes G) = \pi_K(F) \otimes \pi_K(G)$ and $\pi_K(F \parr G) = \pi_K(F) \parr \pi_K(G)$
- If $L + R = J$ and F is a fact of $M(L)$ and G is a fact of $M(R)$, then $\pi_K(F \oplus G) = \pi_{K \cap L}(F) \oplus \pi_{K \cap R}(G)$ and $\pi_K(F \,\&\, G) = \pi_{K \cap L}(F) \,\&\, \pi_{K \cap R}(G)$.

Let $u : L \to J$ be an almost injective function and let F be a fact of $M(L)$. Let $R = u^{-1}(K) \subseteq L$ and let $v : R \to K$ be the restriction of u to R. Using lemma 7, one checks that $\pi_K(u_* F) = v_*(\pi_R(F))$, so by lemma 8, $\pi_K(!_u F) = !_v(\pi_R(F))$ and $\pi_K(?_u F) = ?_v(\pi_R(F))$.

Lemma 10 *Let A be a formula of $\mathsf{LL}(I)$ of domain $J \subseteq I$. Let $K \subseteq J$. Then*

$$A|_K{}^\bullet = \pi_K(A^\bullet) \ .$$

It is a consequence of the observations above.

Definition 11 One says that M is *symmetric* if, for any $J, K \subseteq I$ and any bijection u from J to K, one has $u_*(\bot(J)) = \bot(K)$.

Equivalently, M is symmetric iff for any two *injections* $u, v : J \to I$, one has $u^*\bot = v^*\bot$. This condition is stronger than simply requiring that $u_*\bot = \bot$ for any bijection $u : I \to I$, which would be the most natural definition of symmetry. Consider for instance the case where $P_0 = \{0, 1\}$. Then $P_0^I = \mathcal{P}(I)$, multiplication corresponding to set intersection. Take for $\bot$ the set of all co-infinite subsets J of I (that is, such that $I \setminus J$ is infinite). In that case, $\bot$ is symmetric in the latter sense, but not in the former: take $J \in \bot$ with infinite cardinality. Let $u : J \to I$ be a bijection and let $v : J \to I$ be the inclusion of J into I. Then $v^*\bot = \mathcal{P}(J)$ whereas $u^*\bot$ is the set of all subsets of J which are co-infinite *relative to J*.

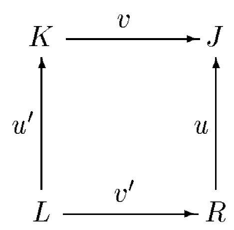
Lemma 12 *Assume that M is symmetric. Let $J, K \subseteq I$ and let $u : J \to K$ be a bijection.*

- $(u_*U)^\bot = u_*(U^\bot)$ *for any subset U of P_0^J.*
- $?_u F = u_* F = !_u F$ *for any fact F of $(P_0^J, \bot(J))$.*

Proof: We just prove the first statement, the second being an immediate consequence. It will be enough to prove that $u_*(U^\bot) \subseteq (u_*U)^\bot$ (indeed, using this inclusion for the set u_*U and for the bijection u^{-1}, one derives the converse inclusion). So let $p \in U^\bot$ and let $q \in U$, we have $(u_*p)(u_*q) = u_*(pq)$, and $pq \in \bot(J)$. We conclude using the symmetry of M that $(u_*p)(u_*q) \in \bot(K)$. ∎

Lemma 13 *Assume that M is symmetric. Let $v : K \to J$ be an injection. If F is a fact of $M(J)$, then v^*F is a fact of $M(K)$ and $(v^*F)^\bot = v^*(F^\bot)$. Moreover, we have the following commutation properties.*

- $v^*(\bot(J)) = \bot(K)$ *and* $v^*(1(J)) = 1(K)$.
- *If F and G are facts of $M(J)$, then $v^*(F \otimes G) = v^*F \otimes v^*G$ and $v^*(F \parr G) = v^*F \parr v^*G$.*
- *If $J = L + R$, if F is a fact of $M(L)$ and G is a fact of $M(R)$, then $v^*(F \mathbin{\&} G) = l^*F \mathbin{\&} r^*G$ and $v^*(F \oplus G) = l^*F \oplus r^*G$ where $l : v^{-1}(L) \to L$ and $r : v^{-1}(R) \to R$ are the injections obtained by restricting v.*
- *If F is a fact of $M(R)$, if $u : R \to J$ is almost injective and if L, u' and v' are such that the following diagram is a pull-back (remember that $v : K \to J$ is an injection),*

  then v' is injective, u' is almost injective, $v^*(!_u F) = !_{u'}(v'^* F)$ and $v^*(?_u F) = ?_{u'}(v'^* F)$.

Proof: Let $v : K \to J$ be an injection. Let $L = v(K) \subseteq J$ and let w be the bijection $K \to L$ induced by v. Then for any $p \in P_0^J$, one has

$$v^*(p) = w^{-1}{}_*(\pi_L(p)) \tag{1}$$

so that v^*F is a fact of $M(K)$ as soon as F is a fact of $M(J)$. For the other statements of the lemma, in view of equation (1) and of the commutation properties of the projection operation with respect to logical constructions on facts, it suffices to prove them in the case where v is a bijection. This is done by applying straightforwardly lemma 12. ∎

As an immediate consequence, we obtain a reindexing lemma which will be essential in the soundness proof.

Lemma 14 *Let A be a formula of $\mathsf{LL}(I)$ of domain $K \subseteq I$. Let $J \subseteq I$ and let $u : K \to J$ be a bijection. If M is symmetric, one has*

$$(u_*A)^\bullet = u_*(A^\bullet)\,.$$

3.2 Soundness

We state and prove a soundness theorem for this phase semantics of $\mathsf{LL}(I)$.

Theorem 15 *Let Γ be a sequence of formulae having all the same domain $J \subseteq I$. If the sequent $\vdash_J \Gamma$ is provable in $\mathsf{LL}(I)$, then the fact $\Gamma^\bullet$ associated to this sequence of formulae in any symmetric product phase model $M = (P_0^I, \bot)$ contains the unit of the monoid P_0^J, that is $1^J \in \Gamma^\bullet$.*

Proof: The proof is of course by induction on the proof π of Γ in $\mathsf{LL}(I)$. The multiplicative cases are completely standard. The additives are handled like in [BE00], with the only difference that we work here in the local phase space $M(J)$. We just deal with the exponential cases.

Assume first that the proof ends with a weakening rule, that is, $\Gamma = (\Delta, ?_{0_J}A)$ with $d(A) = \emptyset$, and π is of the shape

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash_J \Delta\end{array}}{\vdash_J \Delta, ?_{0_J}A}$$

We have to prove that $(\Delta^\bullet)^\perp \subseteq (?_{0_J}A)^\bullet$, and we know by inductive hypothesis that $(\Delta^\bullet)^\perp \subseteq \bot(J)$. So it is sufficient to show that $\bot(J) \subseteq (?_{0_J}A)^\bullet$. But we have $(0_J)_* A^\bullet = \{1^J\}$, as $A^\bullet$ is non-empty, so $(?_{0_J}A)^\bullet = \{1^J\}^\perp = \bot(J)$ and we are done.

Assume now that the proof ends with a contraction rule, that is $\Gamma = (\Delta, ?_l(A|_L), ?_r(A|_R))$, u is an almost injective function from $L + R$ to J and l and r are the restrictions of u to L and R respectively. And π is of the shape

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash_J \Delta, ?_l(A|_L), ?_r(A|_R)\end{array}}{\vdash_J \Delta, ?_u A}$$

It is sufficient to prove that

$$(?_l(A|_L))^\bullet \parr (?_r(A|_R))^\bullet \subseteq (?_u A)^\bullet$$

so by lemma 9, it is sufficient to prove that

$$?_u((A|_L)^\bullet \oplus (A|_R)^\bullet) \subseteq ?_u(A^\bullet)$$

and for this, it is clearly enough to prove that $(A|_L)^\bullet \oplus (A|_R)^\bullet \subseteq A^\bullet$. But we have, by lemma 10,

$$\zeta_L(A|_L)^\bullet = \zeta_L(\pi_L(A^\bullet)) = \varepsilon_L A^\bullet \subseteq A^\bullet$$

and we are done.

Assume now that the last rule of the proof is a dereliction rule. So $\Gamma = (\Delta, ?_u A)$ where u is a bijection from $K = d(A)$ to J, and π is of the shape

$$\dfrac{\vdots\,\pi_1 \atop \vdash_J \Delta, u_* A}{\vdash_J \Delta, ?_u A}$$

It is enough to show that $(u_* A)^\bullet \subseteq (?_u A)^\bullet$, and this results immediately from lemmas 14 and 12.

Last, assume that the last rule of the proof is a promotion rule. So

$$\Gamma = (?_{v\circ u^1} A^1, \ldots, ?_{v\circ u^n} A^n, !_v A)$$

and π is of the shape

$$\dfrac{\vdots\,\pi_1 \atop \vdash_K ?_{u^1} A^1, \ldots, ?_{u^n} A^n, A}{\vdash_J ?_{v\circ u^1} A^1, \ldots, ?_{v\circ u^n} A^n, !_v A}$$

By inductive hypothesis, we have, setting $F_l = A^{l\bullet\perp}$ for $l = 1, \ldots, n$ and $G = A^\bullet$,

$$!_{u^1} F_1 \otimes \ldots \otimes !_{u^n} F_n \subseteq G \ ,$$

that is

$$u^1{}_* F_1 \ldots u^n{}_* F_n \subseteq G \ .$$

So we have

$$v_*(u^1{}_* F_1 \ldots u^n{}_* F_n) \subseteq v_* G \ ,$$

that is

$$(v \circ u^1)_* F_1 \ldots (v \circ u^n)_* F_n \subseteq v_* G$$

hence

$$(v \circ u^1)_* F_1 \ldots (v \circ u^n)_* F_n \subseteq !_v G$$

and we are done. ∎

Of course, this particular phase semantics of $\mathsf{LL}(I)$ is not complete for this system, as it is already not complete for $\mathsf{MALL}(I)$ (see [BE00]). We understand better in the present setting the reason for this incompleteness. Any product phase space validates, for any formula A, the formula $A \multimap !_{\mathrm{Id}} A$, which is not provable in $\mathsf{LL}(I)$. The second author of the present paper has developped an extension of $\mathsf{LL}(I)$ in which this principle is provable, and for which a completeness theorem holds with respect to the symmetric product phase spaces, see [Ehr00].

4 The denotational semantics associated to a symmetric product phase space

Given a symmetric product phase space $M = (P_0^I, \perp)$, we define a category $\mathcal{C}(M)$ of *M-spaces*.

We denote by $\mathcal{F}_M$ the contravariant functor from $\mathcal{I}(I)$ to **Set** which to $J \subseteq I$ associates the set $\mathcal{F}_M(J)$ of all facts of the local phase model $M(J)$ and which, when $J, K \subseteq I$, associates to each injection $u : J \to K$ the map $\mathcal{F}_M(u)$ which to a fact F of $M(K)$ associates the fact u^*F of $M(J)$.

Definition 16 Let $M = (P_0^I, \perp)$ be a symmetric product phase space. An M-space is a pair $X = (|X|, \widehat{X})$ where $|X|$ is a finite or denumerable set (the *web* of X) and $\widehat{X}$ is a natural transformation from the contravariant functor $\mathrm{Fam}_{|X|}$ to the contravariant functor $\mathcal{F}_M$ (both are contravariant functors from $\mathcal{I}(I)$ to **Set**).

Spelling out this definition, for any $J \subseteq I$, we are given a function $\widehat{X}_J : |X|^J \to \mathcal{F}_M(J)$, and moreover, whenever $u : K \to J$ is an injective function, we require that for any $\alpha \in |X|^J$,

$$\widehat{X}_K(u^*\alpha) = u^*(\widehat{X}_J(\alpha))\,.$$

Remark: Since $|X|$ is at most denumerable and since I is denumerable, there exists $J \subseteq I$ and $\alpha \in |X|^J$ such that, for each $a \in |X|$, the set $\alpha^{-1}(a)$ is infinite. Now if K is any subset of I, and if $\beta \in |X|^K$, there exists an injection $u : K \to J$ such that $\beta = u^*(\alpha)$, and so $\widehat{X}_K(\beta) = u^*(\widehat{X}_J(\alpha))$. So it appears that the whole natural transformation $\widehat{X}$ is completely determined by the unique fact $\widehat{X}_J(\alpha)$ for such an "ω-redundant" enumeration α of $|X|$. The problem is of course that there is *a priori* no canonical such enumeration of $|X|$, and that is why an M-space is equipped with a natural transformation, and not simply with a single fact of sufficiently large arity.

Definition 17 Let X be an M-space. A *clique* of X is a subset x of $|X|$ such that, for any $J \subseteq I$ and any $\alpha \in x^J$ (that is, any J-indexed family α of elements of x), one has $1^J \in \widehat{X}_J(\alpha)$. We denote by $\mathrm{Cl}(X)$ the set of all cliques of X.

The set $\mathrm{Cl}(X)$ contains $\emptyset$, is closed under subsets (if $x \in \mathrm{Cl}(X)$ and $y \subseteq x$, then $y \in \mathrm{Cl}(X)$), but has no reason to be closed under directed unions, so that the least fix-point operators which allow usually to accommodate general recursion in denotational semantics will not be available in general. One can mention however that, if $\perp$ satisfies the following property:

$$\text{if } \varepsilon_J p \in \perp \text{ for each finite subset } J \text{ of } I, \text{ then } p \in \perp,$$

then the set of cliques of any M-space is closed under directed unions.

Definition 18 Let X and Y be M-spaces. A morphism from X to Y is a clique of $X \multimap Y$ (see below the definition of this space).

We now show how to associate an M-space to any formula of linear logic. More precisely, for any connector of linear logic, we define a corresponding construction of M-spaces (with the same notations).

If X is an M-space, $X^\perp$ denotes the M-space defined by $|X^\perp| = |X|$ and, when $J \subseteq I$ and $\alpha \in |X|^J$, $\widehat{X^\perp}_J(\alpha) = \widehat{X}_J(\alpha)^\perp$. This defines a natural transformation by lemma 13.

4.1 Additives

One sets $|0| = \emptyset$, and since $\emptyset^J$ is non-empty iff J is empty, one defines entirely $\widehat{0}$ by setting $\widehat{0}_\emptyset(\emptyset) = \{\emptyset\}$ (the unique fact of the local model $M(\emptyset)$). One defines $\top$ in the same way, so that $\top = 0$.

Now let X and Y be M-spaces, one sets $|X \oplus Y| = |X \mathbin{\&} Y| = (\{1\} \times |X|) \cup (\{2\} \times |Y|)$. Let $\gamma \in |X \oplus Y|^J$. As usual, the family γ determines in a unique way two disjoint subsets L and R of J such that $J = L + R$ and two families, $\alpha \in |X|^L$ and $\beta \in |Y|^R$ such that $\gamma = \alpha + \beta$. Then one has $\widehat{X}_L(\alpha) \in \mathcal{F}_M(L)$ and $\widehat{Y}_R(\beta) \in \mathcal{F}_M(R)$, and one sets $\widehat{X \oplus Y}_J(\gamma) = \widehat{X}_L(\alpha) \oplus \widehat{Y}_R(\beta) \in \mathcal{F}_M(J)$.

With the same notations, one sets of course $\widehat{X \mathbin{\&} Y}_J(\gamma) = \widehat{X}_L(\alpha) \mathbin{\&} \widehat{Y}_R(\beta) \in \mathcal{F}_M(J)$.

4.2 Multiplicatives

One sets $|1| = |\bot| = \{*\}$. Then $\widehat{\bot}_J(*^J) = \bot(J)$ and $\widehat{1}_J(*^J) = 1(J)$ (where $*^J$ denotes the J-indexed family which is constantly equal to $*$).

If X and Y are M-spaces, one sets $|X \otimes Y| = |X \parr Y| = |X| \times |Y|$. If $(\alpha, \beta) \in (|X| \times |Y|)^J$, one sets $\widehat{X \otimes Y}_J(\alpha, \beta) = \widehat{X}_J(\alpha) \otimes \widehat{Y}_J(\beta)$ and $\widehat{X \parr Y}_J(\alpha, \beta) = \widehat{X}_J(\alpha) \parr \widehat{Y}_J(\beta)$.

4.3 Exponentials

Let X be an M-space. Then the sets $|!X|$ and $|?X|$ are both equal to the set of all finite multisets of elements of $|X|$. Let $J \subseteq I$, and let $\xi \in |!X|^J$. Let $K \subseteq I$, $u : K \to J$ be an almost injective function and $\alpha \in |X|^K$ be such that, for each $j \in J$, one has $\xi_j = \mathsf{m}(\alpha|_{u^{-1}(j)})$. Such a pair (α, u) always exists and will be called a *representative* of ξ. Then one sets $\widehat{!X}_J(\xi) = !_u(\widehat{X}_K(\alpha)) = (u_*(\widehat{X}_K(\alpha)))^{\bot\bot}$. This definition does not depend on the choice of a representative of ξ. Indeed, let (β, v) be another representative, with $v : L \to J$ almost injective and $\beta \in |X|^L$. Then there exists a bijection $w : L \to K$ such that $u \circ w = v$ and $w^*\alpha = \beta$. We have $u_*(\widehat{X}_K(\alpha)) = v_*(\widehat{X}_L(\beta))$ since, by naturality of $\widehat{X}$, and by the fact that w is a morphism in the category $\mathcal{I}(I)$, one has $\widehat{X}_L(\beta) = w^*(\widehat{X}_K(\alpha))$ (one also uses the fact that $w^* = w_*^{-1}$).

Now we prove that the operation $\widehat{!X}$ so defined is a natural transformation. So let $u : J' \to J$ be an injection, let $\xi \in |!X|^J$, and let (α, v) with $v : K \to J$ and $\alpha \in |X|^K$ be a representative of ξ. Considering $|!X|$ as a commutative monoid (for the addition of finite multisets), and identifying $|X|$ with the subset of $|!X|$ containing the singleton multisets, saying that (α, v) is a representative of ξ simply means that $v_*\alpha = \xi$. Now let $K' \subseteq I$ and $u' : K' \to K$, $v' : K' \to J'$ be such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{\ v\ } & J \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u} \\ K' & \xrightarrow{\ v'\ } & J' \end{array}$$

be a pull-back. We have $u^*\widehat{!X}_J(\xi) = u^*(!_v(\widehat{X}_K(\alpha))) = !_{v'}(u'^*\widehat{X}_K(\alpha))$ by lemma 13. So by naturality of $\widehat{X}$, one has $u^*\widehat{!X}_J(\xi) = !_{v'}(\widehat{X}_{K'}(u'^*\alpha))$. By lemma 7 applied to the monoid $|!X|$, one has $v'_*u'^*\alpha = u^*v_*\alpha = u^*\xi$, so that $(u'^*\alpha, v')$ is a representative of $u^*\xi$. Hence $u^*\widehat{!X}_J(\xi) = \widehat{!X}_{J'}(u^*\xi)$.

4.4 The category of M-spaces

When X and Y are M-spaces, the space $X \multimap Y$ is defined in the usual way: $X \multimap Y = X^\perp \parr Y$. A morphism from X to Y in the category $\mathcal{C}(M)$ is by definition a subset f of $|X \multimap Y|$ such that, for any $J \subseteq I$ and any $(\alpha,\beta) \in f^J$, one has $1^J \in \widehat{X \multimap Y}_J(\alpha,\beta)$, that is $\widehat{X}_J(\alpha) \subseteq \widehat{Y}_J(\beta)$. It is clear that one defines in that way a category, with the usual identity morphism (the diagonal) and the relational composition operation.

One checks easily also that $\top$ is the terminal object of this category. We check that $X \,\&\, Y$, together with the two projections $\pi_1 = \{((1,a),a) \mid a \in |X|\} \subseteq |(X \,\&\, Y) \multimap X|$ and $\pi_2 = \{((2,b),b) \mid b \in |Y|\} \subseteq |(X \,\&\, Y) \multimap X|$ is the cartesian product of the M-spaces X and Y. First, π_1 is a morphism. Indeed, let $\gamma \in (\pi_1)^J$ for some $J \subseteq I$. Then $\gamma = (\delta, \alpha)$, with $\alpha \in |X|^J$, and $\delta \in |X \,\&\, Y|^J$ given by $\delta_j = (1, \alpha_j)$. Then $\widehat{X \,\&\, Y}_J(\delta) = \widehat{X}_J(\alpha)$ by definition of $X \,\&\, Y$, and we are done. Similarly, π_2 is a morphism. Now let Z be an M-space. The pairing of two morphisms $f : Z \to X$ and $g : Z \to Y$ in $\mathcal{C}(M)$ is given by $\langle f, g\rangle = \{(c,(1,a)) \mid (c,a) \in f\} \cup \{(c,(2,b)) \mid (c,b) \in g\}$. One checks that it is indeed a morphism of $\mathcal{C}(M)$. So let $J \subseteq I$ and let $\delta \in \langle f,g\rangle^J$. Then one can write, in a unique way, $\delta = (\gamma, \alpha + \beta)$ for a unique decomposition $J = L + R$, $\alpha \in |X|^L$ and $\beta = |Y|^R$ and $\gamma \in |Z|^J$. One has to show that $\widehat{Z}_J(\gamma) \subseteq \widehat{X}_L(\alpha) \,\&\, \widehat{Y}_R(\beta)$. For this purpose, it suffices to show that $\pi_L(\widehat{Z}_J(\gamma)) \subseteq \widehat{X}_L(\alpha)$ and similarly for R, but by naturality of $\widehat{Z}$, one has $\pi_L(\widehat{Z}_J(\gamma)) = \widehat{Z}_L(\pi_L\gamma)$, and since $(\pi_L\gamma, \alpha) \in f^L$ and f is a morphism in $\mathcal{C}(M)$, we conclude. It remains to check that if h is a morphism $Z \to X \,\&\, Y$ in $\mathcal{C}(M)$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ then $h = \langle f, g\rangle$, but this is obvious. So the category $\mathcal{C}(M)$ is cartesian.

Let $f : X \to X'$ and $g : Y \to Y'$ be two morphisms in $\mathcal{C}(M)$. We show that $f \otimes g = \{((a,b),(a',b')) \mid (a,a') \in f \text{ and } (b,b') \in g\}$ is a morphism $X \otimes Y \to X' \otimes Y'$ in $\mathcal{C}(M)$. So let $J \subseteq I$ and let $((\alpha,\beta),(\alpha',\beta')) \in (f \otimes g)^J$, then we have $\widehat{X}_J(\alpha) \subseteq \widehat{X'}_J(\alpha')$ and $\widehat{Y}_J(\beta) \subseteq \widehat{Y'}_J(\beta')$ since $(\alpha,\alpha') \in f^J$ and $(\beta,\beta') \in g^J$, and therefore $\widehat{X}_J(\alpha) \otimes \widehat{Y}_J(\beta) \subseteq \widehat{X'}_J(\alpha') \otimes \widehat{Y'}_J(\beta')$ as required. Checking that the operation $\otimes$ is functorial and satisfies the required isomorphisms for defining a symmetric monoidal structure on $\mathcal{C}(M)$ is easy. The neutral element for the tensor product is the M-space 1. The monoidal category so defined is easily seen to be closed, the objects of arrows from X to Y being of course $X \multimap Y$. It is also $\star$-autonomous, because $X \multimap \perp$ is isomorphic to $X^\perp$ and $X^{\perp\perp} = X$.

We turn now the exponential ! into a functorial operation from $\mathcal{C}(M)$ to itself. So let $f : X \to Y$ be a morphism in $\mathcal{C}(M)$. One defines as usual $!f \subseteq |!X| \times |!Y|$ as the set of all pairs (μ,ν) of multisets such that there is a finite family $(a_s, b_s)_{s=1,\dots,n}$ of elements of f such that $\mu = [a_1,\dots,a_n]$ and $\nu = [b_1,\dots,b_n]$. Let $J \subseteq I$ and let $(\zeta,\xi) \in (!f)^J$. Then one can find $K \subseteq I$, an almost injective function $u : K \to J$, and two families $\alpha \in |X|^K$ and $\beta \in |Y|^K$ such that $(\alpha,\beta) \in f^K$, (α, u) is a representative of ζ and (β, u) is a representative of ξ. So $\widehat{!X}_J(\zeta) = (u_*\widehat{X}_K(\alpha))^{\perp\perp}$ and $\widehat{!Y}_J(\xi) = (u_*\widehat{Y}_K(\beta))^{\perp\perp}$. One concludes that $\widehat{!X}_J(\zeta) \subseteq \widehat{!Y}_J(\xi)$ since we know that $\widehat{X}_K(\alpha) \subseteq \widehat{Y}_K(\beta)$ as we know that f is a morphism in $\mathcal{C}(M)$. Checking that the operation on morphisms $f \mapsto !f$ is indeed functorial is done like in the category of sets and relations.

We exhibit next the comonad structure of this endofunctor. Let X be an M-space. *Dereliction* is defined as $\mathrm{d}_X = \{([a],a) \mid a \in |X|\} \subseteq |!X| \times |X|$. It is a morphism $!X \to X$ in $\mathcal{C}(X)$. Indeed, let $J \subseteq I$ and let $(\zeta,\alpha) \in \mathrm{d}_X^J$, that is, $\alpha \in |X|^J$ and $\zeta_j = [\alpha_j]$ for each $j \in J$. Then one checks easily that $\widehat{!X}_J(\zeta) = \widehat{X}_J(\alpha)$, for $\mathrm{Id}_* = \mathrm{Id}$ and (α, Id) is obviously a representative of ζ. *Digging* is defined as $\mathrm{p}_X = \{(\mu_1 + \cdots + \mu_n, [\mu_1,\dots,\mu_n])) \mid \mu_1,\dots,\mu_n \in |!X|\} \subseteq |!X| \times |!!X|$, one must prove that p_X is a morphism from $!X$ to $!!X$ in the category $\mathcal{C}(M)$. So let $J \subseteq I$ and let $(\xi, \Xi) \in \mathrm{p}_X^J$. Let $K \subseteq I$, $u : K \to J$ be almost injective and $\zeta \in |!X|^K$ be such that (ζ, u) is a representative of Ξ. Then, by definition of p_X, we have $\xi_j = \sum_{u(k)=j} \zeta_k$ for each $j \in J$. Let $L \subseteq I$ and $v : L \to K$ be such that (α, v) is a representative of ζ. Then one checks easily that $(\alpha, u \circ v)$ is a representative of ξ. Therefore, $\widehat{!X}_J(\xi) = (u_* v_* \widehat{X}_L(\alpha))^{\perp\perp}$. But $v_* \widehat{X}_L(\alpha) \subseteq (v_* \widehat{X}_L(\alpha))^{\perp\perp} = \widehat{!X}_K(\zeta)$. So $\widehat{!X}_J(\xi) \subseteq (u_* \widehat{!X}_K(\zeta))^{\perp\perp} = \widehat{!!X}_J(\Xi)$ and we are done.

The maps d_X and p_X define natural transformations making commutative the usual comonad diagrams (see for instance [Lan71]). Moreover, the canonical bijection between $|!(X \mathbin{\&} Y)|$ and $|!X \otimes !Y|$ is an isomorphism in $\mathcal{C}(M)$ between the M-spaces $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$, due to lemma 9. This isomorphism is of course natural in X and Y.

So the $\star$-autonomous category $\mathcal{C}(M)$, equipped with the comonad !, is a model of linear logic.

Remark: One should be more precise here, invoking typically the work of Bierman [Bie95] who has stated precisely the categorical axioms to be satisfied by a denotational model of linear logic (the convenient notion here seems to be the notion of a *new-Seely category*). The precise checking that these conditions hold involves two kinds of verifications.

- One must exhibit the morphisms required for making the adjunction between $\mathcal{C}(M)$ and the co-Kleisli category of the comonad ! *monoidal*. The required morphisms are present here because they are defined using the canonical isomorphism in $\mathcal{C}(M)$ between $!(X \mathbin{\&} Y)$ and $!X \otimes !Y$.
- One must check the commutation of a number of diagrams. We do not need to check these commutations: we know that they hold because the $\star$-autonomous category of sets and relations (with cartesian product as tensor product and as object of morphisms), together with the comonad of finite multisets, is a *Lafont category* in the sense of [Bie95][7], and because, at the level of webs of M-spaces and of morphisms in $\mathcal{C}(M)$ (which are relations between webs), the operations we define for interpreting linear logic are *exactly the same* as those which make the Lafont category of sets and relations a denotational model of linear logic.

Let us insist on that point which makes the *non uniform* models considered here particularly simple. Given a formula S of ordinary linear logic, the web $|S^*_M|$ of its interpretation S^*_M in $\mathcal{C}(M)$ (defined using inductively the constructions above) does not depend[8] on M, and is equal to the interpretation of S in the purely relational model described in section 2. This also holds for proofs: if π is a proof of a sequent $\vdash \Phi$ in ordinary linear logic, then the interpretation π^*_M that one obtains using the categorical operations described above is just the interpretation $\pi^* \subseteq |S|$ of π in the purely relational model, computed as described in section 6. What we have shown is that, for any symmetric product phase space M, this subset π^* of $|S|$ will always be a clique of S^*_M; each phase space M singles out a certain class of subsets (M-cliques) of the interpretations of formulae in the relational model, and this class contains the definable subsets. The situation is thus completely similar to what happens with logical relations over a fixed semantics, e.g. in [Sie92]: there, by tuning a relation at type 0 (that is, over finite products of the flat domain of natural numbers), one determines various classes of accepted elements (called "invariant" elements in this setting) of the Scott domains interpreting the types. From this viewpoint, the result proven above (each definable element is an M-clique) is the analogue of the so-called "fundamental lemma of logical relations".

The connection between these constructions of spaces in the category $\mathcal{C}(M)$ and the interpretation of $\mathsf{LL}(I)$ formulae in the phase model M is easy to describe. Let A be an $\mathsf{LL}(I)$ formula

[7] We do not know who observed for the first time that the category of sets and relations is a model of linear logic, and we do not know either if this result has ever been published; it is probably a typical piece of folklore in the field.

[8] In ordinary coherence spaces, the coherence relation is used for constructing webs of the exponentials, this is not the case here.

of domain $J \subseteq I$. Then to the underlying linear logic formula $\underline{A}$, we associate its M-space interpretation $\underline{A}^*_M$. As explained above, $|\underline{A}| = |\underline{A}^*_M|$. Moreover, a simple induction on A shows that

$$\widehat{(\underline{A}^*_M)}_J(\langle A\rangle) = A^\bullet .$$

So if the class of symmetric product phase spaces were a complete semantics of $\mathsf{LL}(I)$, proposition 4 (together with lemma 2) would imply that, given a formula S of linear logic, a subset of $|S|$ which is an M-clique for all symmetric product phase spaces M is contained in the interpretation of a proof of S (a form of denotational completeness). The completeness result mentioned at the end of section 3 indicates that such a denotational completeness result might hold for a reasonable extension of linear logic (still to be defined).

5 Example: a non uniform coherence semantics

We shall show how this phase semantics can be used for defining a *non uniform* version of the standard coherence semantics of linear logic. Uniformity is a feature that most denotational models of typed λ-calculi share. It corresponds to the fact that a function can only be applied to an argument which is "accepted" by the model (a clique in the sense of definition 17, in the present setting). In the (multiset-based) coherent semantics for instance, the web of $!X$ is the set of all finite *multicliques* of X and not the set of all finite multisets, like in the category of M-spaces. So for instance the boolean-PCF[9] term

$$\lambda x : \mathrm{Bool}\,.\,\mathsf{if}\ x\ \mathsf{then}\ (\mathsf{if}\ x\ \mathsf{then\ true\ else\ false})\ \mathsf{else}\ (\mathsf{if}\ x\ \mathsf{then\ true\ else\ false})$$

will have different uniform and non-uniform semantics. Intuitively, the non-uniform semantics of this term will contain informations about its behavior when applied to an unreliable "boolean" which takes the value **true** the first time it is used and **false** the second time (in that case, the resulting value is **false**) and also when applied to a "boolean" returning first **false** and then **true** (and in that case, the resulting value is **true**); this information will be absent from the uniform interpretation. In ordinary coherence spaces, this term will just be interpreted as a version of the identity function which uses twice its argument. This example will be studied in section 5.3.

5.1 The product phase space $\mathbf{Coh}_2$

We start with a simple general observation on phase semantics (see [Laf97]).

Let Q and Q' be two commutative monoids and let $h : Q \to Q'$ be a surjective monoid homomorphism. Let $\perp$ be a subset of Q' (so that we consider $(Q', \perp)$ as a phase model). We regard also $(Q, h^{-1}\perp)$ as a phase model. Then for any subset U of Q, one has $U^\perp = h^{-1}((hU)^\perp)$. As a consequence, the map $F \mapsto h^{-1}F$ is a bijective correspondence from the facts of $(Q', \perp)$ to the facts of $(Q, h^{-1}\perp)$, and one has $h^{-1}(F^\perp) = (h^{-1}F)^\perp$ and $h^{-1}(F \otimes G) = (h^{-1}F) \otimes (h^{-1}G)$, for any facts F and G of $(Q', \perp)$.

Let P_0 be the three elements monoid $\{0, 1, \tau\}$, defined by the following equation: $\tau\tau = \tau$. Let $\perp^1 = \{0, 1\} \subseteq P_0$. Then the phase space $(P_0, \perp^1)$ has exactly three facts, namely

- $\overline{\mathsf{C}} = \{0\}$, that we shall call *incoherence*,

[9]Boolean-PCF is a simply typed lambda-calculus with Bool as single ground type, two constants **true** and **false** of type Bool and a conditional construction, with obvious typing and conversion rules. This example can also be carried out in the linear sequent calculus, with Bool represented by the formula $1 \oplus 1$.

- $\mathsf{E} = \{0, 1\}$, that we shall call *neutrality*
- and $\mathsf{C} = \{0, 1, \tau\}$, that we shall call *coherence*.

One checks easily that $\mathsf{C}^\perp = \overline{\mathsf{C}}$ and that $\mathsf{E}^\perp = \mathsf{E}$ and that the tensor and par operations on these facts are given by the following tables.

$\otimes$	$\overline{\mathsf{C}}$	E	C
$\overline{\mathsf{C}}$	$\overline{\mathsf{C}}$	$\overline{\mathsf{C}}$	$\overline{\mathsf{C}}$
E	$\overline{\mathsf{C}}$	E	C
C	$\overline{\mathsf{C}}$	C	C

and

$\invamp$	$\overline{\mathsf{C}}$	E	C
$\overline{\mathsf{C}}$	$\overline{\mathsf{C}}$	$\overline{\mathsf{C}}$	C
E	$\overline{\mathsf{C}}$	E	C
C	C	C	C

Now let n be a non zero integer. Let $h : P_0^n \to P_0$ be given by $h(p_1, \ldots, p_n) = p_1 \ldots p_n$, this function is a surjective monoid homomorphism. So the phase space $(P_0^n, \perp^n)$, where $\perp^n = h^{-1}\perp^1$, has three facts, namely $h^{-1}\mathsf{C}$, $h^{-1}\overline{\mathsf{C}}$ and $h^{-1}\mathsf{E}$, that we simply denote by C, $\overline{\mathsf{C}}$ and E, and the tensor and par operations on these facts are still given by the two tables above.

We define $\perp \subseteq P_0^I$ as follows: an element p of P_0^I belongs to $\perp$ iff, for any family $i_1, \ldots, i_n$ of *pairwise distinct* elements of I, the n-tuple $(p_{i_1}, \ldots, p_{i_n})$ belongs to $\perp^n$, that is, iff the product $p_{i_1} \ldots p_{i_n}$ is different from τ. Then one checks easily that $(P_0^I, \perp)$ is indeed a symmetric product phase space, that we denote by $\mathbf{Coh}_n$.

Continuing along these lines, we would arrive to a n-ary hypergraphical version of coherence spaces, similar to hypercoherences. For the sake of simplicity, we restrict our attention to the case $n = 2$. In that case, an element p of P_0^I belongs to $\perp$ iff, as soon as $p_i = \tau$ for some $i \in I$, one has $p_j = 0$ for all $j \neq i$. We first study the structure of facts in a local space $\mathbf{Coh}_2(J)$ for $J \subseteq I$, and show that these facts admit a simple graphical description.

Definition 19 A *coherence graph* $\mathcal{G}$ on a set of vertices E is given by two disjoint subsets of the set of unordered pairs of *distinct* elements of E called *coherence* and *incoherence* of $\mathcal{G}$. Let $e, e' \in E$ be distinct. We write $e \frown_{\mathcal{G}} e'$ when $\{e, e'\}$ belongs to the coherence of $\mathcal{G}$ and $e \smile_{\mathcal{G}} e'$ when $\{e, e'\}$ belongs to the incoherence of $\mathcal{G}$. We write $e \;\mathsf{n}_{\mathcal{G}}\; e'$ when $\{e, e'\}$ belongs neither to the coherence nor to the incoherence of $\mathcal{G}$, and in that case, we say that the pair $\{e, e'\}$ is neutral.

The following notations are standard in the theory of coherence spaces. Let $e, e' \in E$ be distinct. One writes $e \overset{\frown}{\smile}_{\mathcal{G}} e'$ if $e \frown_{\mathcal{G}} e'$ or $e \;\mathsf{n}_{\mathcal{G}}\; e'$, and one writes $e \asymp_{\mathcal{G}} e'$ if $e \smile_{\mathcal{G}} e'$ or $e \;\mathsf{n}_{\mathcal{G}}\; e'$. A coherence graph $\mathcal{G}$ on E can be completely described by giving any of the following pairs of symmetric relations, subject to the following conditions (again, e and e' are distinct elements of E):

- $\frown_{\mathcal{G}}$ and $\smile_{\mathcal{G}}$ with $e \frown_{\mathcal{G}} e' \Rightarrow (\text{not } e \smile_{\mathcal{G}} e')$
- $\frown_{\mathcal{G}}$ and $\mathsf{n}_{\mathcal{G}}$ with $e \frown_{\mathcal{G}} e' \Rightarrow (\text{not } e \;\mathsf{n}_{\mathcal{G}}\; e')$
- $\smile_{\mathcal{G}}$ and $\mathsf{n}_{\mathcal{G}}$ with $e \smile_{\mathcal{G}} e' \Rightarrow (\text{not } e \;\mathsf{n}_{\mathcal{G}}\; e')$
- $\frown_{\mathcal{G}}$ and $\overset{\frown}{\smile}_{\mathcal{G}}$ with $e \frown_{\mathcal{G}} e' \Rightarrow e \overset{\frown}{\smile}_{\mathcal{G}} e'$
- $\smile_{\mathcal{G}}$ and $\asymp_{\mathcal{G}}$ with $e \smile_{\mathcal{G}} e' \Rightarrow e \asymp_{\mathcal{G}} e'$
- $\overset{\frown}{\smile}_{\mathcal{G}}$ and $\mathsf{n}_{\mathcal{G}}$ with $e \;\mathsf{n}_{\mathcal{G}}\; e' \Rightarrow e \overset{\frown}{\smile}_{\mathcal{G}} e'$
- $\asymp_{\mathcal{G}}$ and $\mathsf{n}_{\mathcal{G}}$ with $e \;\mathsf{n}_{\mathcal{G}}\; e' \Rightarrow e \asymp_{\mathcal{G}} e'$.

Let $J \subseteq I$ and let F be a fact of $\mathbf{Coh}_2(J)$. If $j, j' \in J$ are distinct, we denote by $\pi_{j,j'}$ the projection $P_0^J \to P_0^2$ which maps p to $(p_j, p_{j'})$. Then by lemma 13, we know that $\pi_{j,j'}F$ is a fact of $(P_0^2, \pi_{j,j'}\bot)$. But clearly (since $j \neq j'$), $\pi_{j,j'}\bot = \bot^2$, so that $\pi_{j,j'}F$ can take one of the three different values E, C or $\overline{\mathsf{C}}$. Observe also that, since these tree facts are symmetrical (in the sense that they are invariant under the transposition of the two components of the product P_0^2), one has $\pi_{j,j'}F = \pi_{j',j}F$. So we associate to F the coherence graph $\mathsf{g}(F)$ on J given by $j \frown_{\mathsf{g}(F)} j'$ if $\pi_{j,j'}F = \mathsf{C}$ and $j \smile_{\mathsf{g}(F)} j'$ if $\pi_{j,j'}F = \overline{\mathsf{C}}$ for $j, j' \in J$ with $j \neq j'$. Clearly $j \mathrel{\mathsf{n}_{\mathsf{g}(F)}} j'$ iff $\pi_{j,j'}F = \mathsf{E}$.

Conversely, let $\mathcal{G}$ be a coherence graph on J. If j, j' are two distinct elements of J, we denote by $\varepsilon_{j,j'}$ (resp. $\tau_{j,j'}$) the element of P_0^J which takes the value 0 for all element of J, but for j and j', where it takes the value 1 (resp. τ). We associate to $\mathcal{G}$ the following subset $\mathsf{f}_0(\mathcal{G})$ of P_0^J:

$$\mathsf{f}_0(\mathcal{G}) = \{\tau_{j,j'} \mid j, j' \in J,\ j \neq j' \text{ and } j \frown_{\mathcal{G}} j'\} \cup \{\varepsilon_{j,j'} \mid j, j' \in J, j \neq j' \text{ and } i \mathrel{\mathsf{n}_{\mathcal{G}}} j\}$$

and then we associate to $\mathcal{G}$ the fact $\mathsf{f}(\mathcal{G}) = \mathsf{f}_0(\mathcal{G})^{\bot\bot}$.

Lemma 20 *Let $J \subseteq I$. If F is a fact of $\mathbf{Coh}_2(J)$, then $\mathsf{f}(\mathsf{g}(F)) = F$, and if $\mathcal{G}$ is a coherence graph on J, then $\mathsf{g}(\mathsf{f}(\mathcal{G})) = \mathcal{G}$.*

Proof: We content ourselves with observing that an element p of P_0^J belongs to $\mathsf{f}(\mathcal{G})$ iff, for any distinct $j, j' \in J$,

$$\begin{cases} j \smile_{\mathcal{G}} j' & \Rightarrow \quad \pi_{j,j'}p \in \overline{\mathsf{C}} \\ j \mathrel{\mathsf{n}_{\mathcal{G}}} j' & \Rightarrow \quad \pi_{j,j'}p \in \mathsf{E} \end{cases}$$

Then the proof of the lemma follows easily, using the definition of $\bot$ in terms of $\bot^2 = \mathsf{E}$, and with the help of lemma 8. ∎

Through this bijective correspondence, any operation on facts in the product phase space $\mathbf{Coh}_2(J)$ can be translated into an operation on coherence graphs. We describe now the corresponding operations on coherence graphs (with the usual logical notations). We deal first with the multiplicative and additive connectives.

- Let $J \subseteq I$ and let $\mathcal{G}$ be a coherence graph on J. Then then $\mathcal{G}^\bot$ is the coherence graph on J defined by: $j \frown_{\mathcal{G}^\bot} j'$ if $j \smile_{\mathcal{G}} j'$ and $j \smile_{\mathcal{G}^\bot} j'$ if $j \frown_{\mathcal{G}} j'$.

- Let $J \subseteq I$. The coherence graph $\bot_J$ on J is defined by $j \mathrel{\mathsf{n}_{\bot_J}} j'$ for any $j, j' \in J$ with $j \neq j'$. And the coherence graph 1_J is identical to $\bot_J$.

- Let $J \subseteq I$ and let $\mathcal{G}$ and $\mathcal{H}$ be two coherence graphs on J. Then $\mathcal{G} \otimes \mathcal{H}$ is the coherence graph on J defined as follows: $j \bigcirc_{\mathcal{G}\otimes\mathcal{H}} j'$ iff $j \bigcirc_{\mathcal{G}} j'$ and $j \bigcirc_{\mathcal{H}} j'$, and $j \mathrel{\mathsf{n}_{\mathcal{G}\otimes\mathcal{H}}} j'$ iff $j \mathrel{\mathsf{n}_{\mathcal{G}}} j'$ and $j \mathrel{\mathsf{n}_{\mathcal{H}}} j'$ for $j, j' \in J$ with $j \neq j'$.

- Let $J \subseteq I$ and let $\mathcal{G}$ and $\mathcal{H}$ be two coherence graphs on J. Then $\mathcal{G} \parr \mathcal{H}$ is the coherence graph on J defined as follows: $j \frown_{\mathcal{G}\parr\mathcal{H}} j'$ iff $j \frown_{\mathcal{G}} j'$ or $j \frown_{\mathcal{H}} j'$, and $j \mathrel{\mathsf{n}_{\mathcal{G}\parr\mathcal{H}}} j'$ iff $j \mathrel{\mathsf{n}_{\mathcal{G}}} j'$ and $j \mathrel{\mathsf{n}_{\mathcal{H}}} j'$ for $j, j' \in J$ with $j \neq j'$.

- Both 0 and $\top$ are the unique coherence graph with empty set of vertices.

- Let L and R be two disjoint subsets of I, let $\mathcal{G}$ and $\mathcal{H}$ be coherence graphs on L and R respectively. Then $\mathcal{G} \oplus \mathcal{H}$ is the coherence graph on $J = L + R$ defined as follows (for $j, j' \in L$, with $j \neq j'$): $j \bigcirc_{\mathcal{G}\oplus\mathcal{H}} j'$ iff ($j, j' \in L$ and $j \bigcirc_{\mathcal{G}} j'$) or ($j, j' \in R$ and $j \bigcirc_{\mathcal{H}} j'$), and $j \mathrel{\mathsf{n}_{\mathcal{G}\oplus\mathcal{H}}} j'$ iff ($j, j' \in L$ and $j \mathrel{\mathsf{n}_{\mathcal{G}}} j'$) or ($j, j' \in R$ and $j \mathrel{\mathsf{n}_{\mathcal{H}}} j'$).

- Let L and R be two disjoint subsets of I, let $\mathcal{G}$ and $\mathcal{H}$ be coherence graphs on L and R respectively. Then $\mathcal{G}$ & $\mathcal{H}$ is the coherence graph on $J = L + R$ defined as follows (for $j, j' \in J$, with $i \neq j$): $j \asymp_{\mathcal{G}\&\mathcal{H}} j'$ iff ($j, j' \in L$ and $j \asymp_{\mathcal{G}} j'$) or ($j, j' \in R$ and $j \asymp_{\mathcal{H}} j'$), and $j \mathsf{n}_{\mathcal{G}\&\mathcal{H}} j'$ iff ($j, j' \in L$ and $j \mathsf{n}_{\mathcal{G}} j'$) or ($j, j' \in R$ and $j \mathsf{n}_{\mathcal{H}} j'$).

Now we turn to the exponentials. We need a definition and an easy lemma.

Definition 21 Let let $\mathcal{G}$ be a coherence graph on a subset J of I. One says that $\mathcal{G}$ is a *clique* if for any distinct $j, j' \in J$, one has $j \bigcirc_{\mathcal{G}} j'$. One says that $\mathcal{G}$ is a *star-shaped clique* if $\mathcal{G}$ is a clique and if, moreover, there exists an element $j \in J$ such that $j \frown_{\mathcal{G}} j'$ for any $j' \in J$, with $j \neq j'$.

These definitions are motivated by the following easy lemma.

Lemma 22 *Let $J \subseteq I$ and let F be a fact of* $\mathbf{Coh}_2(J)$.

- $\mathsf{g}(F)$ *is a clique iff* $1^J \in F$
- $\mathsf{g}(F)$ *is a star-shaped clique iff F contains an element p such that $\prod_{j \in J} p_j = \tau$, that is such that $p_j \neq 0$ for all $j \in J$, and there exists $j \in J$ such that $p_j = \tau$.*

Given a coherence graph $\mathcal{G}$ on $J \subseteq I$ and an almost injective function $u : J \to K$ (where $K \subseteq I$), we have to define a coherence graph $!_u\mathcal{G}$ on K. Let F be the fact of $\mathbf{Coh}_2(J)$ corresponding to $\mathcal{G}$ (that is $F = \mathsf{f}(\mathcal{G})$). Then the coherence graph $!_u\mathcal{G}$ is given by $!_u\mathcal{G} = \mathsf{g}(!_u F)$. We describe this graph explicitly.

So let $k, l \in K$ be two distinct elements of K. Let $K_1 = u^{-1}(k)$ and $K_2 = u^{-1}(l)$. We have

$$\pi_{k,l}(!_u F) = \{(\prod_{i \in K_1} p_i, \prod_{i \in K_2} p_i) \mid p \in F\}^{\perp\perp}$$

Let $L = K_1 + K_2$ and let $\mathcal{H}$ be the coherence graph obtained by restricting $\mathcal{G}$ to L. By lemma 22, $k \bigcirc_{!_u\mathcal{G}} l$ iff $\mathcal{H}$ is a clique, and $k \frown_{!_u\mathcal{G}} l$ iff $\mathcal{H}$ is a star-shaped clique. To summarize,

- Let $u : J \to K$ be almost injective (with $J, K \subseteq I$) and let $\mathcal{G}$ be a coherence graph on J. Then $!_u\mathcal{G}$ is the coherence graph on K defined as follows (for $k, l \in K$ with $k \neq l$): $k \bigcirc_{!_u\mathcal{G}} l$ if the restriction of $\mathcal{G}$ to $u^{-1}(\{k, l\})$ is a clique, and $k \frown_{!_u\mathcal{G}} l$ if the restriction of $\mathcal{G}$ to $u^{-1}(\{k, l\})$ is a star-shaped clique.

5.2 Non uniform coherence spaces

We give now a direct description of the category $\mathcal{C}(\mathbf{Coh}_2)$ induced by this symmetric product phase space $\mathbf{Coh}_2$.

Definition 23 A non-uniform coherence space is a triple $E = (|E|, \frown_E, \smile_E)$ where $|E|$ is a finite or denumerable set (the web of E) and $\frown_E$ and $\smile_E$ are two binary symmetric relations on $|E|$ called respectively *coherence* and *incoherence*. The only requirement on these relations is that they must have an empty intersection: one cannot have simultaneously $a \frown_E a'$ and $a \smile_E a'$, when $a, a' \in |E|$.

Observe in particular that we do not require these relations to be anti-reflexive, as in the standard coherence semantics of linear logic. Later, we shall exhibit situations where these relations are neither reflexive, nor anti-reflexive. Coherence graphs and non-uniform coherence space are almost the same notions, they differ only by the fact that, in a non-uniform coherence space, the relations are *not* restricted to pairs of distinct elements of the web.

We adopt for non-uniform coherence spaces exactly the same notational conventions as for coherence graphs, and we observe that, in the same way, they can be specified by giving various pairs of relations on the web.

Definition 24 Let E be a non-uniform coherence space. A *clique* of E is a subset x of $|E|$ such that, for any $a, a' \in x$, one has $a \coh_E a'$.

Observe in particular that if x is a clique of E and if $a \in x$, one must have $a \coh_E a$.

- If E is a non-uniform coherence space, $E^\perp$ is the non-uniform coherence space defined by $|E^\perp| = |E|$, and $a \frown_{E^\perp} a'$ iff $a \smile_E a'$ and $a \smile_{E^\perp} a'$ iff $a \frown_E a'$. So that in particular $a \mathrel{\mathsf{n}}_{E^\perp} a'$ iff $a \mathrel{\mathsf{n}}_E a'$.
- If E and F are non-uniform coherence space, one defines a non-uniform coherence space $E \multimap F$ as follows: $|E \multimap F| = |E| \times |F|$, and when $(a, b), (a', b') \in |E \multimap F|$, one says that $(a, b) \mathrel{\mathsf{n}}_{E\multimap F} (a', b')$ iff $a \mathrel{\mathsf{n}}_E a'$ and $b \mathrel{\mathsf{n}}_F b'$, and one says that $(a, b) \frown_{E\multimap F} (a', b')$ iff $a \smile_E a'$ or $b \frown_F b'$.

Observe that if x is a clique of E and x' is a clique of $E^\perp$, the set $x \cap x'$ can have more than one element. However, if $a, a' \in x \cap x'$, then $a \mathrel{\mathsf{n}}_E a'$.

We define now the category **nuCS** of non-uniform coherence spaces. Its objects are the non-uniform coherence spaces, and if E and F are non-uniform coherence spaces, a morphism from E to F is a clique of the non-uniform coherence space $E \multimap F$ defined above. The identity at E is as usual the diagonal subset $\{(a, a) \mid a \in |E|\}$, and if s is a clique of $E \multimap F$ and t is a clique of $F \multimap G$ (where E, F and G are non-uniform coherence spaces), one defines as usual

$$t \circ s = \{(a, c) \in |E \multimap G| \mid \exists b \in |F| \ (a, b) \in s \text{ and } (b, c) \in t\} .$$

One checks easily that the identity and that $t \circ s$ are cliques in the corresponding non-uniform coherence spaces. When s is a clique of $E \multimap F$, one writes $s : E \multimap F$.

We construct an isomorphism between the category $\mathcal{C}(\mathbf{Coh}_2)$ and the category **nuCS**.

First, given a non-uniform coherence space E, we define for each $J \subseteq I$ a function $\rho_J : |E|^J \to \mathcal{F}_{\mathbf{Coh}_2}(J)$. If $\alpha \in |E|^J$, we define the coherence graph $\mathcal{G}_\alpha$ on J as follows: when $j, j' \in J$ are distinct, the relation between j and j' in $\mathcal{G}_\alpha$ is the same as the relation between α_j and $\alpha_{j'}$ in E. Then we set $\rho_J(\alpha) = \mathsf{f}(\mathcal{G}_\alpha)$. One checks that the family of functions $(\rho_J)_{J\subseteq I}$ is a natural transformation from the functor $\mathrm{Fam}_{|E|}$ to the functor $\mathcal{F}_{\mathbf{Coh}_2}$ (from the category $\mathcal{I}(I)$ to the category **Set**), that is, when $u : K \to J$ is injective one has $u^*\mathsf{f}(\mathcal{G}_\alpha) = \mathsf{f}(\mathcal{G}_{u^*\alpha})$. We denote by E^+ the $\mathbf{Coh}_2$-space $(|E|, \rho)$.

Conversely, let X be a $\mathbf{Coh}_2$-space. We define a non-uniform coherence space X^- as follows. First, $|X^-| = |X|$. Then, let $a, a' \in |X|$. Let $j, j' \in I$ be distinct. Then $\{(j, a), (j', a')\} \in |X|^{\{j,j'\}}$ and so $\widehat{X}_{\{j,j'\}}(\{(j, a), (j', a')\})$ is a fact of the local phase space $\mathbf{Coh}_2(\{j, j'\})$, which is isomorphic to the phase space $(P_0^2, \perp^2)$. So we decide that $a \frown_{X^-} a'$ if $\widehat{X}_{\{j,j'\}}(\{(j, a), (j', a')\}) = \mathsf{C}$ and that $a \smile_{X^-} a'$ if $\widehat{X}_{\{j,j'\}}(\{(j, a), (j', a')\}) = \overline{\mathsf{C}}$. By the naturality requirement on $\widehat{X}$, this definition does not depend on the choice of j and j'.

Lemma 25 *If E is a non-uniform coherence space, then $(E^+)^- = E$ and if X is a* $\mathbf{Coh}_2$*-space then $(X^-)^+ = X$.*

Proof: We just check that $(X^-)^+ = X$. So let $J \subseteq I$ and let $\alpha \in |X|^J$. Denoting by $\mathcal{G}_\alpha$ the coherence graph associated to α in the non-uniform coherence space X^-, it will be enough to show that $\mathcal{G}_\alpha = \mathsf{g}(\widehat{X}_J(\alpha))$. So let $j, j' \in J$ be distinct. Then $j \frown_{\mathcal{G}_\alpha} j'$ holds iff $\alpha_j \frown_{X^-} \alpha_{j'}$, which in turn holds iff $\widehat{X}_{\{j,j'\}}(\{(j,\alpha_j),(j',\alpha_{j'})\}) = \mathsf{C}$. Denoting by u the injection of $\{j,j'\}$ in J induced by the inclusion $\{j,j'\} \subseteq J$, this latter equation holds iff $\widehat{X}_{\{j,j'\}}(u^*\alpha) = \mathsf{C}$, that is, iff $u^*\widehat{X}_J(\alpha) = \mathsf{C}$ by naturality of $\widehat{X}$. Now this latter equation is equivalent to $j \frown_{\mathsf{g}(\widehat{X}_J(\alpha))} j'$, as announced. Of course, the same reasoning applies to $j \smile_{\mathcal{G}_\alpha} j'$ and to $j \, \mathsf{n}_{\mathcal{G}_\alpha} \, j'$. ∎

Moreover, the notions of clique associated to non-uniform coherence spaces and to $\mathbf{Coh}_2$-spaces coincide. More precisely:

Lemma 26 *Let E be a non-uniform coherence space. Let $x \subseteq |E|$. If x is a clique of the non-uniform coherence space E, then x is a clique of the* $\mathbf{Coh}_2$*-space E^+. Let X be a* $\mathbf{Coh}_2$*-space and let $x \subseteq |X|$. If x is a clique of the* $\mathbf{Coh}_2$*-space X, then x is a clique of the non-uniform coherence space X^-.*

The proof is a straightforward verification.

So these two operations define an isomorphism between the categories **nuCS** and $\mathcal{C}(\mathbf{Coh}_2)$. The category **nuCS** inherits, through this isomorphism, the structures and properties which make $\mathcal{C}(\mathbf{Coh}_2)$ a model of linear logic. We describe directly some of the corresponding space constructions.

Let E and F be non-uniform coherence spaces. The tensor product $E \otimes F$ is given by $|E \otimes F| = |E| \times |F|$, and when $(a,b),(a',b') \in |E \otimes F|$, one says that $(a,b) \, \mathsf{n}_{E\otimes F} \, (a',b')$ iff $a \, \mathsf{n}_E \, a'$ and $b \, \mathsf{n}_F \, b'$, and one says that $(a,b) \bigcirc_{E\otimes F} (a',b')$ iff $a \bigcirc_E a'$ and $b \bigcirc_F b'$.

If E and F are non-uniform coherence spaces, one defines $E_1 \,\&\, E_2$ by $|E_1 \,\&\, E_2| = (\{1\} \times |E_1|) \cup (\{2\} \times |E_2|)$, and then, for $(i,a),(j,b) \in |E_1 \,\&\, E_2|$, one says that $(i,a) \, \mathsf{n}_{E_1\&E_2} \, (j,b)$ iff $i = j$ and $a \, \mathsf{n}_{E_i} \, b$, and that $(i,a) \smile_{E_1\&E_2} (j,b)$ iff $i = j$ and $a \smile_{E_i} b$.

If E is a non-uniform coherence space, one defines $!E$ as the non-uniform coherence space having as web the set of *all* finite multisets of elements of $|E|$. This makes the main difference between the non-uniform coherence space semantics and the standard coherence semantics of linear logic.

Definition 27 Let E be a non-uniform coherence space and let μ be a multiset of elements of $|E|$. Let J be a set and let $\alpha \in |E|^J$ be such that $\mathsf{m}(\alpha) = \mu$ (that is, α is an enumeration of μ). Then let $\mathcal{G}_\alpha$ be the coherence graph on J defined by (for $j, j' \in J$ such that $j \neq j'$) $j \frown_{\mathcal{G}_\alpha} j'$ iff $\alpha_j \frown_E \alpha_{j'}$ and $j \smile_{\mathcal{G}_\alpha} j'$ iff $a_j \smile_E a_{j'}$. One says that μ is a *multiclique* if the coherence graph $\mathcal{G}_\alpha$ is a clique, and one says that μ is a *star-shaped multiclique* if $\mathcal{G}_\alpha$ is a star-shaped clique. If $\beta \in |E|^K$ is another enumeration of the same multiset μ, the coherence graphs $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are isomorphic, so this definition does not depend on the choice of the particular enumeration α of μ.

Observe that if $[a_1, \ldots, a_n]$ is a multiclique of E, then $\{a_1, \ldots, a_n\}$ is not necessarily a clique of E (the converse is true). For instance, if $a \in |E|$, then $[a]$ is always a star-shaped multiclique of E, even if a is not coherent with itself. In that latter case, of course, $[a, a]$ is not even a multiclique.

We obtain the following definition of coherence in $!E$ (expressed with many redundancies). Let $\mu, \nu \in |!E|$, then

- $\mu \bigcirc_{!E} \nu$ if $\mu + \nu$ is a multiclique,

- $\mu \frown_{!E} \nu$ if $\mu + \nu$ is a star-shaped multiclique,
- $\mu \, \mathsf{n}_{!E} \, \nu$ if $\mu + \nu$ is a multiclique which is not star-shaped and
- $\mu \smile_{!E} \nu$ if $\mu + \nu$ is not a multiclique.

Just for playing a little with this definition, we check directly that $\mathrm{p}_E = \{(\sum_{l=1}^{n} \mu_l, [\mu_1, \ldots, \mu_n]) \mid \mu_1, \ldots, \mu_n \in |!E|\} \subseteq |!E \multimap !!E|$ is indeed a clique in $!E \multimap !!E$.

let $\mu_1, \ldots, \mu_r, \mu_{r+1} \ldots \mu_n \in |!E|$, and assume that $\sum_{l=1}^{r} \mu_l \bigcirc_{!E} \sum_{l=r+1}^{n} \mu_l$. This simply means that $\sum_{l=1}^{n} \mu_l$ is a multiclique in E. We have to prove that $[\mu_1, \ldots, \mu_r] \bigcirc_{!!E} [\mu_{r+1} \ldots \mu_n]$, that is, that $[\mu_1, \ldots, \mu_n]$ is a multiclique in $!E$. But this holds, since clearly, for $k, l \in \{1, \ldots, n\}$ with $k \neq l$, one has that $\mu_k + \mu_l$ is a multiclique of E, since $\sum_{l=1}^{n} \mu_l$ is a multiclique in E. Assume moreover that $\sum_{l=1}^{r} \mu_l \frown_{!E} \sum_{l=r+1}^{n} \mu_l$. This means that $\sum_{l=1}^{n} \mu_l$ is a star-shaped multiclique in E. Let J be a set and $\alpha \in |E|^J$ be an enumeration of the multiset $\sum_{l=1}^{n} \mu_l$. One can find pairwise distinct subsets $J_1, \ldots, J_n$ of J such that $\sum_{l=1}^{n} J_l = J$ and such that, for each $l \in \{1, \ldots, n\}$, the restriction of α to J_l be an enumeration of μ_l. Let $i \in J$ be such that, for any $j \in J$ with $j \neq i$, one has $\alpha_j \frown_E \alpha_i$. Let k be the unique element of $\{1, \ldots, n\}$ such that $i \in J_k$. Then for any $l \in \{1, \ldots, n\}$ with $l \neq k$, it is clear that $\mu_k + \mu_l$ is a star-shaped multiclique of E, that is $\mu_k \frown_{!E} \mu_l$, and so $[\mu_1, \ldots, \mu_n]$ is a star-shaped multiclique of $!E$, as required.

5.3 Concrete examples

To illustrate the difference between the standard coherence semantics and the non-uniform coherence semantics presented above, we describe a few simple concrete spaces.

The first thing to observe is that, as long as a formula S of linear logic does not contain exponentials, the non-uniform coherence space E associated to S satisfies the following property: for any $a, a' \in |E|$, one has $a \, \mathsf{n}_E \, a'$ if and only if $a = a'$, so that E can be considered as a standard coherence space, and is actually identical to the coherence space associated to S by the usual coherence semantics.

However, as soon as S contains exponentials, its semantics in non-uniform coherence spaces becomes radically different from its standard interpretation in coherence spaces, where, thanks to uniformity, neutrality and equality are identical, even in the presence of exponentials. This is illustrated by the two following examples.

- The non-uniform coherence space E as well as the coherence space E' interpreting the formula $!1$ have the set of all non negative integers $\mathbf{N}$ as web. For $n, m \in \mathbf{N}$, one has $n \frown_{E'} m$ as soon as $n \neq m$. But $n \frown_E m$ if $n + m = 1$, and, in all other cases, $n \, \mathsf{n}_E \, m$. So already in that simple case, n_E is not an equivalence relation (but is reflexive).
- In the non-uniform coherence space E interpreting $!(1 \oplus 1)$, whose web is (in bijection with) $\mathbf{N} \times \mathbf{N}$, one has $(n, n') \smile_E (n, n')$ as soon as $n \neq 0$ and $n' \neq 0$, so in that case, the set $\{(n, n')\}$ is not a clique of E.

The next example illustrates why the possibility for elements of webs of being incoherent with themselves is essential in this non-uniform setting. By Bool, we denote the space of booleans, whose web is $\{\mathsf{true}, \mathsf{false}\}$ with each point neutral with itself and $\mathsf{true} \smile_{\mathrm{Bool}} \mathsf{false}$. The boolean-PCF term

$$t = \lambda x : \mathrm{Bool}\,.\,\mathsf{if}\ x\ \mathsf{then}\ (\mathsf{if}\ x\ \mathsf{then}\ \mathsf{true}\ \mathsf{else}\ \mathsf{false})\ \mathsf{else}\ (\mathsf{if}\ x\ \mathsf{then}\ \mathsf{true}\ \mathsf{else}\ \mathsf{false})$$

will be interpreted as a clique of $!\mathrm{Bool} \multimap \mathrm{Bool}$. For computing this clique, consider first the "linearized" version of this term

$$t' = \lambda x, y : \mathrm{Bool}\,.\, \mathsf{if}\ x\ \mathsf{then}\ (\mathsf{if}\ y\ \mathsf{then}\ \mathsf{true}\ \mathsf{else}\ \mathsf{false})\ \mathsf{else}\ (\mathsf{if}\ y\ \mathsf{then}\ \mathsf{true}\ \mathsf{else}\ \mathsf{false})$$

whose semantics is the following clique in $!\mathrm{Bool} \otimes !\mathrm{Bool} \multimap \mathrm{Bool}$:

$$\{(([\mathsf{true}], [\mathsf{true}]), \mathsf{true}), (([\mathsf{true}], [\mathsf{false}]), \mathsf{false}), (([\mathsf{false}], [\mathsf{true}]), \mathsf{true}), (([\mathsf{false}], [\mathsf{false}]), \mathsf{false})\}\,.$$

The interpretation of t is obtained by composing t' with the contraction morphism $c : !\mathrm{Bool} \multimap !\mathrm{Bool} \otimes !\mathrm{Bool}$, and it is here that the difference between uniformity and non-uniformity appears (the interpretation of t' in uniform and non-uniform coherence spaces are indeed identical). The non-uniform version of c is

$$\begin{aligned} c_{\mathrm{nu}} \ = \ & \{([\mathsf{true}, \mathsf{true}], ([\mathsf{true}], [\mathsf{true}])), ([\mathsf{true}, \mathsf{false}], ([\mathsf{true}], [\mathsf{false}])), \\ & ([\mathsf{false}, \mathsf{true}], ([\mathsf{false}], [\mathsf{true}])), ([\mathsf{false}, \mathsf{false}], ([\mathsf{false}], [\mathsf{false}]))\} \end{aligned}$$

where we have written "$[\mathsf{true}, \mathsf{false}]$" and "$[\mathsf{false}, \mathsf{true}]$" just for pedagogical reasons, but of course, these multisets are equal. The uniform version of c is

$$c_{\mathrm{u}} \ = \ \{([\mathsf{true}, \mathsf{true}], ([\mathsf{true}], [\mathsf{true}])), ([\mathsf{false}, \mathsf{false}], ([\mathsf{false}], [\mathsf{false}]))\}$$

simply because $[\mathsf{true}, \mathsf{false}]$ does not belong to the web of the standard (uniform) coherence space interpretation of $!\mathrm{Bool}$. So the non-uniform interpretation of t is

$$t^* = \{(([\mathsf{true}, \mathsf{true}]), \mathsf{true}), (([\mathsf{true}, \mathsf{false}]), \mathsf{false}), (([\mathsf{false}, \mathsf{true}]), \mathsf{true}), (([\mathsf{false}, \mathsf{false}]), \mathsf{false})\}\,,$$

whereas its uniform interpretation is

$$\{(([\mathsf{true}, \mathsf{true}]), \mathsf{true}), (([\mathsf{false}, \mathsf{false}]), \mathsf{false})\}\,.$$

Although $\mathsf{true} \smile_{\mathrm{Bool}} \mathsf{false}$, the set t^* is a clique, and this is possible (in view of the coherence in $!\mathrm{Bool} \multimap \mathrm{Bool}$) only because $[\mathsf{true}, \mathsf{false}] \smile_{!\mathrm{Bool}} [\mathsf{true}, \mathsf{false}]$.

We have seen in this example a difference between uniformity and non-uniformity which results from the difference between the interpretations of the contraction rule in both setting. Similar example can be obtained using the difference between the uniform and the non-uniform interpretations of the promotion rule. All the other rules are interpreted in the same way in both settings.

6 Appendix: the interpretation of proofs in the category of sets and relations

To each proof π of a sequent in first order propositional linear logic $\vdash \Phi$, we associate a subset of π^* of the set $|\Phi|$ defined in section 2, by induction on π.

Tensor unit: if the proof π is

$$\overline{\vdash 1}$$

then $\pi^* = \{*\}$.

With unit: if the proof π is

$$\overline{\vdash \Phi, \top}$$

then $\pi^* = \emptyset$.

With: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Psi, T\end{array}}{\vdash \Psi, S \,\&\, T}$$

then $\pi^* = \{(c,(1,a)) \mid (c,a) \in \pi_1{}^*)\} \cup \{(c,(2,b)) \mid (c,b) \in \pi_2{}^*)\}$.

Left plus: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array}}{\vdash \Psi, S \oplus T}$$

then $\pi^* = \{(c,(1,a)) \mid (c,a) \in \pi_1{}^*)\}$. And similarly if π ends with a right plus rule.

Par unit: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi\end{array}}{\vdash \Psi, \bot}$$

then $\pi^* = \{(c,*) \mid c \in \pi_1{}^*\}$.

Par: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S, T\end{array}}{\vdash \Psi, S \parr T}$$

then $\pi^* = \{(c,(a,b)) \mid (c,a,b) \in \pi_1{}^*\}$.

Tensor: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ \vdash \Theta, T\end{array}}{\vdash \Psi, \Theta, S \otimes T}$$

then $\pi^* = \{(c,d,(a,b)) \mid (c,a) \in \pi_1{}^*)$ and $(d,b) \in \pi_2{}^*\}$.

Weakening: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi\end{array}}{\vdash \Psi, ?S}$$

then $\pi^* = \{(c,[]) \mid c \in \pi_1{}^*\}$.

Contraction: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, ?S, ?S\end{array}}{\vdash \Psi, ?S}$$

then $\pi^* = \{(c, x+y) \mid (c,x,y) \in \pi_1{}^*\}$ where $x+y$ denotes the sum of the multisets x and y.

Dereliction: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ \vdash \Psi, S\end{array}}{\vdash \Psi, ?S}$$

then $\pi^* = \{(c,[a]) \mid (c,a) \in \pi_1{}^*\}$.

Promotion: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash ?S^1,\ldots,?S^k,S\end{array}}{\vdash ?S^1,\ldots,?S^k,!S}$$

then π^* is the set of all $k+1$-tuples of the shape $(\sum_{j=1}^n x_j^1,\ldots,\sum_{j=1}^n x_j^k,[a_1,\ldots,a_n])$ where $((x_j^1,\ldots,x_j^k,a_j))_{j=1,\ldots,n}$ is any finite family of elements of ${\pi_1}^*$.

The exchange rule does not deserve particular mention.

Cut: if the proof π is

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \vdash \Psi,S\end{array}\qquad\begin{array}{c}\vdots\ \pi_2\\ \vdash \Theta,S^\perp\end{array}}{\vdash \Psi,\Theta}$$

then $\pi^* = \{(c,d) \mid \exists a\,(c,a)\in{\pi_1}^* \text{ and } (d,a)\in{\pi_2}^*\}$.

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