Call-By-Push-Value in system L style

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Abstract. I have cut down the syntax for bilateral (focalised, classical) system L in my style (appended to this note in a slightly modified version with respect to what I had sent before, where the involutive negation is taken seriously as a connective rather than a defined dual). The resulting subsystem is shown equivalent to CBPV in a precise way.

1 A system L syntax for CBPV

Formulas:

$$P ::= P \otimes P \mid P \oplus P \mid {\downarrow}N$$
$$N ::= N \& N \mid P \to N \mid {\uparrow}P$$

Remark 1. When studying polarities without linearity concerns, it is bit odd to still name $\otimes$ the positive conjunction. May be $\wedge^+$ should be used instead. But for the time being I stick with $\otimes$... .

Remark 2. Exactly the same logical system appears (without reference to CBPV) in lecture notes of F. Pfenning on focusing for intuitionistic logic [2]. This independent appearance witnesses the fact that as long as no precision is given on the "kind of effects" we want to model with the monad associated to $F \dashv U$, we are just doing (intuitionistic) polarisation.

Dictionary wrt to P.B. Levy's notation:

value	computation	Σ	$\times$	UN	FP	Π	$P \to N$
positive	negative	$\oplus$	$\otimes$	${\downarrow}N$	${\uparrow}P$	$\&$	$P \to N$ (thought of as $\overline{P}⅋N$)

We have five syntactic categories:

Values	$V ::= x \mid (V,V) \mid inl(V) \mid inr(V) \mid \mu[.]^{\downarrow}.c$	$\Gamma \vdash V : P\,;$
Negative terms	$v := \mu[.].c \mid \mu([fst].c_1, [snd].c_2) \mid \mu[x \cdot [.]].c \mid V^{\uparrow}$	$\Gamma \vdash v : N \mid$
Covalues	$E ::= [.] \mid E[fst] \mid E[snd] \mid [V \cdot E] \mid \tilde{\mu}x^{\uparrow}.c$	$\Gamma\,;\, E : N \vdash [.] : N'$
Positive contexts	$e ::= \tilde{\mu}x.c \mid \tilde{\mu}(x,y).c \mid \tilde{\mu}[inl(x_1).c_1, inr(x_2).c_2] \mid E^{\downarrow}$	$\Gamma \mid e : P \vdash [.] : N$
Commands	$c ::= \langle v \mid E\rangle \mid \langle V \mid e\rangle$	$c : (\Gamma \vdash [.] : N)$

where Γ is a context of positive formulas.

Dictionary wrt to P.B. Levy's notation (continued):

V (value)	M (computation)	K (stack)		
V (value)	v (negative term)	E (covalue)	e (positive context)	c (command)

Typing rules

$$\frac{}{\Gamma, x : P \vdash x : P\,;} \qquad \frac{}{\Gamma\,;\,[.] : N \vdash [.] : N} \qquad \frac{\Gamma \vdash v : N \mid \qquad \Gamma\,;\,E : N \vdash [.] : N'}{\langle v \mid E\rangle : (\Gamma \vdash [.] : N')} \qquad \frac{\Gamma \vdash V : P\,; \qquad \Gamma \mid e : P \vdash [.] : N}{\langle V \mid e\rangle : (\Gamma \vdash [.] : N)}$$

$$\frac{c : (\Gamma \vdash [.] : N)}{\Gamma \vdash \mu[.].c : N \mid} \qquad \frac{c : (\Gamma, x : P \vdash [.] : N)}{\Gamma \mid \tilde{\mu}x.c : P \vdash [.] : N}$$

$$\frac{c : (\mathit{\Gamma} \vdash [.] : N}{\mathit{\Gamma} \vdash \mu[.]^{\downarrow}.c : {\downarrow}N\,;} \qquad \frac{\mathit{\Gamma} \vdash V_1 : P_1\,; \qquad \mathit{\Gamma} \vdash V_2 : P_2\,;}{\mathit{\Gamma} \vdash (V_1, V_2) : P_1 \otimes P_2\,;} \qquad \frac{\mathit{\Gamma} \vdash V_1 : P_1\,;}{\mathit{\Gamma} \vdash \mathit{inl}(V_1) : P_1 \oplus P_2\,;}$$

$$\frac{\mathit{\Gamma} \vdash V : P\,;}{\mathit{\Gamma} \vdash V^{\uparrow} : {\uparrow}P\,|} \qquad \frac{c : (\mathit{\Gamma}, x : P \vdash [.] : N)}{\mathit{\Gamma} \vdash \mu[x \cdot [.]].c : P \to N\,|} \qquad \frac{c_1 : (\mathit{\Gamma} \vdash [.] : N_1) \quad c_2 : (\mathit{\Gamma} \vdash [.] : N_2)}{\mathit{\Gamma} \vdash \mu([\mathit{fst}].c_1, [\mathit{snd}].c_2) : N_1 \& N_2\,|}$$

$$\frac{\mathit{\Gamma}, x : P \vdash [.] : N}{\mathit{\Gamma}\,;\, \tilde{\mu}x^{\uparrow}.c : {\uparrow}P \vdash [.] : N} \qquad \frac{\mathit{\Gamma} \vdash V : P\,; \qquad \mathit{\Gamma}\,;\, E : N \vdash [.] : N'}{\mathit{\Gamma}\,;\, [V \cdot E] : P \to N \vdash [.] : N'} \qquad \frac{\mathit{\Gamma}\,;\, E_1 : N_1 \vdash [.] : N}{\mathit{\Gamma}\,;\, E_1[\mathit{fst}] : N_1 \& N_2 \vdash [.] : N}$$

$$\frac{\mathit{\Gamma}\,;\, E : N \vdash [.] : N'}{\mathit{\Gamma} \,|\, E^{\downarrow} : {\downarrow}N \vdash [.] : N'} \qquad \frac{c : (\mathit{\Gamma}, x_1 : P_1, x_2 : P_2 \vdash [.] : N)}{\mathit{\Gamma} \,|\, \tilde{\mu}(x_1, x_2).c : P_1 \otimes P_2 \vdash [.] : N} \qquad \frac{c_1 : (\mathit{\Gamma}, x_1 : P_1 \vdash [.] : N) \quad c_2 : (\mathit{\Gamma}, x_2 : P_2 \vdash [.] : N)}{\mathit{\Gamma} \,|\, \tilde{\mu}[\mathit{inl}(x_1).c_1, \mathit{inr}(x_2).c_2] : P_1 \oplus P_2 \vdash [.] : N}$$

Operational semantics:

$$\begin{array}{l}
\langle V \mid \tilde{\mu}x.c\rangle \to c[V/x] \\
\langle \mu[.].c \mid E\rangle \to c[E/[.]] \\
\langle (V_1, V_2) \mid \tilde{\mu}(x_1, x_2).c\rangle \to c[V_1/x_1, V_2/x_2] \\
\langle \mu[x \cdot \alpha].c \mid [V, E]\rangle \to c[V/x, E/\alpha] \\
\langle \mathit{inl}(V_1) \mid \tilde{\mu}[\mathit{inl}(x_1).c_1, \mathit{inr}(x_2).c_2])\rangle \to c_1[V_1/x_1] \\
\langle \mu([\mathit{fst}].c_1, [\mathit{snd}].c_2)) \mid E_1[\mathit{fst}]\rangle \to c_1[E_1/[.]] \\
\langle \mu[.]^{\downarrow}.c \mid E^{\downarrow}\rangle \to c[E/[.]] \\
\langle V^{\uparrow} \mid \tilde{\mu}x^{\uparrow}.c\rangle \to c[V/x]
\end{array}$$

2 Translations

2.1 From CBPV

(read “let V (resp. v, v_1, E,…) be the translation of V (resp M, M_1, K,…)”)

x	$\rightsquigarrow$	x
return V	$\rightsquigarrow$	$V^{\uparrow}$
thunk M	$\rightsquigarrow$	$\mu[.]^{\downarrow}.\langle v \mid [.]\rangle$
$\mathit{\Sigma}$ introduction	$\rightsquigarrow$	$\mathit{inl}, \mathit{inr}$
(V, V')	$\rightsquigarrow$	(V, V')
$\lambda\{1.M_1, 2.M_2\}$	$\rightsquigarrow$	$\mu([\mathit{fst}].\langle v_1 \mid [.]\rangle, [\mathit{snd}].\langle v_2 \mid [.]\rangle)$
$\lambda x.M$	$\rightsquigarrow$	$\mu[x \cdot [.]].\langle v \mid [.]\rangle$
let V be $x.M$	$\rightsquigarrow$	$\mu[.].\langle V \mid \tilde{\mu}x.\langle v \mid [.]\rangle\rangle$
M_1 to $x.M_2$	$\rightsquigarrow$	$\mu[.].\langle v_1 \mid \tilde{\mu}x^{\uparrow}.\langle v_2 \mid [.]\rangle\rangle$
force V	$\rightsquigarrow$	$\mu[.].\langle V \mid [.]^{\downarrow}\rangle$
pm V as $\{(1, x_1).M_1, (2, x_2).M_2\}$	$\rightsquigarrow$	$\mu[.].\langle V \mid \tilde{\mu}[\mathit{inl}(x_1).\langle v_1 \mid [.]\rangle, \mathit{inr}(x_2).\langle v_2 \mid [.]\rangle]\rangle$
pm V as $(x, y).M$	$\rightsquigarrow$	$\mu[.].\langle V \mid \tilde{\mu}(x, y).\langle v \mid [.]\rangle\rangle$
î‘M	$\rightsquigarrow$	$\mu[.].\langle v \mid [.][\mathit{fst}]\rangle$
V‘M	$\rightsquigarrow$	$\mu[.].\langle(v \mid [V \cdot [.]]\rangle$
nil	$\rightsquigarrow$	$[\cdot]$
$[\cdot]$ to $x.M :: K$	$\rightsquigarrow$	$\tilde{\mu}x^{\uparrow}.\langle v \mid E\rangle$
$\hat{1} :: K$	$\rightsquigarrow$	$E[\mathit{fst}]$ (idem $\hat{2}$, snd)
$V :: K$	$\rightsquigarrow$	$[V \cdot E]$

2.2 To CBPV

The three categories e, c, v are translated to computations M, while V, E of course translate to values and stacks. For contexts e, the translation is parameterised by a variable x (the place-holder of e in the sequent). The translation makes use of the dismantling $M \bullet K$ (or read-back) of a state (M, K) as a computation (beginning of Section 3 of [1]).

$$
\begin{array}{l}
x^\dagger = x \\
(\mathit{inl}(V))^\dagger = (\hat{1}, V^\dagger) \quad \text{(idem } \mathit{inr}) \\
(V_1, V_2)^\dagger = ((V_1)^\dagger, (V_2)^\dagger) \\
(\mu[.]^\downarrow.c)^\dagger = \text{thunk } c^\dagger \\
\\
(\mu[.].c)^\dagger = c^\dagger \\
(\mu([\mathit{fst}].c_1, [\mathit{snd}].c_2))^\dagger = \lambda\{1.(c_1)^\dagger, 2.(c_2)^\dagger\} \\
(\mu[x \cdot [.]].c)^\dagger = \lambda x.c^\dagger \\
(V^\uparrow)^\dagger = \text{return } V^\dagger \\
\\
[.]^\dagger = \text{nil} \\
E[\mathit{fst}]^\dagger = \hat{1} :: E^\dagger \quad \text{(idem } \mathit{snd}) \\
[V \cdot E]^\dagger = V^\dagger :: E^\dagger \\
(\tilde{\mu}x^\uparrow.c)^\dagger = [\cdot] \text{ to } x.c^\dagger :: \text{nil} \\
\\
(\tilde{\mu}x.c)^\dagger_x = c^\dagger \\
(\tilde{\mu}(x_1, x_2).c)^\dagger_x = \text{pm } x \text{ as } (x_1, x_2).c^\dagger \\
(\tilde{\mu}[\mathit{inl}(x_1).c_1, \mathit{inr}(x_2).c_2])^\dagger_x = \text{pm } x \text{ as } \{(1, x_1).(c_1)^\dagger_x, (2, x_2).(c_2)^\dagger_x\} \\
(E^\downarrow)^\dagger_x = (\text{force } x) \bullet E^\dagger \\
\\
\langle v \mid E\rangle^\dagger = v^\dagger \bullet E^\dagger \\
\langle V \mid e\rangle^\dagger = e^\dagger_x[V/x]
\end{array}
$$

2.3 Equivalence

One checks easily that the two systems simulate each other. Note that the steps in Levy's machine are somewhat more informative in the system L (analysis to be completed…: switch from CBN mode to CBV mode and conversely). For example, we have that

$$\langle \text{return } V \mid [\cdot] \text{ to } x.M :: K\rangle \to \langle N[V/x] \mid K\rangle$$

becomes

$$\langle V^\uparrow \mid (\tilde{\mu}x.\langle v \mid E\rangle)^\uparrow\rangle \to \langle V \mid \tilde{\mu}x.\langle v \mid E\rangle\rangle \to \langle v[V/x] \mid E\rangle$$

(of the shape $\langle v \mid E\rangle \to \langle V_1 \mid e_1\rangle \to \langle v_2 \mid E_2\rangle$)

To get that the translations are inverse to each other, we need the η-rules (cf. appendix).

3 Categorical interpretation

We use $\mathbf{P}$ and $\mathbf{N}$ for Levy's $\mathcal{C}$ and $\mathcal{D}$, in conformity with the syntax of formulas. We write $\mathbf{0}[P, N]$ where Levy would write $\mathcal{O}_A(\underline{B})$, to suggest a profunctor view.

The idea is to interpret the five categories V, v, E, e, c in $\mathbf{P}$, $\mathbf{N}$, $\mathbf{N}^{op}$, $\mathbf{P}^{op}$ and $\mathbf{0}$, respectively.

Using action notations as proposed by Marcelo, the interpretation of the judgements goes as follows:

$$
\begin{array}{ll}
\Gamma \vdash V : P\,; & \mathbf{P}[\Gamma, P] \\
\Gamma \vdash v : N \mid & \mathbf{N}[\Gamma \lfloor 1_{\mathbf{N}}, N] \\
\Gamma\,;\, E : N_1 \vdash [.] : N_2 & \mathbf{N}^{op}[N_2, \Gamma \lfloor N_1] \\
\Gamma \mid e : P \vdash [.] : N & \mathbf{P}^{op}[1_{\mathbf{P}} \rfloor N, \Gamma \otimes P] \\
c : (\Gamma \vdash [.] : N) & \mathbf{O}[\Gamma, N]
\end{array}
$$

See the discussion on the adjunction $\uparrow \dashv \downarrow$ at the end of the appendix.

A Focalising system L in full bilateral form

$$
\begin{aligned}
P &::= X \mid P \otimes Q \mid P \oplus Q \mid \neg N \mid {\downarrow} N \\
N &::= \overline{X} \mid N \parr N \mid N \& N \mid \neg P \mid {\uparrow} P \\
A &::= P \mid N
\end{aligned}
$$

In sequents, Γ stands for $\ldots, x^+ : P, \ldots, x^- : N, \ldots$, and Δ stands for $\ldots, \alpha^+ : P, \ldots, \alpha^- : N, \ldots$.

Commands	$c ::= \langle v^+ \mid e^+ \rangle \mid \langle v^- \mid e^- \rangle$
Expressions	$v^+ ::= V \mid \mu\alpha^+.c$
	$v^- ::= x^- \mid \mu\alpha^-.c \mid \mu\alpha^{+\uparrow}.c \mid \mu[\alpha_1^-, \alpha_2^-].c \mid \mu(\alpha_1^-[\mathit{fst}].c_1, \alpha_2^-[\mathit{snd}].c_2) \mid \mu x^{+\neg}.c$
Values	$V ::= x^+ \mid (V, V) \mid \mathit{inl}(V) \mid \mathit{inr}(V) \mid v^{-\downarrow}$
Contexts	$e^- ::= E \mid \tilde{\mu} x^-.c$
	$e^+ ::= \alpha^+ \mid \tilde{\mu} x^+.c \mid \tilde{\mu} x^{-\downarrow}.c \mid \tilde{\mu}(x^+, y^+).c \mid \tilde{\mu}[\mathit{inl}(x_1^+).c_1, \mathit{inr}(x_2^+).c_2]$
Covalues	$E ::= \alpha^- \mid [E, E] \mid E[\mathit{fst}] \mid E[\mathit{snd}] \mid e^{+\uparrow} \mid V^{\neg}$

We can factorise a few rules using the following mergings:

$$v ::= v^+ \mid v^- \qquad \alpha ::= \alpha^+ \mid \alpha^- \qquad e ::= e^+ \mid e^- \qquad x ::= x^+ \mid x^-$$

$$\frac{}{\Gamma, x^+ : P \vdash x^+ : P \,;\, \Delta} \qquad \frac{}{\Gamma \mid \alpha^+ : P \vdash \alpha^+ : P, \Delta} \qquad \frac{}{\Gamma \,;\, \alpha^- : N \vdash \alpha^- : N, \Delta} \qquad \frac{}{\Gamma, x^- : N \vdash x^- : N \mid \Delta}$$

$$\frac{\Gamma \vdash v : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle v \mid e \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{c : (\Gamma \vdash \alpha : A \,,\, \Delta)}{\Gamma \vdash \mu\alpha.c : A \mid \Delta} \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu} x.c : A \vdash \Delta} \qquad \frac{\Gamma \vdash V : P \,;\, \Delta}{\Gamma \vdash V : P \mid \Delta} \qquad \frac{\Gamma \,;\, E : N \vdash \Delta}{\Gamma \mid E : N \vdash \Delta}$$

$$\frac{\Gamma \vdash v : N \mid \Delta}{\Gamma \vdash v^{\downarrow} : {\downarrow} N \,;\, \Delta} \qquad \frac{\Gamma \vdash V_1 : P_1 \,;\, \Delta \qquad \Gamma \vdash V_2 : P_2 \,;\, \Delta}{\Gamma \vdash (V_1, V_2) : P_1 \otimes P_2 \,;\, \Delta} \qquad \frac{\Gamma \vdash V_1 : P_1 \,;\, \Delta}{\Gamma \vdash \mathit{inl}(V_1) : P_1 \oplus P_2 \,;\, \Delta} \qquad \frac{\Gamma \vdash V : P \,;\, \Delta}{\Gamma \,;\, V^{\neg} : \neg P \vdash \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha^+ : P, \Delta)}{\Gamma \vdash \mu\alpha^{+\uparrow}.c : {\uparrow} P \mid \Delta} \qquad \frac{c : (\Gamma \vdash \alpha_1^- : N_1 \,,\, \alpha_2^- : N_2 \,,\, \Delta)}{\Gamma \vdash \mu[\alpha_1^-, \alpha_2^-].c : N_1 \parr N_2 \mid \Delta} \qquad \frac{c_1 : (\Gamma \vdash \alpha_1^- : N_1 \,,\, \Delta) \quad c_2 : (\Gamma \vdash \alpha_2^- : N_2 \,,\, \Delta)}{\Gamma \vdash \mu(\alpha_1^-[\mathit{fst}].c_1, \alpha_2^-[\mathit{snd}].c_2) : N_1 \& N_2 \mid \Delta} \qquad \frac{c : (\Gamma, x : P \vdash \Delta}{\Gamma \vdash \mu x^{+\neg}.c : \neg P \mid \Delta}$$

$$\frac{\Gamma \mid e : P \vdash \Delta}{\Gamma \,;\, e^{\uparrow} : {\uparrow} P \vdash \Delta} \qquad \frac{\Gamma \,;\, E_1 : N_1 \vdash \Delta \qquad \Gamma \,;\, E_2 : N_2 \vdash \Delta}{\Gamma \,;\, [E_1, E_2] : N_1 \parr N_2 \vdash \Delta} \qquad \frac{\Gamma \,;\, E_1 : N_1 \vdash \Delta}{\Gamma \,;\, E_1[\mathit{fst}] : N_1 \& N_2 \vdash \Delta}$$

$$\frac{c : (\Gamma, x^- : N \vdash \Delta)}{\Gamma \mid \tilde{\mu} x^{-\downarrow}.c : {\downarrow} N \vdash \Delta} \qquad \frac{c : (\Gamma, x_1^+ : P_1, x_2^+ : P_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}(x_1^+, x_2^+).c : P_1 \otimes P_2 \vdash \Delta} \qquad \frac{c_1 : (\Gamma, x_1^+ : P_1 \vdash \Delta) \quad c_2 : (\Gamma, x_2^+ : P_2 \vdash \Delta)}{\Gamma \mid \tilde{\mu}[\mathit{inl}(x_1^+).c_1, \mathit{inr}(x_2^+).c_2] : P_1 \oplus P_2 \vdash \Delta}$$

Operational semantics:

$$\begin{array}{l}
\langle V \mid \tilde{\mu}x^{+}.c\rangle \to c[V/x^{+}]\\
\langle \mu\alpha^{-}.c \mid E\rangle \to c[E/\alpha^{-}]\\
\langle v^{-} \mid \tilde{\mu}x^{-}.c\rangle \to c[v^{-}/x^{-}]\\
\langle \mu\alpha^{+}.c \mid e^{+}\rangle \to c[e^{+}/\alpha^{+}]\\
\langle (V_1,V_2) \mid \tilde{\mu}(x_1^{+},x_2^{+}).c\rangle \to c[V_1/x_1^{+},V_2/x_2^{+}]\\
\langle \mu[\alpha_1^{-},\alpha_2^{-}].c \mid [E_1,E_2]\rangle \to c[E_1/\alpha_1^{-},E_2/\alpha_2^{-}]\\
\langle \mathit{inl}(V_1) \mid \tilde{\mu}[\mathit{inl}(x_1^{+}).c_1,\mathit{inr}(x_2^{+}).c_2]\rangle \to c_1[V_1/x_1^{+}]\\
\langle \mu(\alpha_1^{-}[\mathit{fst}].c_1,\alpha_2^{-}[\mathit{snd}].c_2) \mid E_1[\mathit{fst}]\rangle \to c_1[E_1/\alpha_1^{-}]\\
\langle v^{-\downarrow} \mid \tilde{\mu}x^{-\downarrow}.c\rangle \to c[v^{-}/x^{-}]\\
\langle \mu\alpha^{+\uparrow}.c \mid e^{+\uparrow}\rangle \to c[e^{+}/\alpha^{+}]\\
\langle \tilde{\mu}x^{+\neg}.c \mid V^{\neg}\rangle \to c[V/x^{+}]
\end{array}$$

For such full bilateral sequents, the interpretation might look as follows (Γ's positive and Δ's negative): $\Gamma_1,\Delta_1 \vdash V : P\,;\,\Gamma_2,\Delta_2$ could live in $\mathbf{P}[\Gamma\rfloor\Delta,P]$, where $\Gamma=\Gamma_1\otimes\neg\Delta_2$ and Δ is defined similarly using the tensor in $\mathbf{N}$. Etc… (not sure of what I am writing here...).

A.1 Two syntactical adjunctions

In this section, we give evidence that

$$\begin{array}{ll}
\downarrow\dashv\uparrow & \text{at the level of positive contexts and negative terms}\\
\uparrow\dashv\downarrow & \text{at the level of covalues and values}
\end{array}$$

As we shall see, the first adjunction is in some sense more primitive than the second, since our choice of pattern-matching notation for the syntax of system L, which favours the right invertibility of $\uparrow$, allows to define the pattern-matching notation suggested by the second adjunction as syntactic sugar, but *not conversely*.

We start with the first adjunction, which is mediated by command judgements. We exhibit the inverse syntactic isomorphisms. We need two η-rules (which express invertibility):

$$\begin{array}{ll}
v^{-}=\mu\alpha^{+\uparrow}.\langle v^{-}\mid\alpha^{+\uparrow}\rangle & (\text{for } \Gamma\vdash v^{-}:\uparrow P\mid\Delta)\\
e^{+}=\tilde{\mu}x^{-\downarrow}.\langle x^{-\downarrow}\mid e^{+}\rangle & (\text{for } \Gamma\mid e^{+}:\downarrow N\vdash\Delta)
\end{array}$$

$$\begin{array}{ccc}
\Gamma\vdash v^{-}:\uparrow P\mid\Delta & v^{-} & \mu\alpha^{+\uparrow}.c\\
 & \downarrow & \uparrow\\
c:(\Gamma\vdash\alpha^{+}:P,\Delta) & \langle v^{-}\mid\alpha^{+\uparrow}\rangle & c\\[1em]
\Gamma\mid e^{+}:\downarrow N\vdash\Delta & e^{+} & \tilde{\mu}x^{-\downarrow}.c\\
 & \downarrow & \uparrow\\
c:(\Gamma,x^{-}:N\vdash\Delta) & \langle x^{-\downarrow}\mid e^{+}\rangle & c
\end{array}$$

so that putting these isos together we obtain isos between

$$\Gamma,x^{-}:N\vdash v^{-}:\uparrow P\mid\Delta \qquad c:(\Gamma,x^{-}:N\vdash\alpha^{+}:P,\Delta) \qquad \Gamma\mid e^{+}:\downarrow N\vdash\alpha^{+}:P,\Delta$$

Before turning to the second adjunction, we define syntactic sugar (for which the choice of notation will be justified by the second adjunction). We *define*:

$$\begin{array}{l}
\mu\alpha^{-\downarrow}.c=(\mu\alpha^{-}.c)^{\downarrow}\\
E^{\downarrow}=\tilde{\mu}x^{-\downarrow}.\langle x^{-}\mid E\rangle\\
\tilde{\mu}x^{+\uparrow}.c=(\tilde{\mu}x^{+}.c)^{\uparrow}\\
V^{\uparrow}=\mu\alpha^{+\uparrow}.\langle V\mid\alpha^{+}\rangle
\end{array}$$

With the following derived typing rules:

$$\frac{c : (\Gamma \vdash \alpha^- : N, \Delta)}{\Gamma \vdash \mu\alpha^{-\downarrow}.c : {\downarrow}N \,;\, \Delta} \qquad \frac{\Gamma \,;\, E : N \vdash \Delta}{\Gamma \mid E^{\downarrow} : {\downarrow}N \vdash \Delta} \qquad \frac{c : (\Gamma, x^+ : P \vdash \Delta)}{\Gamma \,;\, \tilde{\mu}x^{+\uparrow}.c : {\uparrow}P \vdash \Delta} \qquad \frac{\Gamma \vdash V : P \,;\, \Delta}{\Gamma \vdash V^{\uparrow} : {\uparrow}P \mid \Delta}$$

We need two new η-rules, which again express invertibility, but this time in relation with the second adjunction, and which are "by value" (cf. $\lambda x.Vx$ in CBV λ-calculus):

$$\begin{array}{ll} V = \mu\alpha^{-\downarrow}.\langle V \mid \alpha^{-\downarrow}\rangle & (\text{for } \Gamma \vdash V : {\downarrow}N \,;\, \Delta) \\ E = \tilde{\mu}x^{+\uparrow}.\langle x^{+\uparrow} \mid E\rangle & (\text{for } \Gamma \,;\, E : {\uparrow}P \vdash \Delta) \end{array}$$

(note that expanding the macros, these η-rules are uglier and less suggestive, e.g., $V = (\mu\alpha^-.\langle V \mid \tilde{\mu}x^{-\downarrow}.\langle x \mid \alpha\rangle\rangle)^{\downarrow}$)

We are now ready to exhibit the second adjunction (again mediated by command judgements)

$$\begin{array}{ccc} \Gamma \vdash V : {\downarrow}N \,;\, \Delta & V & \mu\alpha^{-\downarrow}.c \\ & \downarrow & \uparrow \\ c : (\Gamma \vdash \alpha^- : N, \Delta) & \langle V \mid \alpha^{-\downarrow}\rangle & c \\ \Gamma \,;\, E : {\uparrow}P \vdash \Delta & E & \tilde{\mu}x^{+\uparrow}.c \\ & \downarrow & \uparrow \\ c : (\Gamma, x^+ : P \vdash \Delta) & \langle x^{+\uparrow} \mid E\rangle & c \end{array}$$

and putting this together, we get the chain of isos so that putting these isos together we obtain isos between

$$\Gamma, x^+ : P \vdash V : {\downarrow}N \,;\, \Delta \qquad c : (\Gamma, x^+ : P \vdash \alpha^- : N, \Delta) \qquad \Gamma \,;\, E : {\uparrow}P \vdash \alpha^- : N, \Delta$$

Notice that in the design choices, I applied Guillaume's guide-line to associate pattern)matching to the invertibility of the right adjoint.

Could we have taken the defined syntax, tailored to the second adjustment, as primitive? Well, one recovers easily $v^{-\downarrow}$ defined as $(\mu\alpha^{-\downarrow}.c)^{\downarrow}$, but for $\tilde{\mu}^{-\downarrow}.c$ we cannot form $(\tilde{\mu}x^-.c)^{\downarrow}$ because $\tilde{\mu}x^-.c$ is not a covalue.

Now, what happens when cutting down to CBPV (or to LLP, or to intuitionistic logic) is that there is no space to express the first adjunction: there are no sequents in which there is a variable x of negative type in the (left) context, and similarly no sequents with a variable α of positive type on the right (only $[.] : N$ is available). But the macros make sense (and are not macros anymore in the cut-down version) and the second adjunction persists. One sees only the second one

Note that this situation corresponds to the following categorical fact. Let $F \dashv G$ be an adjunction between $\mathbf{C}$ and $\mathbf{C}'$. Then this induces an adjunction $\underline{G} \dashv \underline{F}$ between the Kleisli category of the induced monad T and the coKleisli category of the induced comonad D. It goes like this. The functors $\underline{F}$ and $\underline{G}$ act as F and G on objects. A Kleisli morphism from Gd to c is a morphism from Gd to GFc which by adjunction corresponds to a coKleisli morphism for d to Fc. On morphisms, the definition of $\underline{F}$ goes as follows. Given $f : c_1 \to GFc_2$, we define $\underline{F}f$ by composing the the counit at Fc_1 with the morphism from $Fc_1 \to Fc_2$ obtained by adjunction.

References

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