LMFI Cours Algèbre Homotopique et catégories supérieures

Final examination, Monday, April 4 2022 (9am-12am)

You may answer the questions in French or in English. The examination consists of two problems. I do not expect that you'll have the time necessary to treat both with equal attention, though I would expect you to tackle both, and at least one in some depth.

I) In my lectures, I only mentioned that, parallel to strictification of monoidal categories, there is a similar "splitification" of fibrations. In the course notes (CoursB, p.26), I describe it, but you should not need to refer to the notes, as I recall here the definition given there. The goal of the problem is to show the equivalence with another definition, and to show, using this other definition, that we indeed get a split fibration from a fibration (resp. comprehension structure) and a faithful functor.

Here are the two definitions. We start from a Grothendieck fibration $p : \mathbb{E} \to \mathbb{B}$, and we define a new fibration (on the same basis) $p' : \mathbb{E}' \to \mathbb{B}$ (resp. $p'' : E'' \to \mathbb{B}$) as follows:

A) $Obj(\mathbb{E}')$ consists of the pairs (X, ϕ), where $\phi : \mathtt{dom}_X \to p$, where $\mathtt{dom}_X$ is the domain fibration $\mathbb{B}/X \to \mathbb{B}$. We require ϕ to be a morphism of fibrations, i.e. to preserve cartesian morphisms, which amounts to say that $\phi(f : g_1 \to g_2)$ is cartesian for every morphism in $\mathbb{B}/X$. We set $p'(X, \phi) = X$. For morphisms, $\mathbb{E}'[(X, \phi), (Y, \psi)]$ consists of the pairs (t, μ) where $t \in \mathbb{B}[X, Y]$ and μ is a natural transformation from ϕ to $\psi \circ \mathtt{dom}_t$ over $\mathbb{B}$, i.e. $\mu_s : \phi(s) \to \psi(t \circ s)$ and $p(\mu_s) = id_Z$ for all $s : Z \to X$.

B) $Obj(\mathbb{E}'')$ consists of the pairs $(A, -_A)$ where $A \in Obj(E)$ and $-_A$ is a *local cleavage* for A, i.e., a choice, for each $(Y, u : Y \to p(A))$, of a cartesian lifting $u_A : A[u] \to A$ in $\mathbb{E}$. One sets $p''(A, -_A) = p(A)$. Morphisms of $\mathbb{E}''$ are just morphisms between the underlying objects of $\mathbb{E}$ (thus discarding the local cleavage information).

I.1) Translate carefully from one definition to the other (and conversely). Does your analysis give rise to an isomorphism of fibrations (or just to an equivalence) between p' and p''? For completing the answer to this question, you will need Question I.2. On the way, prove that p''' is a fibration.

I.2) Show that, for an arbitrary fibration $p : \mathbb{E} \to \mathbb{B}$, if $g \circ f$ and g are cartesian, so is f (cf. pullback lemma!).

I.3) We recall that a (cloven) fibration is called *split* when composites of chosen cartesian morphisms are chosen cartesian morphisms, i.e., when

$$(u : X \to Y\,,\, v : Y \to Z\,,\, p(C) = Z) \;\Rightarrow\; (C[v][u] = C[v \circ u] \text{ and } v_C \circ u_{C[v]} = (v \circ u)_C).$$

Also, one requires, for all A over X, $A[id] = A$ and $(id_X)_A = id_A$. Show that the fibration p'' above is split (and, before that, exhibit a cleavage for it).

I.4) Exhibit the associated comprehension structure (again in style (B)).

I.5) Exhibit the fibration functor from p to p'' over $\mathbb{B}$ and show that it is faithful.

I.6) The construction of p' arises as a right adjoint to a forgetful functor from split fibrations to fibrations. Can you give evidence for this adjunction? Here, style (A) might be more natural (it was indeed synthesised in this way by Giraud).

I.7) There is also a left adjoint. Can you sketch its construction (again in style (A))?

II) Let A be a fixed type (throughout the problem). A globular context is a context constructed by the following rules: (i) $x : A$ is a globular context, (ii) if $(\Gamma_1, x : B, \Gamma_2)$ is a globular context, then so is $(\Gamma_1, x : B, \Gamma_2, y : B)$, (iii) if $(\Gamma_1, x : B, \Gamma_2, y : B, \Gamma_3)$ is a globular context, then so is $(\Gamma_1, x : B, \Gamma_2, y : B, \Gamma_3, f : x =_B y)$, where y, f are variables that have not appeared so far.

II.1) Show that globular contexts Γ are well-formed contexts in type theory (i.e. $\Gamma\ \mathsf{ctx}$ can be proved, assuming $\vdash A : U$).

II.2) Show how to define a globular set G_Γ from a globular context. [Recall the category $\mathbf{G}$ of globes from CoursC. Objects are natural numbers, and morphisms are (iterated) source and target satisfying (contravariantly) "source of source = source of target" and "target of source = target of target".]

II.3) Show that every finite globular set G such that $G_0 \neq \emptyset$ arises in this way.

II.4) Can you adjust the definition of globular context so as to account for all finite globular sets (and revisit questions II.2 and II.3, mutatis mutandis)?

II.5) We define the equivalence relation $\sim$ between globular contexts as the smallest one such that $(\Gamma_1, u_1 : B_1, u_2 : B_2, \Gamma_2) \sim (\Gamma_1, u_2 : B_2, u_1 : B_1, \Gamma_2)$ provided the two hand sides in this base case of $\sim$ are globular contexts. Show that, for any two globular contexts Γ, Δ, we have $G_\Gamma = G_\Delta$ if and only if $\Gamma \sim \Delta$.

II.6) For G a globular set (i.e. $G \in \mathbf{Set}^{\mathbf{G}^{op}}$), we define the partial (strict) order $\triangleleft$ on globes of G (i.e., elements of G_n for some n) as the transitive closure of the following relation:

$$s(x) \triangleleft x \qquad\qquad x \triangleleft t(x).$$

Display the order on G_Γ, for two or three of your favourite globular contexts that yield a pasting diagram π (viewed as the associated globular set $\hat{\pi}$).

II.7) Same question with one or two globular contexts Γ of your choice such that G_Γ is *not* (coming from) a pasting diagram.

The next goal of the exercise is to show that pasting diagrams are characterised among finite globular sets G as those for which $\triangleleft$ is a linear (or total) order, i.e., such that we can write $\bigcup_{n\geq 0} G_n = \{a_1 \triangleleft \cdots \triangleleft a_n\}$.

II.8) Take your favourite Batanin tree (e.g. the one in my notes). Can you exhibit the $\triangleleft$ order on it and verbalise it in terms of nodes and sectors?

II.9) (A bit more difficult) Given a finite globular set G such that $\triangleleft$ is a total order, show (by some induction) how to associate with it a Batanin tree.

II.10) We come back to syntax. The goal is to find out canonical representatives for $\sim$-equivalence classes. The following device consists in applying transitions to pointed globular contexts, i.e., pairs of a context, and of a highlighted item in it. Those pairs are written formally as $(\Gamma \searrow x : B)$. Here are the rules:

$$\begin{array}{l} (\Gamma \searrow x : B) \longrightarrow (\Gamma, y : B, f : (x =_B y) \searrow f : (x =_B y)) \quad (y, f \text{ new names}) \\ (\Gamma \searrow f : (x =_B y)) \longrightarrow (\Gamma \searrow y : B) \end{array}$$

The initial state of the device is $(x : A) \searrow (x : A)$. Experiment with this device, exhibiting its non-determinism (different transitions may be possible from a given state). What is the good notion of terminal state for the device? Can you give a topological intuition behind the first rule above?

II.11) With that notion of terminal state, show that each accepted context (i.e. the first component Γ of a final state) reads as a pasting diagram π (i.e. $G_\Gamma = \hat{\pi}$), and that this establishes a bijection (which we denote by $\pi \mapsto \Gamma^\pi$) between pasting diagrams and accepted contexts. Show moreover that reading an accepted context from left to right exactly displays the order $\triangleleft$.

II.12) Give evidence that your examples in Question II.7 are not obtainable via the device.

II.13) Show that for every globular context Γ such that $G_\Gamma = \hat{\pi}$ for some pasting diagram π, we have $\Gamma \sim \Gamma^\pi$, and that if π_1 and π_2 are pasting diagrams such $\Gamma^{\pi_1} \sim \Gamma^{\pi_2}$, then $\pi_1 = \pi_2$.

II.14) Illustrate the previous question with an example: take a pasting diagram π, describe it as a globular context Γ that is not in the form produced by the device, and transform it to Γ^π by successive steps of $\sim$.