LMFI Cours Algèbre Homotopique et catégories supérieures

House work, February 2021

You may answer the questions in French or in English. You can either send a manuscript or a typed file. Don't forget to mention your name + student number + university + name of master program on the file, and to mention whether you intend to validate the course or if you just want to get a feed-back (in which case giving your name + master program is enough). Please return your copies **by February 28 evening**.

The four problems are of inequal length. I do not necessarily expect that you'll have the time necessary to treat the four problems with equal attention. I would prefer you to treat one of the longer problems I or II and one of the shorter problems III and IV in some depth than to do a little bit of everything superficially.

I) The goal of this problem is to prove that there exists exactly one non-identity isomorphism (in **Cat**) from Δ to Δ. Recall the simplicial category Δ from the course notes (Lecture 2 on simplicial sets). You may treat the following questions somewhat in parallel: progress in one can help for the others.

1) Find out the natural candidate for this isomorphism and check that it is indeed an isomorphism.

Therefore, we have at least one non-identity isomorphism. The rest of the problem focuses on uniqueness. Let $\rho : \Delta \to \Delta$ be an isomorphism.

2) Show that any isomorphism between two categories must preserve limits and colimits (try to find a one-line argument!)

3) Show that ρ is necessarily identity-on-objects, i.e., $\rho([n]) = [n]$ for all n (hint: focus on the action of ρ on $\Delta([0],[n])$).

4) Show that each object $[n]$ of Δ is the colimit of a diagram $F : I \to \Delta$ such that $Fi \in \{[0],[1]\}$ for all $i \in I$.

5) Show that ρ is entirely determined by its action on all homsets $\Delta([0],[n])$ and $\Delta([1],[n])$.

6) Show that there are only two choices for ρ (hint: focus on the commutative triangles $f \circ d^1$ and $f \circ d^0$, where f ranges over $\Delta([1],[n])$ and d^1, d^0 are the two maps in $\Delta([0],[1])$).

II) Recall from the course notes (Lecture 2) that the functor $\Delta^\bullet_{top} : \Delta \to \mathbf{Top}$ maps $[n]$ to

$$\Delta^n_{top} = \{(t_0, \ldots, t_n) \in \mathbb{R} \mid (\forall i \; t_i \geq 0) \text{ and } \Sigma_{i \in [n]} t_i = 1\}.$$

The t_i's are called barycentric coordinates. The functor part of $\Delta^\bullet_{top}$ is defined as follows, for $\alpha : [m] \to [n]$: $\Delta^\alpha_{top}(t_0, \ldots, t_m) = (t'_0, \ldots, t'_n)$, where $t'_i = \Sigma_{\{j \mid \alpha(j)=i\}} t_j$ for all $i \in [n]$. We use the notation $\mathbf{t}$ for $(t_0, \ldots, t_n)$ and $\alpha \cdot \mathbf{t}$ for $\Delta^\alpha_{top}(t_0, \ldots, t_m)$.

1) Check that these data indeed define a functor from Δ to **Top**.

2) A point $(t_0, \ldots, t_n)$ is called interior in Δ^n_{top} if $t_i > 0$ for all i. Justify this terminology from the point of view of topology.

3) Show that if $\alpha : [n] \to [p]$ is surjective and $\mathbf{t}$ is interior in Δ^n_{top}, then $\alpha \cdot \mathbf{t}$ is interior in Δ^p_{top}.

4) Let $\mathbf{t}$ be an arbitrarily chosen interior point of Δ^m_{top}. Show that the map

$$\alpha \mapsto (\alpha \cdot \mathbf{t}) : \Delta([m],[n]) \to \Delta^n_{top}$$

is injective. (Hint: consider the smallest i at which two distinct $\alpha, \beta : [m] \to [n]$ differ.)

In the following questions, we shall introduce and use a different system of coordinates for the points in the topological simplex Δ^n_{top}.

5) Show that there is a bijection from Δ^n_{top} to the set of $(n+2)$-tuples $(s_{-1}, s_0, \ldots, s_{n-1}, s_n)$ such that

$$0 = s_{-1} \leq s_0 \leq \cdots \leq s_{n-1} \leq s_n = 1$$

We call s_i the i-th sum coordinate (this name provides a hint!).

6) Show that under this new description of Δ^n_{top}, we have, for all generators s^i and d^i of Δ (renamed here σ^i and δ^i to avoid notational clashes):

$$\begin{array}{l}(\sigma^i \cdot (s_{-1}, \ldots, s_{n+1})) = (s_{-1}, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n+1}) \\ (\delta^i \cdot (s_{-1}, \ldots, s_{n-1})) = (s_{-1}, \ldots, s_{i-1}, s_{i-1}, \ldots, s_{n-1})\end{array}$$

Show that for arbitrary $\alpha : [n] \to [p]$ and $u \in \Delta^n_{top}$ and positive real number s, if s occurs as a sum coordinate of $\alpha \cdot u$, then it occurs as a sum coordinate of u.

7) Show that the interior points are the tuples such that $0 < s_0 < \cdots < s_{n-1} < 1$, i.e., are the points whose sum coordinates are all distinct.

8) Show that if $u \in \Delta^m_{top}$ is interior, then for any $[n]$, the map $\alpha \mapsto (\alpha \cdot u)$ is a bijection from $\Delta([m],[n])$ to the set of elements of Δ^n_{top} whose sum coordinates are all among the sum coordinates of u.

9) Show that if $u \in \Delta^m_{top}$ is interior and if $\alpha : [m] \to [n]$ is such that the i-th sum coordinate of u does not occur in $\alpha \cdot u$, then there exists β such that $\alpha = \beta \sigma^i$.

Recall that the topological realisation of a simplicial set is defined (as a set) by:

$$|X| = (\Sigma_{n \in \mathbb{N}} (X_n \times \Delta^n_{top})) / \sim$$

where $\sim$ is the smallest equivalence relation containing $(X\alpha x, u) \sim (x, (\alpha \cdot u))$, for all $\alpha : [m] \to [n]$, $x \in X_n$ and $u \in \Delta^m_{top}$. We write $[x, v]$ for the equivalence class of (x, v).

10) Show that each equivalence class in $|X|$ has a representative (x, v) such that x is non-degenerate and v is interior (hints: use Eilenberg-Zilber's lemma and Question 3).

11) Show that $|\Delta^n|$ is in bijection with Δ^n_{top} (recall that Δ^n, not to be confused with Δ^n_{top}, is the image of $[n]$ under the Yoneda embedding).

12) Recall that products of simplicial sets are defined pointwise: $(X \times Y)_n = X_n \times Y_n$. The projections $X \times Y \to X$ and $X \times Y \to Y$ induce maps $|X \times Y| \to |X|$ and $|X \times Y| \to |Y|$. Spell out these maps. Combining with the previous question, we have a map $h : |\Delta^p \times \Delta^q| \to \Delta^p_{top} \times \Delta^q_{top}$. Spell out this map h explicitly, and justify how it was synthesised.

13) Show that h is a bijection. We provide the following hints. Let $u \in \Delta^p_{top}$ and $v \in \Delta^q_{top}$ (in sum coordinates format), and consider the set of positive reals appearing as sum coordinates in u or in v. This gives rise to a an internal point w in some Δ^r_{top} (cf. Question 7). Use then Question 8 to define the candidate g for being the inverse of h. For proving the harder side $g \circ h = id$, use Questions 10 and 9.

(One can then use a compactness argument to prove that h is in fact a homeomorphism. One can then exploit this isomorphism to show that, more generally, topological realisation preserves products.)

III) We consider the set $\mathbb{R}_+ = \{x \in \mathbb{R} | \, x \geq 0\}$ of positive real numbers, considered as a category with the order reversed, i.e., there is a morphism $r \to s$ exactly when $s \leq r$.

1) Show that, setting $r \otimes s = r + s$, $\mathbb{R}_+$ is a symmetric monoidal category.

2) Find out what the notion of $\mathcal{V}$-category boils down to when $\mathcal{V}$ is a monoidal category that is a preorder (i.e., for all v, w, $\mathcal{V}(v, w)$ has at most one element).

3) Instantiate this further in the case where $\mathcal{V} = \mathbb{R}_+$. To be more suggestive, write X for the collection of objects and $d(x, y)$ for the hom-object of x and y. Contrast this with the notion of metric space (X, d).

4) What does the underlying category of an $\mathbb{R}_+$-category boil down to?

5) Make explicit what an $\mathbb{R}_+$-functor from (X, d) to $\underline{\mathbb{R}_+}$ is (remember from the course notes, Lecture 3, that a monoidal-closed category $\mathcal{V}$ gives rise to a $\mathcal{V}$-category $\underline{\mathcal{V}}$).

6) Make explicit what a tensored $\mathbb{R}_+$-category is. Can you find an example?

IV) Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ be tensored $\mathcal{V}$-categories.

1) Show that if F is a $\mathcal{V}$-functor from $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$, then there are natural maps $\kappa : v \otimes Fm \to F(v \otimes m)$ in the underlying category $\mathcal{N}$ satisfying some compatibilities with the canonical isomorphisms $\phi : I \otimes m \to m$ and $\psi : v \otimes (w \otimes m) \to (v \otimes w) \otimes m$. Spell out these compatibilities (cf. the definition of lax monoidal functor given in the course notes), and prove that κ satisfies them.

2) Conversely, show that an ordinary functor F together with such natural compatible maps κ gives rise to a $\mathcal{V}$ functor whose underlying functor is F.

3) Show that the correspondences established in the previous two questions are inverse.

4) Show that if the underlying functor F is a left adjoint, then κ is an isomorphism. (Hint: you may refer to p. 19 of the course notes, Lecture 3, instantiate the series of isomorphisms at the bottom of this page with $n = F(v \otimes m)$, and prove that the iso from $v \otimes Fm$ to $F(v \otimes m)$ obtained via these isomorphisms from the identity morphism on $F(v \otimes m)$ is in fact equal to κ as synthesised in Question 1.)