Lecture 0 Categories (reminders + few complements)

- limits and colimits:

● If I has a terminal object 1 and $F : I \to \mathbb{C}$,

then $\boxed{\operatorname{colim} F = F1}$ (- colimiting cocone $\lambda_i = F(c \to 1)$ - every cocone γ has component id at 1)

● Representable functors $\mathbb{C}(-, c)$ take colimits to limits

+ covariant version: the functors $\mathbb{C}(c, -)$ preserve limits.

It is a matter of packaging the information.

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr] & \\ F_i & & F_j \end{tikzcd}$$

First, there is an adjunction situation

$$\Delta : \mathbb{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbb{C}^{I} : \lim$$

(Δ : constant / diagonal $\Delta c = i \mapsto c$)

$\mathbb{C}^{I}(\Delta c, F) = \mathbb{C}(c, \lim F)$

We have to prove $(I \xrightarrow{F} \mathbb{C} \xrightarrow{\mathbb{C}(c,-)} \mathrm{Set})$

$\mathbb{C}(a, F)$ for what

$$\mathbb{C}(c, \lim F) = \lim(\mathbb{C}(c, F_-))$$

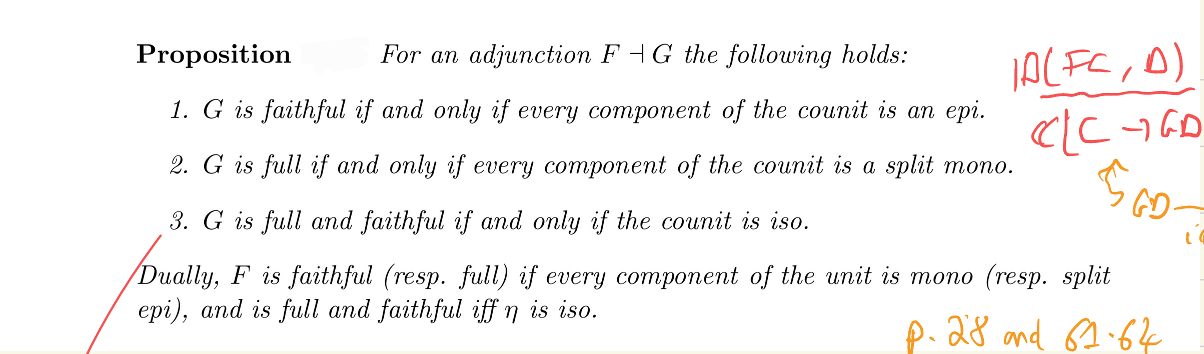
$f \overset{*}{\mapsto} \tilde{f} : \Delta c \to F$ $\quad$ $(\lambda f \lambda i . [\tilde{f}_i : c \to F_i)]$

swap !

$$\begin{tikzcd} f \in \mathbb{C}(c, \lim F) \arrow[d, "\lambda_i = f \mapsto \tilde{f}_i"] \\ \mathbb{C}(c, F_i) \end{tikzcd}$$

$(\lambda i \lambda f . \tilde{f}_i)$

● Exercise: Prove $\operatorname{colim}(\mathbb{C}(-, d)) = \{*\}$ (see also lecture 1 and p. 7)



Proposition *For an adjunction $F \dashv G$ the following holds:*

1. *G is faithful if and only if every component of the counit is an epi.*
2. *G is full if and only if every component of the counit is a split mono.*
3. *G is full and faithful if and only if the counit is iso.*

Dually, F is faithful (resp. full) if every component of the unit is mono (resp. split epi), and is full and faithful iff η is iso.

$$\frac{\mathbb{D}(FC, D)}{\mathbb{C}[C \to GD)}$$

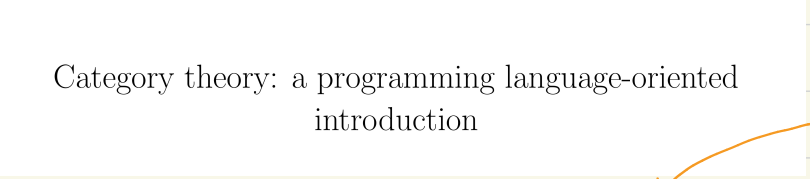
$FGD \xrightarrow[\varepsilon]{} D$

$GD \xrightarrow[id]{} GD$

p. 28 and 61-64

Proof left as exercise

[for a proof illustrating string diagrams, see
my notes



Category theory: a programming language-oriented
introduction

now also on
curien.galene.org

www.irif.fr/~curien/categories-pl.ps)

We then say that $F \dashv G$ is a $\boxed{\text{reflection}}$

Exercise A monad T is called idempotent if μ is iso.
Show that if T is idempotent, then every T-algebra $(C, \alpha : TC \to C)$ is s.t. α is iso.

Exercise Show that all reflections are monadic (↳ monadic in the terminology of Métayer's lecture notes)

Solutions next page

(a) For any monad T, the T-algebra adjunction is terminal among adjunctions
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
with the same underlying monad $T = GF$.

Recall the comparison functor $K : \mathbb{D} \to \mathbb{C}^T$: $K(d) = (Gd, G\varepsilon_d)$, and we have $UK = G$ $KF = I$, where
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathbb{C}^T \arrow[l, bend left, "U"] \end{tikzcd}$$
, $Ic = (Tc, \mu_c)$ $(\mu_c = G\varepsilon_{Fc})$.

(b) We now suppose G f.f., or, equivalently, ε iso. Then

(i) G faithful, $G = UK \Rightarrow K$ faithful

(ii) We prove K full as follows. Let $h \in \mathbb{C}^T(Kd, Kd')$, i.e.
$$\begin{tikzcd} Gd \arrow[r, "h"] & Gd' \\ {} \arrow[u, "G\varepsilon_d"] \arrow[r, "GFh"'] & {} \arrow[u, "G\varepsilon_{d'}"'] \end{tikzcd}$$
which reads (thanks to ε_d iso)
$$h = G\varepsilon_{d'} \circ GFh \circ (G\varepsilon_d)^{-2} = \overset{/\!/\,K(-)}{G}(\varepsilon_{d'} \circ Fh \circ \varepsilon_d^{-1})$$

(iii) To prove essential surjectivity of K, preparation needed.

(c) An idempotent monad is a monad T s.t. $\mu : TT \to T$ is iso.

(Under the assumption G f.f., we get immediately that GF is idempotent.)

In an idempotent monad, every algebra $(c, \alpha : Tc \to c)$ is s.t. α is iso.

(i) $\alpha\eta_c = id_c \Rightarrow \alpha$ split epi

(ii) $\alpha\eta = id$, $\mu \circ T\eta = id$, μ iso $\Rightarrow T\alpha$ iso (since $T\eta$ iso and $T\alpha T\eta = id$)

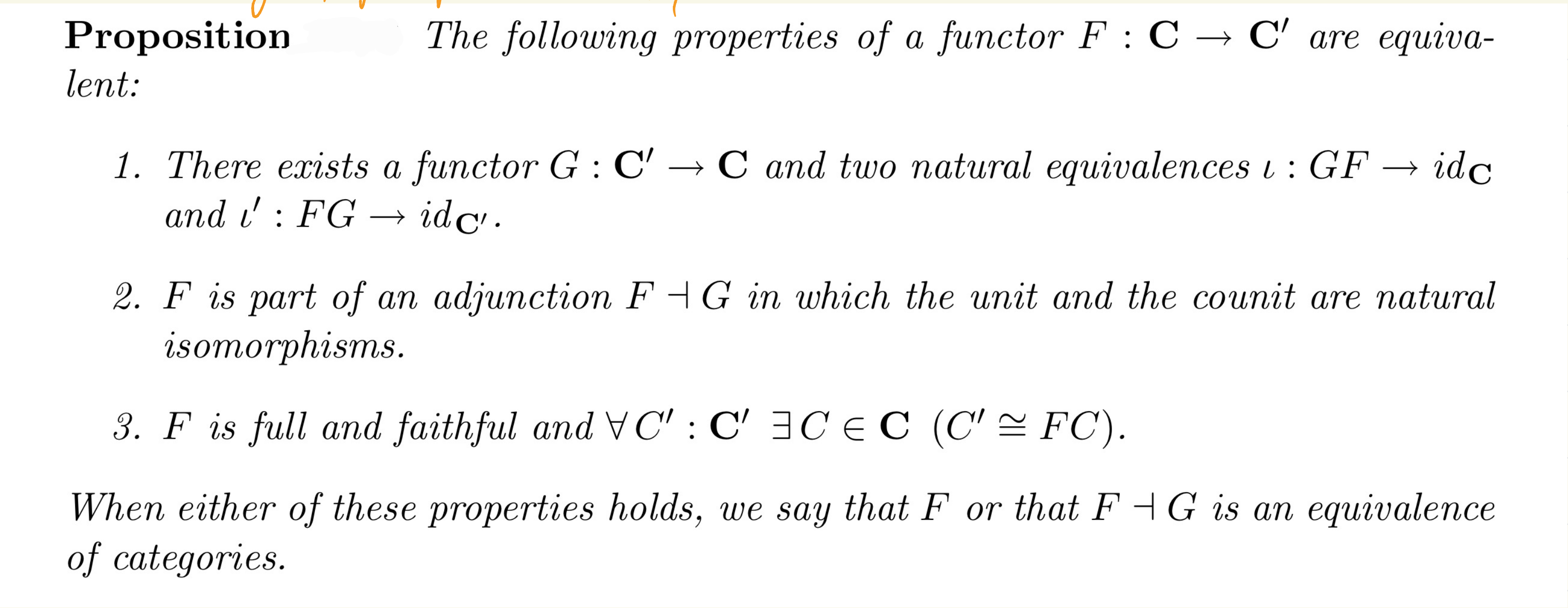
(iii) $T\alpha$ iso, $\mu \circ \eta T = id \Rightarrow \alpha$ mono (by naturality, $\eta_c\alpha = \underbrace{T\alpha \circ \overset{iso}{\eta_{Tc}}}_{iso \Rightarrow mono}$)

Back to (b)(iii). Since the monad T is idempotent, every object $(c, \alpha : Tc \to c)$ of $\mathbb{C}^T$ is s.t. α is iso, hence η_c is iso (since $\alpha\eta_c = id$) and is an iso from (c, α) to $K(Fc) = (GFc, \mu_c)$, since
$$\begin{tikzcd} Tc \arrow[r, "\alpha"] \arrow[d, "T\eta_c"'] \arrow[dr, "id"] & c \arrow[d, "\eta_c"] \\ TTc \arrow[r, "\mu_c"'] & Tc \end{tikzcd}$$
α / η_c inverses

monad law

Equivalences of categories

Categories-pl p.28 and 64-66



Proposition *The following properties of a functor $F : \mathbf{C} \to \mathbf{C}'$ are equivalent:*

1. *There exists a functor $G : \mathbf{C}' \to \mathbf{C}$ and two natural equivalences $\iota : GF \to id_{\mathbf{C}}$ and $\iota' : FG \to id_{\mathbf{C}'}$.*
2. *F is part of an adjunction $F \dashv G$ in which the unit and the counit are natural isomorphisms.*
3. *F is full and faithful and $\forall C' : \mathbf{C}' \; \exists C \in \mathbf{C} \; (C' \cong FC)$.*

When either of these properties holds, we say that F or that $F \dashv G$ is an equivalence of categories.

Presheaf categories are cartesian closed

- Cartesian = terminal object + finite products = finite products
- cartesian closed = cartesian + natural bijections

$$\frac{C(c\times d, e)}{C(c, d \Rightarrow e)} \qquad (-\times d) \dashv (d \Rightarrow\ )$$

The counit is often called

$$\text{eval} : \underline{(d \Rightarrow e)} \times d \to e$$

internal hom

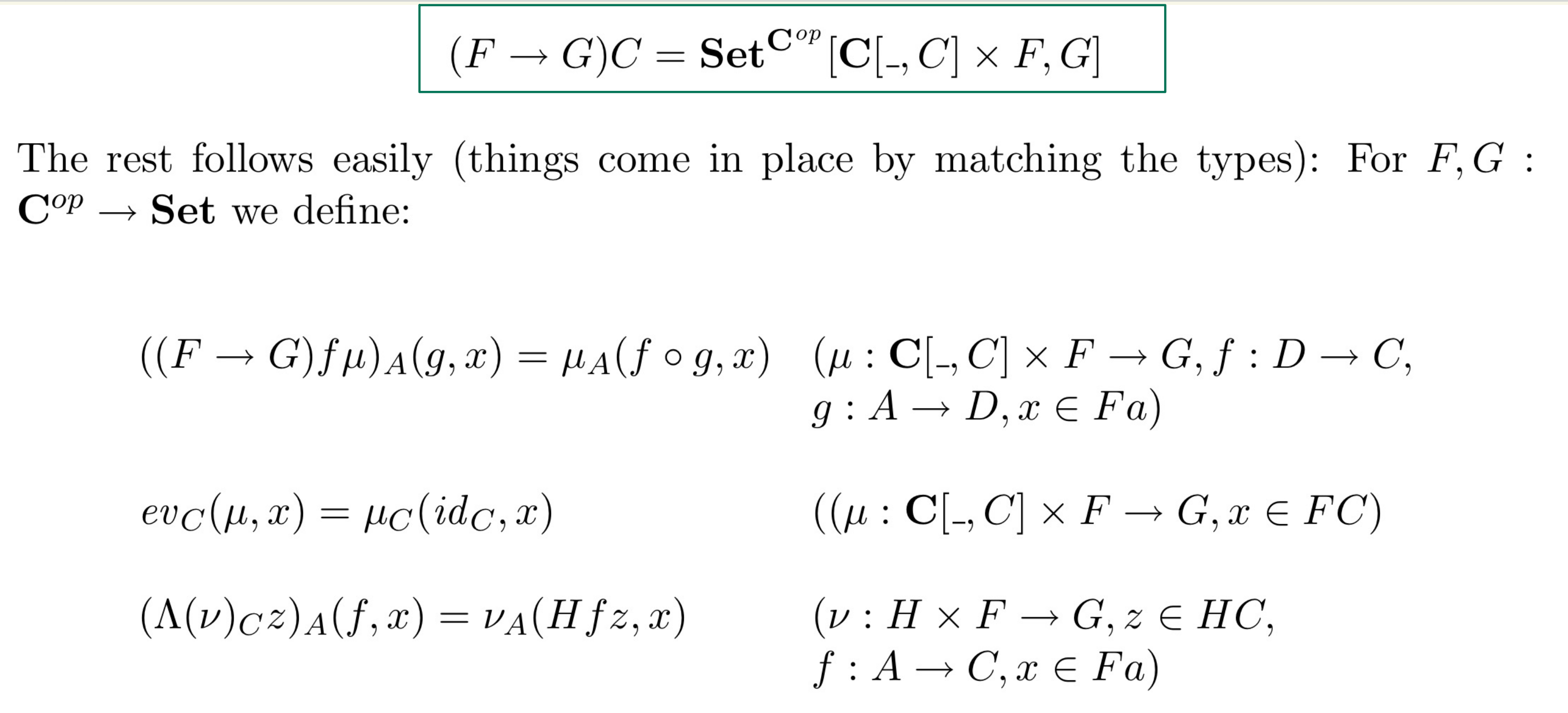
Theorem For any category C, the presheaf category $\hat{C}$ $= Set^{C^{op}}$ is cartesian closed.

Proof Products are pointwise (as all limits, cf Métayer's notes)

The internal hom is synthesised via Yoneda: we must have:

$$Set^{C^{op}}(C(-,c), \underline{(F \to G)})$$

Yoneda $\wr$ $\qquad$ $\cong$ adjunction



$$\boxed{(F \to G)C = \mathbf{Set}^{\mathbf{C}^{op}}[\mathbf{C}[\_, C] \times F, G]}$$

The rest follows easily (things come in place by matching the types): For $F, G : \mathbf{C}^{op} \to \mathbf{Set}$ we define:

$$((F \to G) f \mu)_A(g, x) = \mu_A(f \circ g, x) \quad (\mu : \mathbf{C}[\_, C] \times F \to G, f : D \to C, g : A \to D, x \in Fa)$$

$$ev_C(\mu, x) = \mu_C(id_C, x) \quad ((\mu : \mathbf{C}[\_, C] \times F \to G, x \in FC)$$

$$(\Lambda(\nu)_C z)_A(f, x) = \nu_A(Hfz, x) \quad (\nu : H \times F \to G, z \in HC, f : A \to C, x \in Fa)$$

Ends and coends

Consider a functor $F: \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$ and the following data (that looks like a cone, and is called a wedge)

$c \mapsto \lambda c$ s.t.

$\forall f : c \to d$

$$\begin{tikzcd}
 & d \arrow[dl, "\lambda c"'] \arrow[dr, "\lambda d"] & e \arrow[l, "\tilde{\gamma}"'] \arrow[dll, "\gamma"] \arrow[d] \\
F(c,c) \arrow[dr, "Fcf"] & & F(d,d) \arrow[dl, "Ffd"] \\
 & F(c,d) &
\end{tikzcd}$$

An end is a universal such wedge, i.e. $\forall e, \gamma \ \exists! \ \tilde{\gamma}$

Notation : $\int_c F(c,c)$

$\int_c H$, where $H = \mathbb{D}(F-, G-)$

A typical example of end

$F, G : \mathbb{C} \to \mathbb{D}$ $\qquad \mathbb{D}^{\mathbb{C}}[F,G] = \int_c \mathbb{D}[Fc, Gc]$

$\mu \qquad \mu_c$

$f : c \to d$

$$\begin{tikzcd}
 & \mu \in \mathbb{D}^{\mathbb{C}}(F, D) \arrow[dl] \arrow[d, "-c"] \arrow[drr, "-d"] & & \\
\mu_c \arrow[dr] & \mathbb{D}(Fc, Gc) \arrow[d, "\mathbb{D}(Fc, Gf)"] & \mu_d \arrow[dl] & \mathbb{D}(Fd, Gd) \\
 & \mathbb{D}(Fc, Gd) & &
\end{tikzcd}$$

$$Gf \circ \mu_c = \mu_d \circ Ff$$

$$\begin{tikzcd}
{} \arrow[r, "Ff"] \arrow[d, "\mu_c"'] & {} \arrow[d, "\mu_d"] \\
{} \arrow[r, "Gf"'] & {}
\end{tikzcd}$$

- When F is a honest functor $C \to D$ disguised as $C^{op} \times C \xrightarrow[\pi_2]{} C \xrightarrow[F]{} D$,

  then
  $$\int_c F\pi = \lim F$$

$$\begin{tikzcd}
 & d \arrow[dl] \arrow[dr] & \\
F\pi(c,c) = Fc \arrow[rr] \arrow[d, "id"'] & & F\pi(d,d) = d \\
F\pi(c,d) \arrow[urr] & &
\end{tikzcd}$$

- The end can also be described as a limit, specifically as an equaliser:

$$\begin{tikzcd}
 & \int_c F(c,c) \arrow[dl, "\lambda c"] \arrow[dr, "\lambda d"] & \\
F(c,c) \arrow[dr, "Fcf"] & & F(d,d) \arrow[dl, "Ffd"] \\
 & F(c,d) &
\end{tikzcd}$$

$$\boxed{\int_c F(c,c) = \lim \left(\prod_c F(c,c) \rightrightarrows \prod_{c,d} \prod_{f : c \to d} F(c,d) \right)}$$

where

$$\xrightarrow[c,d,f]{} \;=\; \prod_c F(c,c) \xrightarrow{\text{projection}} F(c,c) \xrightarrow{Fcf} F(c,d)$$

$$\xrightarrow[c,d,f]{} \;=\; \prod_c F(c,c) \xrightarrow{\text{projection}} F(d,d) \xrightarrow{Ffd} F(c,d)$$

- Dually, cowedges, and coends, notation

$$\int^c F(c,c)$$

Path components of a category

sets as discrete categories = only identities

- The inclusion $\subseteq : \mathrm{Set} \to \mathrm{Cat}$ has both a left and a right adjoint.

  - The right adjoint ist $C \mapsto \mathrm{Ob}\, C$

  Indeed, the only information in a functor $(\subseteq A) \to C$ is its object part.

  - The left adjoint is $C \mapsto \pi_0 C$ where $\pi_0 C$ is the quotient of $\mathrm{Ob} C$ by the equivalence $\simeq$ generated by the pairs (c,d) such that $\exists f : c \to d$, i.e., e.g.

$$\underline{c \to \to \leftarrow \to \leftarrow \leftarrow \to d}$$

  Indeed, if $F : C \to \subseteq A$, then $Ff = id$ for all f forces the object part of F (again the only information of F) to factor through $\simeq$

- The elements of $\pi_0 C$ are called <u>path components</u>. If C has exactly one path component, we say that C is <u>connected</u>

<u>Exercise</u> Let $X : C \to \mathrm{Set}$ (covariant presheaf). Show that $\boxed{\mathrm{colim}\, X = \pi_0(\mathrm{el} X)}$

↑ cf. lecture 1 p.10

Hints
- define the component at c of the colimiting cocone by

  $\underset{\substack{\Uparrow \\ X_c}}{\lambda_c x} = [(c,x)]$ ← equiv. class

- Note that for the generating case $(c,x) \simeq (c', X_f x)$, with $\overset{f}{c \to c'}$, we have, for any cocone

$$\begin{tikzcd}
 & m & \\
X_c \arrow[ur, "\lambda_c"] \arrow[rr, "X_f"] \arrow[d, Rightarrow] & & X_{c'} \arrow[ul, "\lambda_{c'}"'] \arrow[d, Rightarrow] \\
x & & X_f x
\end{tikzcd}$$

(λ, viewed as $\underset{c}{\bigsqcup} X_c \to m$, factors throug $\pi_0(\mathrm{el} X)$)

$\mathrm{Ob}(\mathrm{el} X)$

Final functors

Riehl CHT section 8.3

Definition A functor $K : \mathcal{C} \to \mathcal{D}$ is **final** if for any functor $F : \mathcal{D} \to \mathcal{M}$, the canonical map

If λd (d in D) is a cocone for F, then λKc (c in C) is a cocone for FK

$$\operatorname*{colim}_{\mathcal{C}} FK \xrightarrow{\cong} \operatorname*{colim}_{\mathcal{D}} F$$

is an isomorphism, both sides existing if either does.

Lemma *A functor $K : \mathcal{C} \to \mathcal{D}$ is final if and only if for each $d \in \mathcal{D}$, the slice category d/K is non-empty and connected.*

see lecture 1, p.3 : $\mathrm{Ob}\ d/K = \{(c, f : d \to Kc)\}$

Proof ● "If" direction. It is enough to show that any cocone (μ_c) for FK extends to a cocone (λ_d) for F.

$$\begin{tikzcd} & m \\ FKc \arrow[ur, "\mu_c"] & \end{tikzcd}$$

● Claim If $(c, f) \simeq (c', f')$ in d/K, then

cf. p.7 ; i.e. in the same path component

$$\boxed{\mu_c \circ Ff = \mu_{c'} \circ Ff'}$$

It is enough to consider the generating clause of $\simeq$:

Suppose thus

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Kc \arrow[rr, "Kh"] & & Kc' \end{tikzcd}$$

$\exists h$

Then we have

$$\begin{tikzcd} & Fd \arrow[dl, "Ff"'] \arrow[dr, "Ff'"] & \\ FKc \arrow[rr, "FKh"] \arrow[dr, "\mu_c"'] & & FKc' \arrow[dl, "\mu_{c'}"] \\ & m & \end{tikzcd}$$

functoriality

cocone

● By ~~assumption~~ claim †, we can define $\lambda d = \mu_c \circ Ff$ for a choice of (c, f).

● "Only if" direction. We have $d/K = \mathrm{el}\, \mathcal{D}(d, K-)$. Then

rephrasing

$$\pi_0(d/K) = \pi_0(\mathrm{el}\, \mathcal{D}(d, K-)) \cong \mathrm{colim}(\mathcal{D}(d, K-)) \cong \mathrm{colim}(\mathcal{D}(d, -)) \cong \{*\}$$

exercise p.7 K final exercise p.1

Example For $t : \mathbb{1} \to \mathcal{D}$ (t terminal in $\mathcal{D}$), we have $d/K \simeq \mathbb{1}$, and hence $t : \mathbb{1} \to \mathcal{D}$ is final, implying $\mathrm{colim}\, F \simeq Ft$ (cf. p.1)

Detecting representable functors

A representable functor $C^{op} \to$ Set is a functor F together with an object c_0 of C and a natural isomorphism

$$\kappa : C(-, c_0) \to F$$

Lemma F is representable if and only if there exists an object c_0 of C and an element x_0 of Fc_0 such that the natural transformation $\lambda^{c_0,x_0} : C(-,c_0) \to F$ defined by

$$(\lambda^{c_0,x_0})_c(\underset{c \xrightarrow{f} c_0}{f}) = \underbrace{Ff x_0}_{\in Fc} \quad \text{is iso}$$

Proof One direction is obvious (qui peut le plus peut le moins).
Conversely, suppose that $\kappa : C(-, c_0) \to F$ is given.
This data is equivalent (Yoneda lemma) to the data of some $x_0 \in Fx_0$, and κ is entirely determined by x_0, i.e. $\kappa = \lambda^{c_0,x_0}$.

We say that c_0, x_0 are universal for F

Useful abbreviations

TFAE = the following are equivalent

a.k.a = also known as