Lecture 1 Kan Extensions

Given
$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ D & \end{tikzcd}$$
find universal
$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{E} \\ D \arrow[ur, bend right, "\mathrm{Lan}_K F"'] & \end{tikzcd}$$

i.e., for all G, γ
$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Downarrow\gamma"] & \mathcal{E} \\ D \arrow[ur, bend right, "G"'] & \end{tikzcd}$$

there exists a unique
$$\begin{tikzcd} & \mathcal{E} \\ D \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right=50, "G"'] \arrow[ur, phantom, bend right=20, "\Downarrow"] & \end{tikzcd}$$
s.t.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Downarrow\gamma"] & \mathcal{E} \\ D \arrow[ur, bend right, "G"'] & \end{tikzcd} = \begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{E} \\ D \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right=50, "G"'] \arrow[ur, phantom, bend right=20, "\Downarrow"] & \end{tikzcd}$$

If, for K fixed $\mathrm{Lan}_K F$ exists, then all this is encapsulated in an adjunction

$$\boxed{\mathrm{Lan}_K : \mathcal{E}^C \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E}^D \quad K^* = G \mapsto G \circ K}$$

<u>Exercise</u> Exhibit the counit of this adjunction.

Symmetrically, if K^* has a right adjoint, we say that we have right Kan extensions:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Uparrow\varepsilon"] & \mathcal{E} \\ D \arrow[ur, "\mathrm{Ran}_K F"'] & \end{tikzcd}$$

s.t. $\forall\ G, \delta\ \exists!\ G \to \mathrm{Ran}_K F$

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Uparrow\delta"] & \mathcal{E} \\ D \arrow[ur, "G"'] & \end{tikzcd} = \begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Uparrow\varepsilon"] & \mathcal{E} \\ D \arrow[ur, "\mathrm{Ran}_K F"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\Uparrow"'] & \end{tikzcd}$$

If we name $\Uparrow$ as δ', this means $\varepsilon \circ \delta' K = \delta$

We want to be able to compute these extensions by means of colimits and limits.

Heuristics

phattened as π (depending on K only)

$$\begin{tikzcd} ?(K,d) \arrow[r, "\pi^d"] & C \arrow[r, "F"] \arrow[d, "K"] & \mathcal{E} \\ & D & \end{tikzcd}$$

$d \in$

$$\mathrm{Ran}_K F(d) = \lim F\pi^d$$

If such a formula exists, then it also applies to

$$\begin{tikzcd} ?(K,d) \arrow[r, "\pi"] & C \arrow[r, "F"] \arrow[d, "K"] & \mathcal{E} \arrow[r, "{\mathcal{E}(e,-)}"] & Set \\ & D \arrow[urr] & & \end{tikzcd}$$

$(c,f) \mapsto c$

$e \in$ $\qquad \mathcal{E}(e,-)F$

$d \in$

i.e. $\mathrm{Ran}_K(\mathcal{E}(e,F))d =$

$\lim(\mathcal{E}(e,-)F\pi) =$

$\mathcal{E}(e, \lim(F\pi)) =$ $\quad$ $\mathcal{E}(e,-)$ preserves limits

$\underline{\mathcal{E}(e, \mathrm{Ran}_K F(d))}$

Yoneda

$Set^D(D(d,-), \mathrm{Ran}_K(\mathcal{E}(e,F))) =$

$= \underline{Set^C(D(d,K), \mathcal{E}(e,F))}$

Def of Ran_K

Thus, a morphism from e to $\mathrm{Ran}_K F(d)$ is equivalent to the data of maps $\lambda_c : D(d, Kc) \to \mathcal{E}(e, Fc)$, which provide a cone

$f \in$

$$\begin{tikzcd} & e \arrow[d, "\lambda_{c,f}"] \\ F\pi(c,f) \arrow[r, phantom, "="] & Fc \end{tikzcd}$$

indexed over pairs $(c, f : d \to Kc)$

a.k.a. d/K

objects of $\boxed{?(K,d) = d \downarrow K}$ $\to D$

$(d\downarrow K)[(c,f),(c',f')] = \{h \mid$

$$\begin{tikzcd} & d \arrow[dl] \arrow[dr] & \\ Kc \arrow[rr, "Kh"'] & & Kc' \end{tikzcd}$$

$\}$

$\pi(c,f) = c$

Finally, we have synthesized our candidate!

Step 1 : we synthesised $d \downarrow K$ and $\pi^d : (d\downarrow K) \to C$

Step 2 : we prove that if $\lim F\pi^d$ (for all π^d) exists, it is indeed a right Kan extension.

We look for $\varepsilon_c : (\mathrm{Ran}_K F)(Kc) \to Fc$

Let $\lambda^d : \underset{\substack{\| \\ (\mathrm{Ran}_K F) d}}{\lim(F\pi)} \to F\pi$ be the limiting cone

Specialising to $d = Kc$, we note that $Kc \downarrow K \ni (c, id : Kc \to Kc)$

and we define $\varepsilon_c = \lambda^{Kc}_{(c, id)} : (\mathrm{Ran}_K F)(Kc) \to F\pi(c, id) = Fc$

One then checks that it works!

(*)

$\boxed{\begin{array}{l}\text{Here we have proved, for } \pi^d : d\downarrow K \to C \text{, that} \\ \text{if } \lim(F\pi^d) \text{ exists for all } d\text{, then } \mathrm{Ran}_K F \text{ exists} \\ \text{and } \mathrm{Ran}_K F d = \lim(F\pi^d)\end{array}}$

In particular, if $\mathcal{E}$ is complete $\mathrm{Ran}_K F$ always exists

But more can be said.

We say that $L : \mathcal{E} \to \mathcal{F}$ preserves right Kan extensions if $\mathrm{Ran}_K(LF) = L(\mathrm{Ran}_K F)$, and that a right Kan extension is pointwise if it is preserved by all $\mathcal{E}(e,-)$

Then if $\lim(F\pi^d)$ exists for all d, $\mathrm{Ran}_K F$ is in fact pointwise: indeed, since $\mathcal{E}(e,-)$ preserves limits

$$\underset{\substack{\|\,(*) \\ \mathrm{Ran}_K(\mathcal{E}(e,F))d}}{\lim(\mathcal{E}(e,-)F\pi)} = \mathcal{E}(e, \underbrace{\lim(F\pi)}_{\substack{\|\,(*) \\ \mathrm{Ran}_K F d}})$$ exists, and

Even more can be said!

If $\mathrm{Ran}_K F$ is a pointwise right Kan extension, then $\lim(F\pi^d)$ exists (and hence $\mathrm{Ran}_K F d = \lim(F\pi^d)$)

The proof is a variation on our synthesis above:

$$\mathrm{Ran}_K(\mathcal{E}(e,F))d = \lim(\mathcal{E}(e,-)F\pi) = \mathcal{E}(e, \lim(F\pi)) = \mathcal{E}(e, \mathrm{Ran}_K F(d))$$

$\mathcal{E}(e,-)$ preserves limits

Yoneda

$$\mathrm{Set}^D(D(d,-), \mathrm{Ran}_K(\mathcal{E}(e,F))) = \mathrm{Set}^C(D(d,K), \mathcal{E}(e,F))$$

Def of Ran_K

We replace the right column with $\mathrm{Ran}_K(\mathcal{E}(e,F))d = \mathcal{E}(e, \mathrm{Ran}_K F(d))$ (pointwise)

In other words the formula $\mathrm{Ran}_K F d = \lim(F\pi^d)$ characterises pointwise right Kan extensions:

A pair
$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ D & \end{tikzcd}$$
admits a pointwise Kan extension if and only if $\lim(F\pi^d)$ exists for all object d of D, and then $\mathrm{Ran}_K F d = \lim(F\pi^d)$

Dually, here is the formula for Lan_K :

$$\boxed{(\mathrm{Lan}_K F)d = \operatorname{colim}(F\pi)}$$

where $\pi : K\downarrow d \to C$ and $\mathrm{Ob}(K\downarrow d) = \{(c, f : Kc \to d)\}$

a.k.a K/d

- Size issues : for these colimits and limits to exist, we need $K\downarrow d$ and $d\downarrow K$ small, which is guaranteed if C is small and D is locally small.

- Packaging : We have

contravariant covariant

$$\boxed{\begin{array}{l} \mathrm{Lan}_K F d = \operatorname*{colim}_{(c,f)} Fc = \displaystyle\int^{c} \underbrace{D(Kc,d)\cdot Fc}_{\text{coproduct of } |D(Kc,d)| \text{ copies of } Fc} \\[2em] \text{Dually } (\mathrm{Ran}_K F)d = \displaystyle\int_{c} \underbrace{Fc^{D(d,Kc)}}_{\text{product of } |D(d,Kc)| \text{ copies of } Fc} \end{array}}$$

Copower

$A \cdot c$

Set $\quad$ object

$\# A$ - fold coproduct of c

$c^A = \# A$ - fold product of c

power

Special case $D = 1$

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "!"'] & \mathcal{E} \\ 1 \arrow[ur, "G"'] & \end{tikzcd}$$

Then a functor G is an object of $\mathcal{E}$ and a nat. transf. $\delta : G! \to F$ is a cone over F.
Therefore $\mathrm{Ran}_! F = \lim F$

$$\mathrm{Lan}_! F = \mathrm{colim}\, F$$

(here we refer only to the universal property of Kan extensions)
For another such example, see adjunctions next page)

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "id"'] & \mathcal{E} \\ C & \end{tikzcd}$$

Special case $D = C$ and $K = id$. Then trivially $\mathrm{Ran}_K F = F = \mathrm{Lan}_K F$ (with $\varepsilon, \eta = id$), as they satisfy the defining universality property. Therefore we have

Furthermore, these are absolute (and a fortiori pointwise) Kan extensions.

$$Fd \simeq \mathrm{Ran}_{id} F d = \int_c Fc^{C(d,c)}$$

$$Fd = \mathrm{Lan}_{id} F d = \int^c C(c,d) \cdot Fc$$

This is called co-Yoneda lemma

If furthermore $\mathcal{E} = Set$, then

$$Fd = \int_c Fc^{C(d,c)}$$

$$= \int_c Set(C(d,c), Fc)$$

$$= Set^C(C(d,-), F)$$

We retrieve Yoneda lemma!

$$Fd = \int^c C(c,d) \cdot Fc$$

$$= \int^c C(c,d) \times Fc$$

$$\simeq \int^c Fc \times C(c,d)$$

$$\Rightarrow F = \int^c Fc \cdot C(c,-)$$

This is called the density formula
(F limit of representables)

Proposition 3.3.2 *The following properties are equivalent, for $K : \mathbf{A} \to \mathbf{C}$ and $R : \mathbf{C} \to \mathbf{A}$:*

$$\begin{tikzcd} C \arrow[r, "id"] \arrow[d, "K"'] & C \\ D & \end{tikzcd}$$

absolute Kan extension

1. *$R \dashv K$, with ϵ as counit.*

2. *(R, ϵ) is a right Kan extension of id along K that is preserved by all functors.*

3. *(R, ϵ) is a right Kan extension of id along K that is preserved by K.*

Proof. (1) $\Rightarrow$ (2). Let $\mu : SK \to id$. We look for $\mu' : S \to R$ such that $\epsilon \circ \mu' K = \mu$. Writing this equality as a string diagram, and plugging the unit η, we see that μ' is necessarily equal to $\mu R \circ S\eta$ and that this definition of μ' fits. Let now $F : \mathbf{A} \to \mathbf{A}'$, and let $\mu : HK \to F$. We see likewise that $\mu R \circ H\eta$ is the unique transformation from H to FR that fits.

(3) $\Rightarrow$ (1). We apply the assumption that $(KR, K\epsilon)$ is a Kan extension, with $id : \mathbf{C} \to \mathbf{C}$. This yields a transformation η that satisfies one of the two laws of adjunction. For the second one, we observe that $\nu = \epsilon R \circ R\eta$ satisfies the condition $\epsilon \circ \nu K = \epsilon$ and hence $\nu = id$ by uniqueness.

Absolute Kan extensions

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \arrow[r, "G"] & \mathcal{F} \\ \mathcal{D} \arrow[ur, "\mathrm{Ran}_K F"'] \arrow[u, phantom, "\Uparrow \varepsilon"] & & \end{tikzcd}$$

(idem for absolute left Kan extensions)

absolute whenever for all G

$$\mathrm{Ran}_K(GF) = G \circ \mathrm{Ran}_K F$$

ε $\quad$ $G\varepsilon$

<u>Exercise</u> Suppose we have

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[d, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend left, "G"] \arrow[d, "K"] \\ \mathcal{M}' & \mathcal{N}' \end{tikzcd}$$

Show that if $\mathrm{Ran}_H(KF)$ and $\mathrm{Lan}_K(HG)$ are absolute,

then
$$\begin{tikzcd} \mathcal{M}' \arrow[r, bend left, "\mathrm{Ran}_H(KF)"] \arrow[r, phantom, "\perp"] & \mathcal{N}' \arrow[l, bend left, "\mathrm{Lan}_K(HG)"] \end{tikzcd}$$

Proposition: If K is full and faithful, and if Ran_K is given by the formula $(\mathrm{Ran}_K F)d = \lim(F\pi^d)$, then the "counit" ε is iso [we can even take $\varepsilon = \mathrm{id}$].

Proof We observe that the information that K is f. and f. is equivalent to:

$$\forall c \quad (c, \mathrm{id} : Kc \to Kc) \text{ is initial in } Kc \downarrow K$$

$$\begin{tikzcd} \cdot \arrow[r, "F"] \arrow[d, "K"'] & \cdot \\ \cdot \arrow[ur, "\mathrm{Ran}_K F"'] & \end{tikzcd}$$

$$\begin{tikzcd} (c, \mathrm{id}) \arrow[d, "g"] \\ (c', f) \end{tikzcd} \qquad \begin{tikzcd} & Kc \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ Kc \arrow[rr, "Kg"'] & & Kc' \end{tikzcd}$$

i.e. a morphism $(c, \mathrm{id}) \to (c', f)$ is a g s.t. $Kg = f$

Therefore we have $(\mathrm{Ran}_K F)(Kc) = \lim(F\pi^{Kc}) = F\pi(c, \mathrm{id}) = Fc$ and the $\underbrace{\text{component at } (c, \mathrm{id}) \text{ of the limiting cone}}_{\varepsilon_c}$ is $\mathrm{id} : Fc \to Fc$ □

- Dually, if K is full and faithful, and if Lan_K is given by the formula

$$(\mathrm{Lan}_K F)d = \mathrm{colim}(F\pi^d)$$

then η is iso and can be taken $= \mathrm{id}$.

(equivalently, K f. and f. $\Rightarrow$ Lan_K, Ran_K f. and f.)

Thus, specialising to $D = \hat{C}$ and $K = y$ (full and faithful by Yoneda!)

$\underbrace{\hat{C}}_{= Set^{C^{op}}}$

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "y"'] \arrow[dr, phantom, "\cong"] & \mathcal{E} \\ \hat{C} \arrow[ur, "Lan_y F"'] & \end{tikzcd}$$

In this case, we have that $y \downarrow X \simeq el(X)$ a.k.a. C/X ($\mathcal{E}_X$ dans les notes de Métayer)

$(c, f : yc \to X) \rightsquigarrow (c, x \in Xc)$ Yoneda lemma

So we can write $\boxed{\begin{aligned}(Lan_y F)X &= \underset{(c, x \in Xc)}{colim}(F\pi) \\ &= \int^c Xc . Fc\end{aligned}}$ $\boxed{\Sigma c : C . Xc}$

On the other hand, we have a natural functor $NF : \mathcal{E} \to \hat{C}$:

$NF e c = \mathcal{E}[Fc, e]$ NERVE

<u>Proposition</u>: $Lan_y F \dashv NF$

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, bend left, "NF"] \\ \hat{C} \arrow[ur, bend left, "Lan_y F"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

Proof: A morphism $(Lan_y F)X \to e \simeq$ a cocone $F\pi \to e$

$\simeq \forall c, x \in Xc$ a morphism $Fc \to e$

$\simeq$ a natural transformation $X \to NFe$ $\square$

We note, as a consequence, that $Lan_y F$ preserves colimits

Remark: The proposition can be strengthened as follows: the pair

$C \xrightarrow{F} \mathcal{E}$, $C \xrightarrow{y} \hat{C}$ admits a pointwise left Kan extension iff NF has a left adjoint.

But there is more to it!

We can further specialise to

$$\begin{tikzcd} C \arrow[r, "y"] \arrow[d, "y"'] & \hat{C} \\ \hat{C} & \end{tikzcd}$$

We observe that $NyX_c = \hat{C}[y_c, X) \simeq X_c$

Hence $Ny \simeq \mathrm{Id}$ (naturally isomorphic)

It follows that $\mathrm{Lan}_y y \dashv \mathrm{Id}$ (and $\mathrm{Lan}_y y$ exists)

So we have $\mathrm{Lan}_y y \simeq \mathrm{Id}$

This is the density formula $\forall X \in \hat{C}$

$$\boxed{X \simeq \operatorname*{colim}_{(c,\, x \in X_c)} y_c}$$

: every presheaf is a colimit of representables

or $X \simeq \int^c X_c \cdot y_c$ ie pointwise $\boxed{X_d = \int^c X_c \cdot C(d,c)}$

cf. co-Yoneda lemma

Definition If

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "F"'] & D \\ D \arrow[ur, "id"'] \arrow[u, phantom, "\Downarrow id"] & \end{tikzcd}$$

is a pointwise left Kan extension

ie. $\forall d \in D \quad d = \operatorname{colim}(F\pi^d)$

we say that F is dense.

We have shown above that y is dense

Exercise Show that F dense $\Leftrightarrow$ NF full and faithful.

(Hint: note that a natural transformation $NFd \to NFd'$ gets repackaged as a morphism of cocones over $F\pi^d$.)

Proposition : Let $K : C \to D$ be dense and full and faithful, and suppose we have a pointwise left Kan extension $\mathrm{Lan}_K F$. Then an alternative description of $\mathrm{Lan}_K F$ is the following: $\mathrm{Lan}_K F$ is the unique functor $\bar{F} : D \to \mathcal{E}$ preserving all colimits of the form $\mathrm{colim}(K\pi^d)$ and extending F.

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ D \arrow[ur, "\bar{F}"'] \arrow[u, phantom, ""] & \end{tikzcd} \quad (\cong)$$

(which exist by density of K)

Moreover, the adjunction $\mathrm{Lan}_K \dashv K^*$ restricts to an equivalence of categories

$$\boxed{\mathrm{Lan}_K : \mathcal{E}^C \underset{\perp}{\rightleftarrows} \mathcal{E}^D_c \ K^*}$$

full subcategory of $\mathcal{E}^D$ of the functors preserving those colimits

Proof. Since K is full and faithful, we have $\mathrm{Lan}_K F\, K = F$, and

$$\mathrm{Lan}_K F(\mathrm{colim}\, K\pi^d) \underset{\text{density}}{=} \mathrm{Lan}_K F d \underset{\text{formula}}{=} \mathrm{colim}(F\pi^d) = \mathrm{colim}(\mathrm{Lan}_K F\, K\pi^d)$$

Conversely given $\bar{F}$, we get $\bar{F}d \underset{\text{density}}{=} \bar{F}(\mathrm{colim}\, K\pi^d) \underset{\text{preservation}}{=} \mathrm{colim}(\bar{F}K\pi^d)$

$$= \mathrm{colim}(F\pi^d) = \mathrm{Lan}_K F\, d.$$

For the last part of the statement, we note that, starting from $\bar{F}$:

$$\bar{F}d \underset{\text{density}}{=} \bar{F}(\mathrm{colim}(K\pi^d)) \underset{\bar{F} \in \mathcal{E}^D_c}{=} \mathrm{colim}(\bar{F}K\pi^d) \underset{\text{def}}{=} \mathrm{Lan}_K(\bar{F}K)\, d$$

$\Rightarrow$ the counit $\mathrm{Lan}_K(\bar{F}K) \to \bar{F}$ is iso.

Back to y: when $K = y$, we have the further information that $\mathrm{Lan}_y F$ is a left adjoint, hence it preserves all colimits. This leads to the following specialisation and strengthening of the previous proposition

cocontinuous

Proposition Given $F : C \to \mathcal{E}$, it is equivalent to have

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{E} \\ \hat{C} \arrow[ur] \arrow[u, phantom, shift left=4, "="] & \end{tikzcd}$$

(a) a left adjoint to $N_F : \mathcal{E} \to \hat{C}$

(b) a pointwise left Kan extension $\mathrm{Lan}_y F$

(c) a unique functor $\overline{F} : D \to \mathcal{E}$ preserving all colimits of the form $\mathrm{colim}(y\pi^{\phi})$ and extending F

(d) a unique cocontinuous functor extending F

Proof (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) hold by previous results.

And we have $(a) \underset{\text{left adj.}}{\Rightarrow} (d) \Rightarrow (c)$.

Taking left Kan extensions is a left Kan extension

i.e. $\mathrm{Lan}_K = \mathrm{Lan}_{K'} K''$ $(K : C \to D)$ for some K', K''

Consider the nerve construction applied to $C \xrightarrow{K} D \xrightarrow{y} \hat{D}$

We have $N(y \circ K) \cong (K^{op})^* : \hat{D} \to \hat{C}$ $\quad N(y \circ K) : \hat{D} \to \hat{C}$

Proof: $N(y \circ K)\underset{\in \hat{D}}{Y}\, c = \hat{D}(y(Kc), Y) \underset{\text{Yoneda}}{=} Y(Kc) = ((K^{op})^* Y)c$

$C^{op} \xrightarrow{K^{op}} D^{op} \xrightarrow{Y} \mathrm{Set}$

It follows that their left adjoints are isomorphic functors

$$\mathrm{Lan}_K X = \mathrm{Lan}_y (y \circ K) X$$

$$\begin{tikzcd} C^{op} \arrow[r, "X"] \arrow[d, "K"'] & \mathrm{Set} \\ D^{op} \arrow[ur, "\mathrm{Lan}_K X"'] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "K"] \arrow[d, "y"'] & D \arrow[r, "y"] & \hat{D} \\ \hat{C} \arrow[urr, "\mathrm{Lan}_y(y \circ K)"'] & & \end{tikzcd}$$

Exercise, check this directly
(similar to the two versions of Co-Yoneda)

Note the "reification" of the arrow X into the object X $\quad X \in \hat{C}$

[leaving y implicit, this yields "$N\_ = \_^*$". As a matter of fact, one finds in the literature the following notation:

$K_!$ for Lan_K $\qquad F^*$ for NF

K_* for Ran_K $\qquad F_!$ for $\mathrm{Lan}_y F$

(even if they meet only in the above situation).]