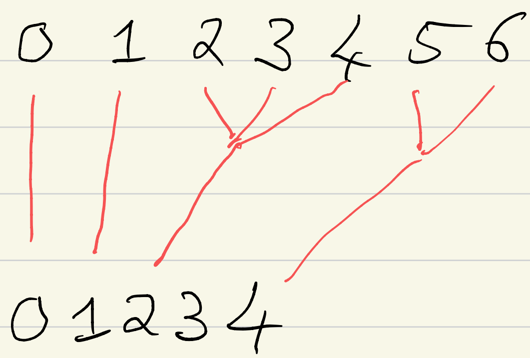
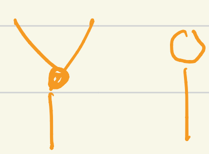
Lecture 2 Simplicial sets

The category Δ has as objects the finite ordinals $[n] = \{0 < 1 \cdots < n\}$ (in part. $[0] = \{0\}$)

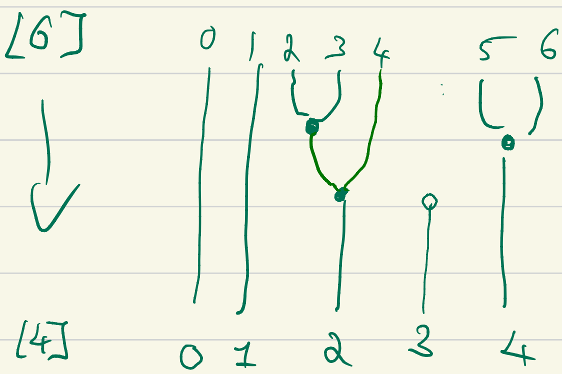
Morphisms are (weakly) monotone functions:



⇝ two "graphical" generators

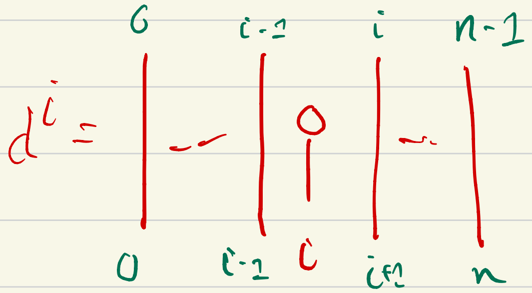


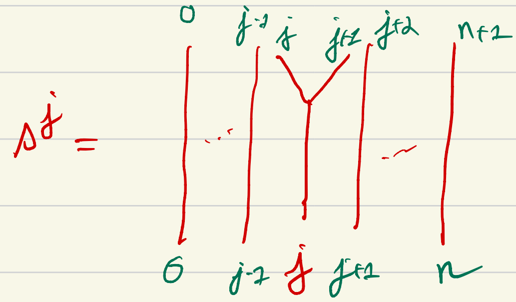
(cf. μ an η of a monad)



Presenting Δ by generators and relations

$$d^i : \mathbf{n-1} \to \mathbf{n} \qquad 0 \le i \le n \qquad \text{(cofaces)}$$
$$s^j : \mathbf{n+1} \to \mathbf{n} \qquad 0 \le j \le n \qquad \text{(codegeneracies)}$$

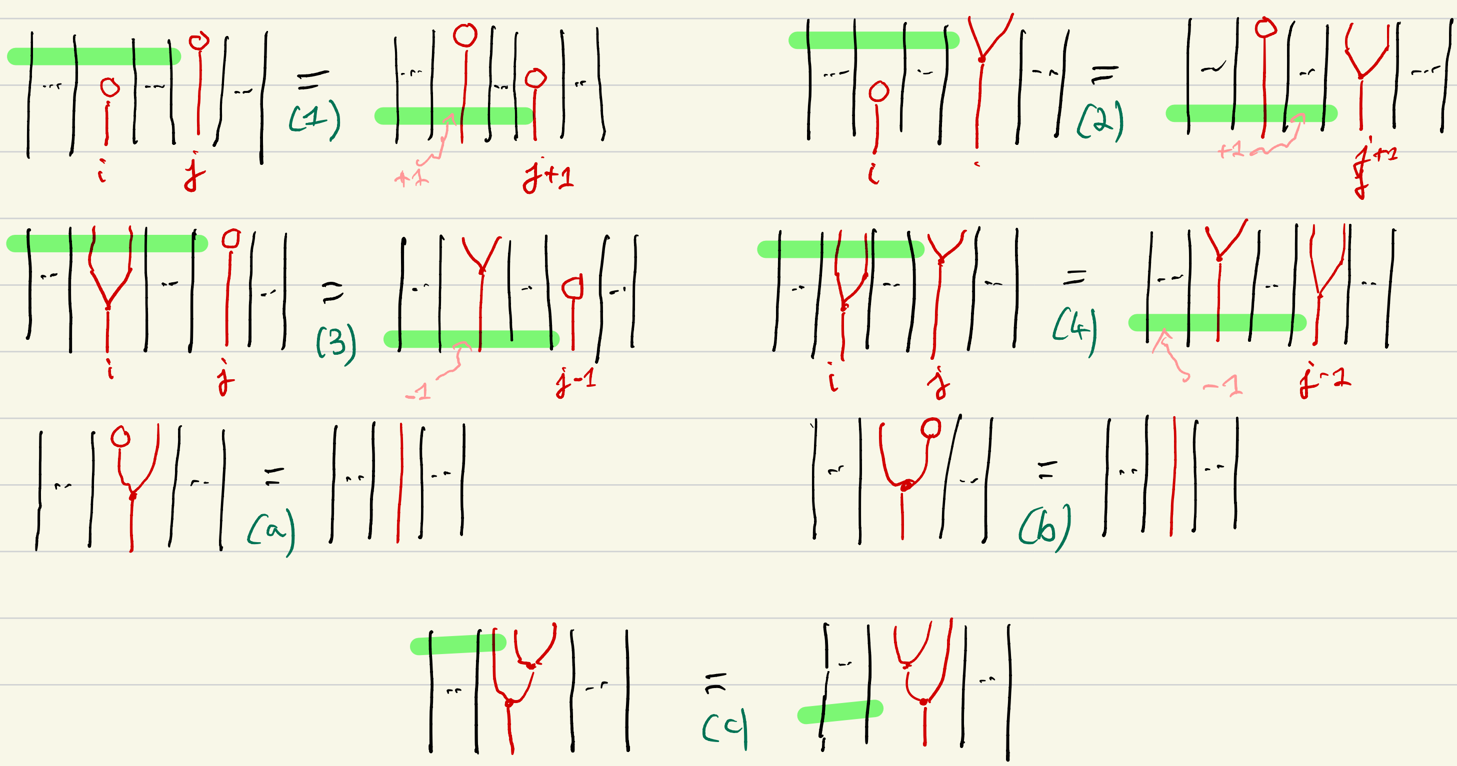




$$\begin{cases} d^i d^j = d^{j+1} d^i & (i \le j) \quad (1) \\ d^i s^j = s^{j+1} d^i & (i \le j) \quad (2) \\ s^j d^j = 1 = s^j d^{j+1} & \quad (a), (b) \\ s^i d^j = d^{j-1} s^i & (i < j-1) \quad (3) \\ s^i s^j = s^{j-1} s^i & (i < j) \quad (4)\ (i < j-1),\ (c)\ (i = j-1) \end{cases}$$

(1) – (4) are "naturality conditions" = "pulling strings differently"

(a) – (c) are the monad / monoid laws



Proving that this presentation is indeed a presentation of Δ

- Orient all equations from left to right and prove that they form a confluent and terminating rewriting system $\rightsquigarrow$ uniqueness of normal forms

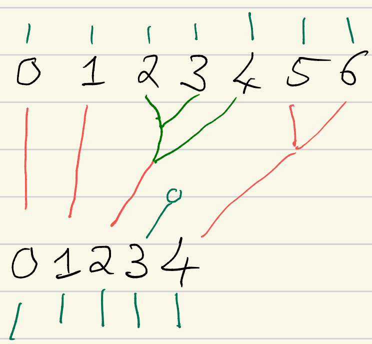
  (see next page)

- Observe that the normal forms are of the form

  $u_1 \dots u_p$, together with a strictly decreasing map $[1,p] \to [n]$ (for some n), $j \mapsto i_j$

  s.t. for all j $u_j = d^{i_j}$ or $u_j = \underbrace{s^{i_j} \dots s^{i_j}}_{q \text{ times, for some } q}$

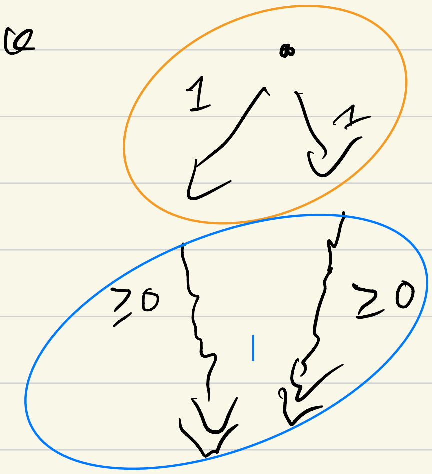
- Observe that these normal forms encode precisely the descriptions p.1:



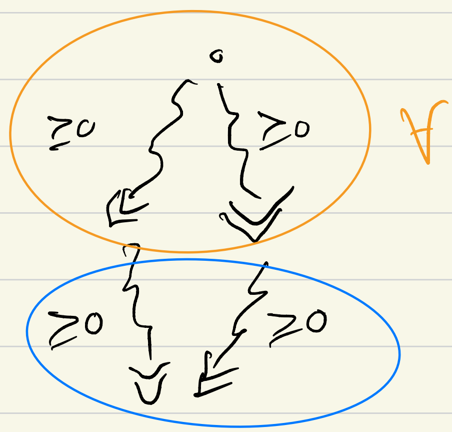
$\rightsquigarrow$ $d^3(s^2 s^2)$

Rewriting theory background

- local confluence



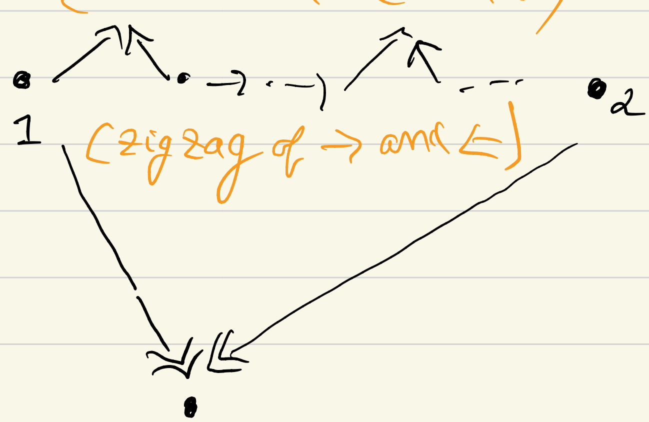
- confluence :



- strong normalisation $\nexists$ $\cdot \to \cdot \to \cdot \cdots \cdot \to \cdot \cdots$ (infinite)
- normal form : $\cdot \not\to$ ($\nexists$)

- local confluence + strong normalisation ⇒ confluence (Newman lemma)

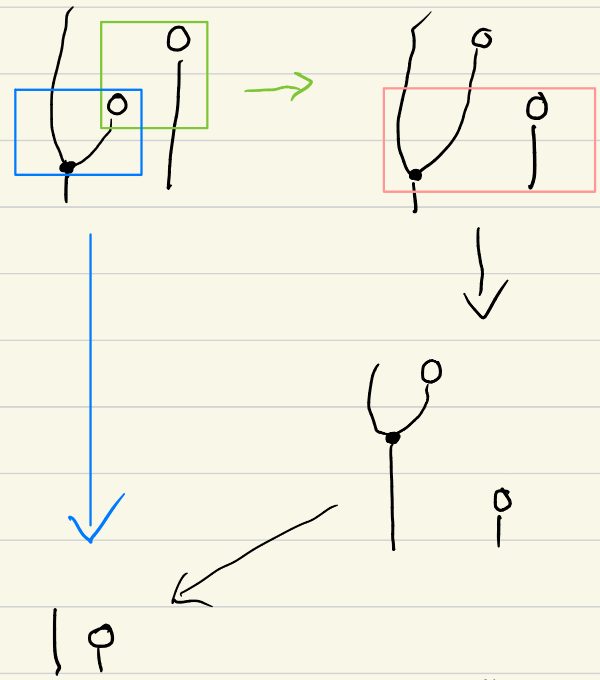
- If confluence + strong norm., then

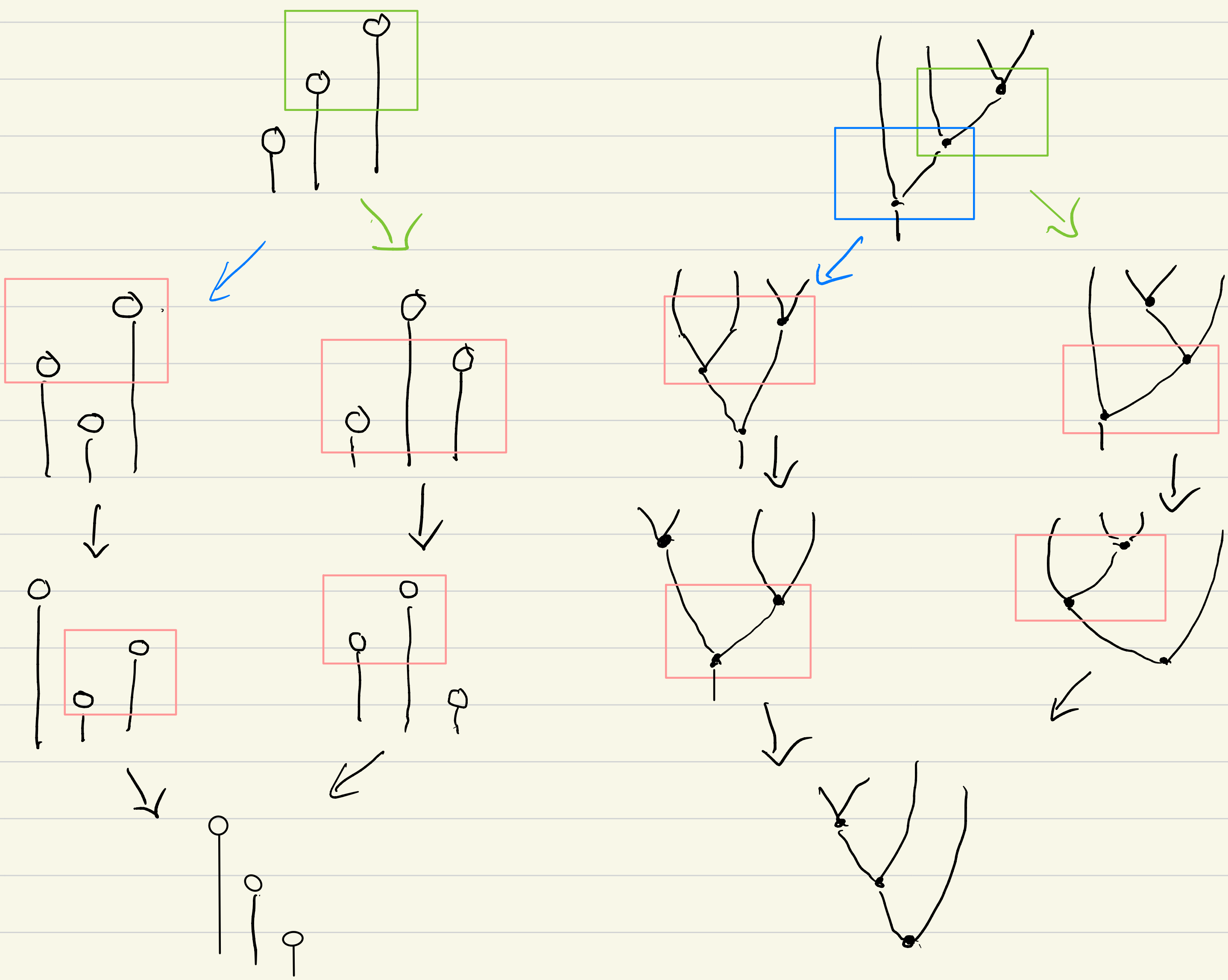


⇒ $\cdot_1$ and $\cdot_2$ have the same normal form.

Local confluence checking

For example





Simplicial sets

> More generally, a simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$

A simplicial set is a presheaf $X : \Delta^{op} \to Set$

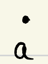
The category of simplicial sets is $sSet = Set^{\Delta^{op}}$ ← also written $\widehat{\Delta}$

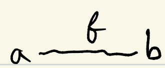
Terminology: An element of X_n is called an <u>element of X over n</u>, or an <u>n-simplex</u>

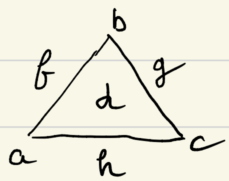
> Notation: we write $d_i x$ for $X d^i x$ and $s_i x$ for $X s^i x$
> Hence $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$

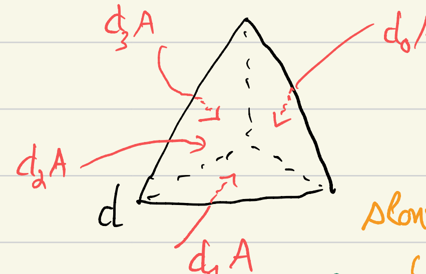
Drawing the elements of a simplicial set: ← Anticipating topological realisation!

$a \in X_0$ $\qquad$ $f \in X_1$ $\qquad$ $\alpha \in X_2$ $\qquad$ $A \in X_3$ tetrahedron









where $a = d_1 f$, $b = d_0 f$

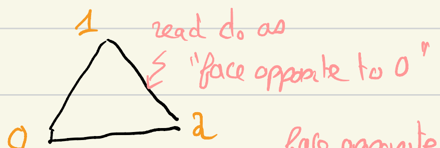
where $f = d_2 \alpha$, $g = d_0 \alpha$, $h = d_1 \alpha$, $a = d_1 f = d_1 h$ ...

etc...

slowly next page: $b = d_{\{0,2\}} \alpha$, $d = d_{\{1,2,3\}} A$

|| a decoration of

|| a decoration of



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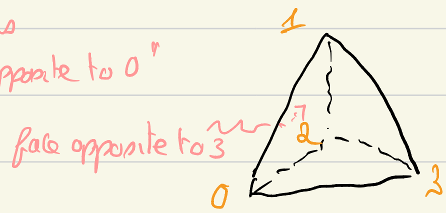


Thus (repeated) d_i's take iterated faces.
= injections

$X f x \in X_m$

$f : [m] \to [n]$ $\qquad$ $x \in X_n$

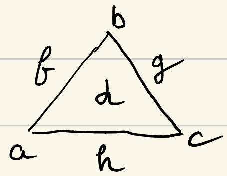
$Xfx \leftarrow x$

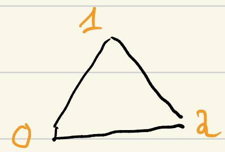
result of transporting x along f.

> A handy notation for iterated faces: for $f : [m] \to [n]$ injective we write
> $$Xfx = d_{[n]/Im(f)} x$$

$d_I x$ = (decoration of) the face obtained by intersecting the codimension 1 faces of x opposite to $i \in I$

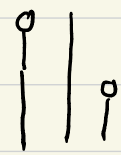
Combinatorics of taking faces



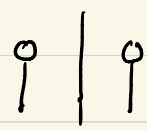


We have

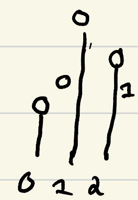
$b = d_0 f = d_0 d_2 \alpha$

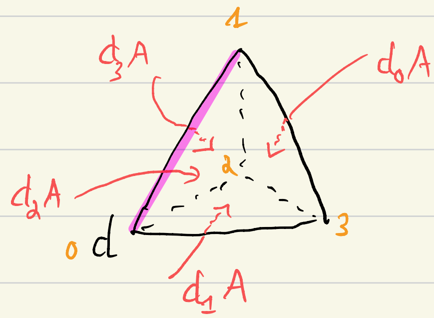


$b = d_{\{0,2\}} \alpha$



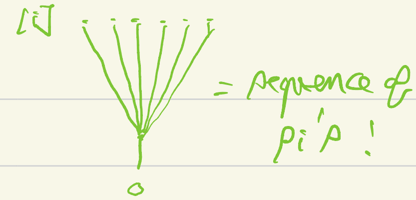
$b = d_1 g = d_1 d_0 \alpha$



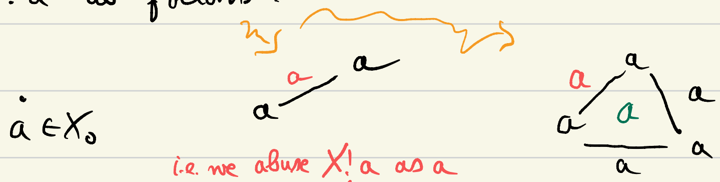


$= d_{\{2,3\}} A$

Degeneracies



What about s_i ? Let $a \in X_0$ and let $!_i$ be the unique map $[i] \to [0]$. Then we draw $X!a$ as follows:



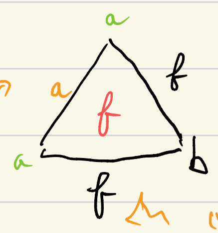
i.e. we abuse $X!_1 a$ as a

Note that abuses are consistent: $d_2 a = d_2 X!_2 a = X(!_2 \circ d^2) a = X!_1 a = a$

Other example: let $f \in X_1$ then we draw $s_0 f \in X_2$ as follows:

(heuristic for s_i : glueing copies of f along their i-th face)

indeed $d_2 s_0 f = s_0 d_1 f = a$ [illegible]



(abusing again $s_0 f$ as f)

indeed $d_1 s_0 f = f$ since [illegible] $= ||$

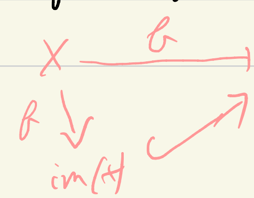
"contains no s_i"

Definition An n-simplex x is called degenerate if there exists a surjection ($\neq$ id) $f : [n] \to [m]$ (hence $m < n$) and an m-simplex y s.t. $x = Xf y$.

(the above examples are degenerate simplices)

Exercise Prove that one can replace "surjection $\neq$ id" by "any $f : [n] \to [m]$ s.t. $m < n$" (hint: use the epi-mono factorisation in Δ, showing on the way that it exists and is inherited from the one in Set).

Exercise Show that if x is non-degenerate, and f, y are such that $x = Xf y$, then f is injective. (hint: use again the epi-mono factorisation)



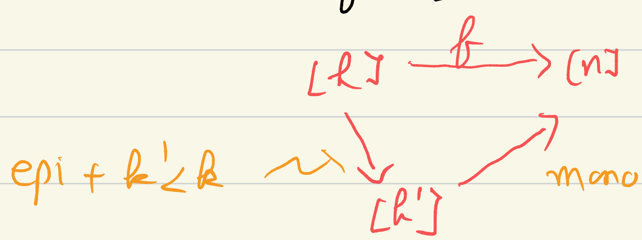
The standard simplex Δ^n

Δ^n is the simplicial set y_n $\quad (y : \Delta \to \hat{\Delta})$

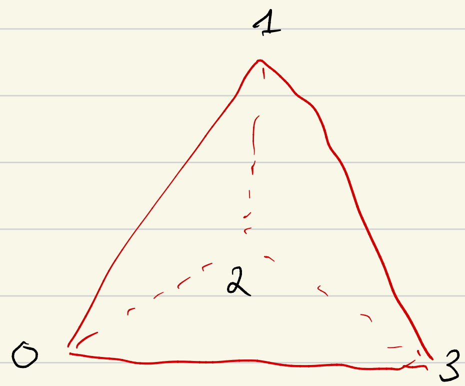
Explicitly: $\quad (\Delta^n)_m = \Delta(m, n)$

Exercise: Show that the non-degenerate simplices of Δ^n are exactly the injective maps.

Hint: note that $f : [k] \to [n]$ is non-injective iff



Therefore, say, Δ^3 has a unique non-degenerate 3-simplex (increasing injections = identities), justifying to draw it as



(iterated) ✓

Its faces are given by the subsets of $\{0,1,2,3\}$

(see also simplicial complexes below)

Eilenberg – Zilber's lemma

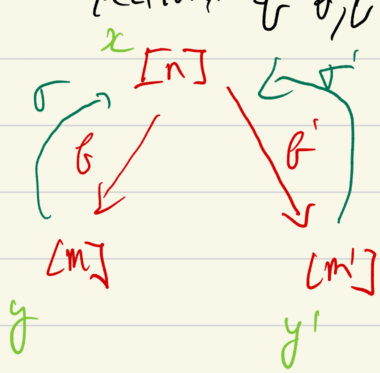
Proposition Let $X \in \hat{\Delta}$ and $x \in X_n$. Then there exists a unique pair (f, y) s.t.

- $f : [n] \to [m]$ is surjective
- $y \in X_m$ is non-degenerate
- $Xf\, y = x$

Proof Existence: If x is non-degenerate, take $f = id$, $y = x$.
If x is degenerate, then we apply iteratively the definition of being degenerate (induction on n).

Uniqueness: We use easy properties of Δ:
- if $f : [m] \to [n]$ is surjective, then it admits a section (any set-theoretic section is automatically monotone):
- Moreover, f is characterised by its set of sections

Suppose (f, y) and (f', y') satisfy the assumptions. Take σ, σ' sections of f, f', resp.:



We have $y = X(f\sigma)y = X\sigma(Xf\,y) = X\sigma\, x$ (section; assumption)

$X(f'\sigma)y' = X\sigma\, x = y$ (section'; just proved; non-degenerate!)

$\Rightarrow f'\sigma$ mono (exercise prev. page) $\Rightarrow m \leq m'$ $\Rightarrow m = m'$ (symmetry)

sections$(f) \subseteq$ sections(f') $\Leftarrow$ $f'\sigma = id$ $\Leftarrow$ $f'\sigma$ bijective (ordinal) and $\boxed{y = y'}$

$= $ (by symmetry)

$\boxed{f = f'} \Leftarrow$

Simplicial complexes

Simplicial complexes are simpler than simplicial sets, but less flexible.

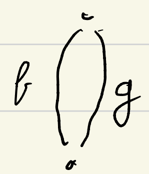
A simplicial complex is a pair (V, C), where

V is a set of vertices (the 0-simplices), and where

$C \subseteq \mathcal{P}^{fin}(V)$, and

- $x \in C \Rightarrow x \neq \emptyset$
- $\forall v \in V \ \{v\} \in C$
- $\emptyset \neq y \subseteq x \in C \Rightarrow y \in C$

The elements of C of cardinal n are the n-simplices.



simplicial set

circle as

simplicial complex

$V = \{v_0, v_1, v_2\}$

$C = \{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}\}$

A morphism $\Sigma : (V, C) \to (V', C')$ of simplicial complexes is a function $\Sigma : V \to V'$ such that $\forall A \in C \ \Sigma(A) \in C'$.

Relating simplicial complexes and simplicial sets

We make a little twist on both side, to establish a reflection.

- We consider ordered simplicial complexes, which are (V,C) + a total order on V. This affects the morphisms only! we accept only those $\xi : (V,C) \to (V',C')$ such that $\xi : V \to V'$ is monotone (= non-decreasing).

- We consider 0-simplices-ordered simplicial sets, which are X + a total order on X_0. Again, this affects the morphisms only: we accept only those $f : X \to Y$ such that $f_0 : X_0 \to Y_0$ is monotone.

We denote these two categories by **Scpx** and **Sset**.
We write TO for the category of totally ordered sets and monotone functions.

From Scpx to Sset. We define, for $(V,C) \in$ Scpx $\leadsto V$

(if $\{v_0, v_1\} \in C$, then $v_0 v_0 v_1 \in (X_C)_2$)

$(X_C)_n = \{\varphi \in TO[[n],V] \mid \varphi([n]) \in C\}$ (note that $(X_C)_0$ is naturally ordered)

For $f : [m] \to [n]$, we set $(X_C)f : (X_C)_n \to (X_C)_m : \varphi \mapsto \varphi \circ f$
(note that $(\varphi \circ f)[m] = \varphi(f[m]) \subseteq \varphi[n] \in C$, hence $(\varphi \circ f)[m] \in C$

In English, simplices are lists $(v_0 \cdots v_n)$ where there may be repetitions = degenerate simplices!

For $d^i = (0 \cdots \hat{i} \cdots m+1) : (v_0 \cdots v_{m+1}) \mapsto (v_0 \cdots \hat{v_i} \cdots v_{m+1})$
For $s^i = (0, \cdots i\, i \cdots m) : (v_0 \cdots v_m) \mapsto (v_0 \cdots v_i\, v_i \cdots v_m)$

For $\xi : (V,C) \to (V',C')$ we set $(\xi^*)_n \varphi = \xi \circ \varphi$ (note: $\xi(\varphi[n])$ — $\varphi[n] \in C$, $\xi(\varphi[n]) \in C'$)

Thus $(V,C) \mapsto X_C$, $\xi \mapsto \xi^*$ defines a functor

Scpx-to-Sset : Scpx $\to$ Sset

From $\underset{\sim}{\text{Sset}}$ to $\underset{\sim}{\text{Scpx}}$. Let X be a simplicial set. We define (V_X, C_X) as follows:

$V_X = X_0$ (with its total order)

$C_X = \{ \text{Vert}(n,x) \mid x \in X_n \} = \text{Vert}(\text{elt}(X))$, where

$\text{Vert}(n,x) = \{ Xfx \mid f : [0] \to [n] \}$

Claim: C_X is closed under subsets.

Proof: We introduce notation: $\begin{cases} \text{if } f : [0] \to [n] \text{ then } \underline{f} = f(0) \\ \text{if } j \in [n], \text{ then } \bar{j} : [0] \to [n] = 0 \mapsto j \end{cases}$ (more generally, for any set)

Let $A \subseteq \text{Vert}(n,x)$. Let $A' = \{ j \mid X\bar{j}x \in A \}$

Let $\iota : [m] \to [n]$ be s.t. ι is injective and its image is A'.

Then $\text{Vert}(m, X\iota x) = \{ Xg(X\iota x) \mid g : [0] \to [m] \}$ ($= X(\iota f)x$)

$= \{ Xfx \mid f \text{ factors through } \iota \}$

$= \{ Xfx \mid \underline{f} \in A' \} = \{ Xfx \mid Xfx \in A \} = A$

Let $u : X \to Y$. Then we have $u_0 : X_0 \to Y_0$.

Claim: u_0 is a morphism. Indeed, we have $u_0(\text{Vert}(n,x)) \subseteq \text{Vert}(n, u_n(x))$:

$$\begin{array}{ccc} x \in X_n & \longrightarrow & Y_n \ni u_n(x) \\ \downarrow Xf & & \downarrow Yf \\ X_0 & \xrightarrow{u_0} & Y_0 \end{array}$$

$Xfx \in \text{Vert}(n,x) \subseteq$; $\exists f, x$; w

Thus we have a functor $\text{Sset-to-Scpx} : \underset{\sim}{\text{Sset}} \to \underset{\sim}{\text{Scpx}}$:

$X \mapsto (X_0, C_X)$, $u \mapsto u_0$

Theorem The functors Sset-to-Scpx and Scpx-to-Sset form a reflection, i.e. Sset-to-Scpx : $\underset{\sim}{\text{Sset}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underset{\sim}{\text{Scpx}}$: Scpx-to-Sset

and Scpx-to-Sset is full and faithful.

Proof Full and faithfulness follows from

$$\Phi(v) = \underline{\Phi \circ \bar{v}} = \underline{\Phi_0^{*} \bar{v}}$$

i.e. we can retrieve Φ from Scpx-to-Sset$(\Phi) = \Phi^{*}$.

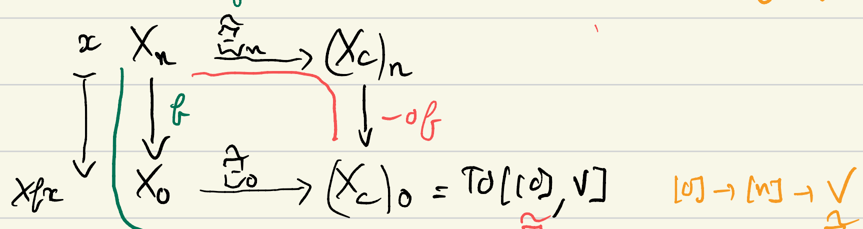
For X and (C,V), we need to establish

$$\frac{\Phi : X_0 \to V \quad \text{st } \forall (n,x) \in el(X) \;\; \Phi(\mathrm{Vert}(n,x)) \in C}{\tilde{\Phi}_n : X_n \to (X_C)_n \quad \text{natural}}$$

- Given $\tilde{\Phi}$, define $\Phi(x_0) = \underline{\tilde{\Phi}_0(x_0)}$. We have to check $\Phi(\mathrm{Vert}(n,x)) \in C$, which follows from the following claim

  Claim $\underline{\Phi(\mathrm{Vert}(n,x))} \subseteq (\tilde{\Phi}_n x)[n]$: $\in C$ by def. of X_C

  $v \in$



$\bar{v} = \tilde{\Phi}_n(x) \circ f \Leftarrow v \in \tilde{\Phi}_n(x)[n]$

more precisely $v = \tilde{\Phi}_n(x)(\underline{f})$

- Given Φ, (a) the bijection forces $\tilde{\Phi}_0$: $\tilde{\Phi}_0(x_0) = \overline{\Phi(x_0)}$

(b) **Claim** Any $\tilde{\Phi}$ is determined by its $\tilde{\Phi}_0$ component: Choose n and $i \in [n]$: put $f = \bar{i}$. The diagram reads as $\tilde{\Phi}_n(x)(i) = \underline{\tilde{\Phi}_0(X \bar{i} x)}$ $\left(= \Phi \bar{i} x \text{ by (a)}\right)$

(c) By (b)+(a), we have to set $\boxed{\tilde{\Phi}_n(x)(i) = \Phi(X \bar{i} x)}$

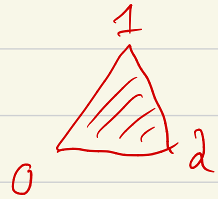
Easy: for $f : [m] \to [n]$ and $i \in m$; we get ┐ $i \mapsto \Phi(X f \bar{i} x)$
└ $i \mapsto \Phi(X \overline{fi} x)$ $=$ (since $f \circ \bar{i} = \overline{fi}$)

Boundaries $\partial\Delta^n$ and Horns Λ^n_i

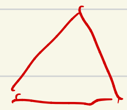
We give 3 definitions, from the more intuitive to the "less intuitive".

- As simplicial complexes:

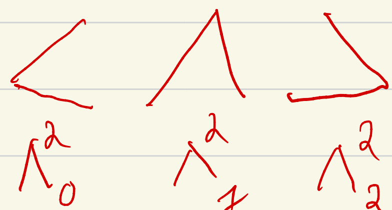
$\Delta^n = (V, \mathcal{P}^*(V))$ $\quad (V = \{v_0 \dots v_n\})$



$\partial\Delta^n = (V, \mathcal{P}^*(V) / \{V\})$



$\Lambda^n_k = \partial\Delta^n / \{v_0, \dots, \hat{v_k}, \dots v_n\}$



- www.math.univ-paris13.fr/~vallette → Cours Homotopy theories p.60

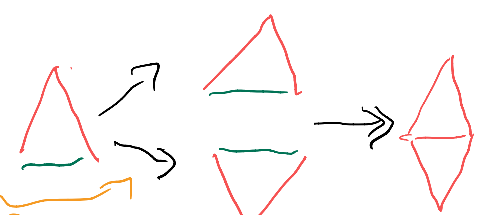
◇ *Le bord $\partial\Delta^n$ du simplexe standard est égal au coégalisateur suivant :*

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{0\leqslant l\leqslant n} \Delta^{n-1} \twoheadrightarrow \partial\Delta^n ,$$

où le morphisme en haut à gauche envoie la copie indicée par $i < j$ sur celle indicée par j via δ_i, le morphisme en bas à gauche envoie la copie indicée par $i < j$ sur celle indicée par i via δ_{j-1}

◇ *Le k^e-cornet Λ^n_k de dimension n est égal au coégalisateur suivant :*

$$\coprod_{0\leqslant i<j\leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0\leqslant l\leqslant n \\ l\neq k}} \Delta^{n-1} \twoheadrightarrow \Lambda^n_k ,$$



avec les mêmes morphismes. See p.27 for the real topological meaning!

- www.math.uni-bonn.de/people/fhebestr/Higher%20Cats/Land-scriptx.pdf p.12

(1) The boundary $\partial\Delta^n$ is the subsimplicial set of Δ^n whose k-simplices consist of the non-surjective maps $[k] \to [n]$.

(2) For any subset $S \subseteq [n]$ the S-horn $\Lambda^n_S \subseteq \Delta^n$ consists of those k-simplices $f\colon [k] \to [n]$ where there exists a $i \in [n] \setminus S$ such that i is not in the image of f.

We then set $\Lambda^n_j = \Lambda^n_{\{j\}}$

<u>Exercise</u> Show that these three definitions are equivalent.

Hint: for the first definition, use Scpx-to-Sset.

Spines

The third definition of horns / boundaries makes it clear that Λ^n_i, $\partial\Delta^n$ are subsimplicial sets (i.e. levelwise inclusions that are natural) of Δ^n.

Another important family are the spines $I^n \subseteq \Delta^n$. Again, the simplest definition is via simplicial complexes:

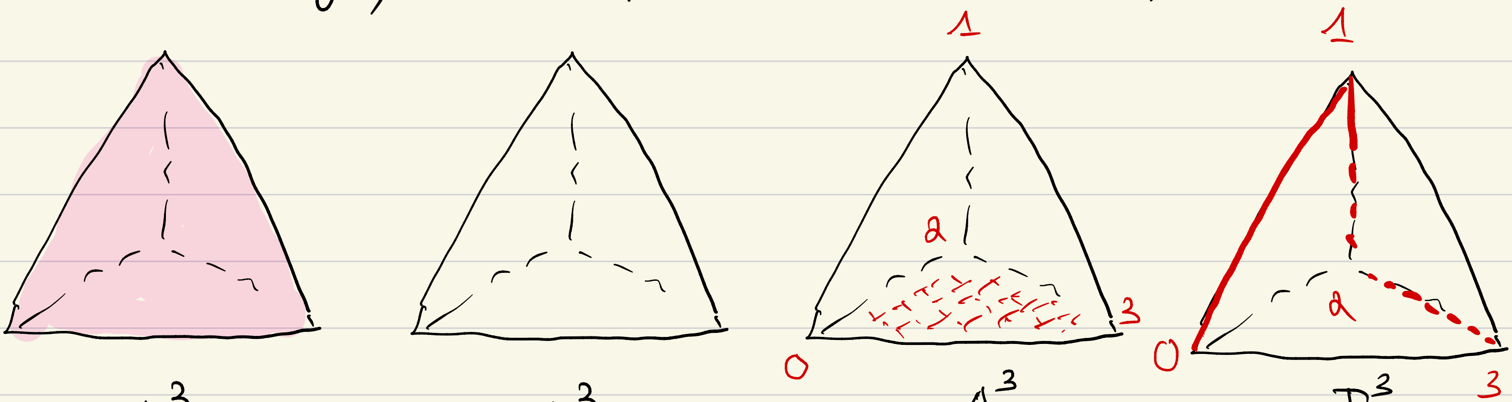
$$I^n = \big(\{v_0, \dots v_n\}, \{\{v_0\}, \dots \{v_n\}, \{v_0, v_1\}\{v_1, v_2\} \cdots \{v_{n-1} v_n\}\}\big)$$

Exercise: Show that spines are equivalently defined by

The spine $I^n \subseteq \Delta^n$ is given by those k-simplices $f\colon [k] \to [n]$ whose image is either of the form $\{j\}$ or $\{j, j+1\}$. (from Land's notes p.12)

Give also a definition via a coequaliser.

In summary, and in pictures (dimension 3):



Δ^3 (everything) cf. 3-ball

$\partial\Delta^3$ (four Δ^2 glued) cf. 2-sphere

Λ^3_2 (three Δ^2, glued) " all but the marked one, opposite to

I^3

Remark: We have $\boxed{I^2 = \Lambda^2_1}$

Nerve of a category

There is a natural full and faithful functor $\leq : \Delta \to Cat$

$[0, \dots, n] \mapsto 0 \to 1 \to 2 \to \cdots \to n$

(pre)orders as categories

Recall the nerve functor from Lecture 1:

$N_{\leq} : Cat \to \widehat{\Delta}$ (abbreviated as N)

$(NC)_n = Cat(\leq[n], C)$ (abbreviated as $Cat([n], C)$)

$c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \cdots c_{n-1} \xrightarrow{f_{n-1}} c_n$ = sequence of n composable morphisms

Notation $(f_0, \dots, f_{n-1})$

In particular $(NC)_0 = Ob\, C$ $\quad \cdot\, c_0$

Face maps: $d_i\left(\cdots \xrightarrow{f_{i-1}} \cdot \xrightarrow{f_i} \cdots \right) = \cdots \xrightarrow{f_i \circ f_{i-1}} \cdots$ ($0 < i < n$: "inner")

$d_0\left(c_0 \xrightarrow{f_0} c_1 \cdots \right) = c_1 \cdots \qquad d_n\left(\cdots \xrightarrow{f_n} \right) = \cdots$

Degeneracies $s_i\left(\cdots \cdot \cdots \right) = \cdots \cdot \xrightarrow{id} \cdot \cdots$

Exercise: Prove that those are indeed the maps $Cat(\leq d^i, C)$, $Cat(\leq s^i, C)$

(Hint: spell out the action of $\leq$ on d^i and s^i).

(Visual hint: $\leq \begin{pmatrix} 0 & & 1 \\ | & \circ & | \\ 0 & 1 & 2 \end{pmatrix} (0 \to 1) = 0 \to 1 \to 2$)

and therefore, $\leq : \Delta \to Cat$ is dense (cf. exercise p. 12 of lecture 1)

Exercise: Show that $N : Cat \to \widehat{\Delta}$ is full and faithful.

(Hint: note that $NC_0 = Ob\, C$, $NC_1 = Mor\, C$.)

Since N is f. and f., this raises the question of ...

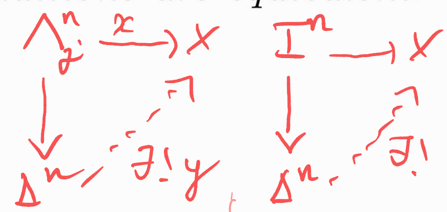
Which simplicial sets are nerves of categories?

Nerve theorem

also called liftings — Land, p. 14–15

For a simplicial set X, the following three conditions are equivalent.

(1) X has unique extensions for $\Lambda^n_j \to \Delta^n$ if $0 < j < n$.
(2) X has unique extensions for $I^n \to \Delta^n$ for $n \geq 2$.
(3) X is isomorphic to the nerve of a category.



inner horns (contrary = outer horns)

Notation $y|\Lambda^n_j$

We shall make a slightly sharper statement and prove it at the end of the lecture using material introduced next:

- description of the left adjoint of N (from sSet to Cat)
- skeleton and coskeleton

Here is a preview of the sharper statement. We write

- $(1)^+$ for (1) where the inner restriction is removed for $n \geq 4$

Then we shall prove $(2) \Rightarrow (3) \Rightarrow (1)^+ \Rightarrow (1) \Rightarrow (2)$

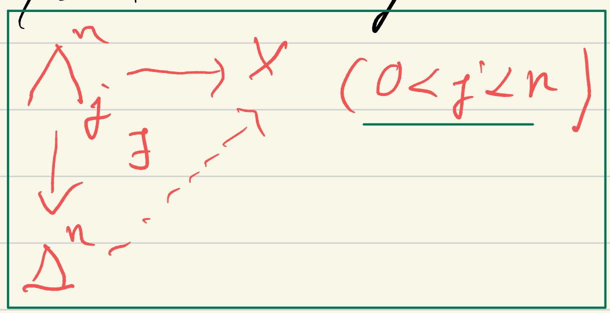
Exercise: Show that if (1) holds for all horns $(0 \leq j \leq n)$, then X is isomorphic to the nerve of a groupoid, and conversely.

A review of two approaches to higher categories

The two characterisations of the theorem on p. 14, when weakened, yield notions of *weak category*.

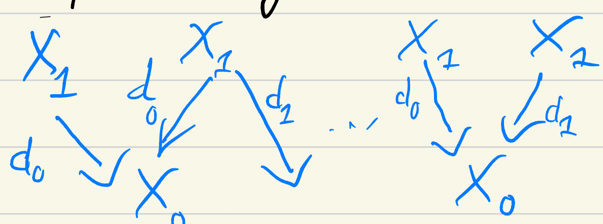
- A simplicial set satisfying (1) "minus uniqueness" is called a *quasicategory*. Explicitly:

A quasicategory $(\infty, 1)$ is a simplicial set X admitting liftings for all inner horns:



- Weakening (2) leads to another approach, due to *Rezk*. We first reformulate (2).

Exercise • Show that $\hat{\Delta}(I^n, X)$ is equivalently described as the colimit $\underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n \text{ times}}$ of

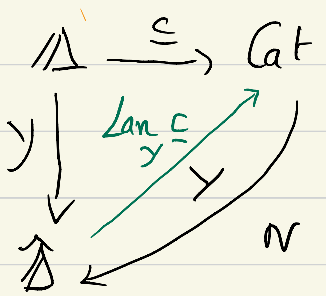


- and that (2) can be reformulated as (strict Segal condition)
The natural map $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is iso.

By relaxing this iso to be only a weak equivalence (in a suitable model category context) of simplicial sets (and hence taking X to be a bisimplicial set $\Delta^{op} \to \hat{\Delta}$ $(\Delta^{op} \times \Delta^{op} \to \underline{\Delta})$) we get *Rezk's* notion of *Segal space* ...

The other way around: from simplicial sets to categories

Consider



Since Cat has all colimits, we can apply the formula.

$$\mathrm{Lan}_y c\, X = \underset{(n,\, x\in X_n)}{\mathrm{colim}}\; c[n]$$

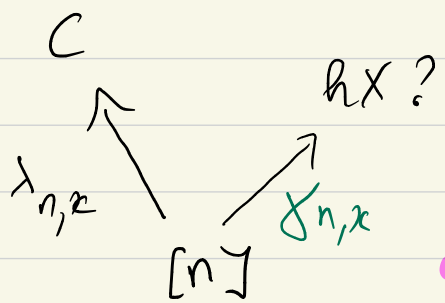
also written hX

a "homotopy category" — "realisation of X as a category"

It turns out that hX has an alternative description, that we can synthesise: A cocone λ for the functor

$$(n,x) \mapsto [n] : \mathrm{el}(X) \to \mathrm{Cat}$$

with vertex C provides in particular



- a map $\lambda_0 : X_0 \to \mathrm{Ob}\, C$ ($\lambda_{0,-}$)
- a map $\lambda_1 : X_1 \to \mathrm{Mor}\, C$
- such that (naturality of λ) $\lambda_1(a \xrightarrow{a} a) = \mathrm{id}_{\lambda_0(a)}$ $(\forall a \in X_0)$

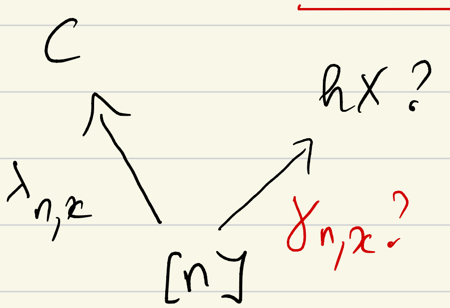
It is a good starting point for hX!

$\boxed{\mathrm{Ob}(hX) = X_0}$ $\quad \mathrm{Mor}(hX) \overset{?}{=} X_1 \quad$ $\boxed{\forall a,\ \mathrm{id}_a = a \xrightarrow{a} a}$

too naive!

- we need to define a colimiting cone γ
- we need to compose morphisms!

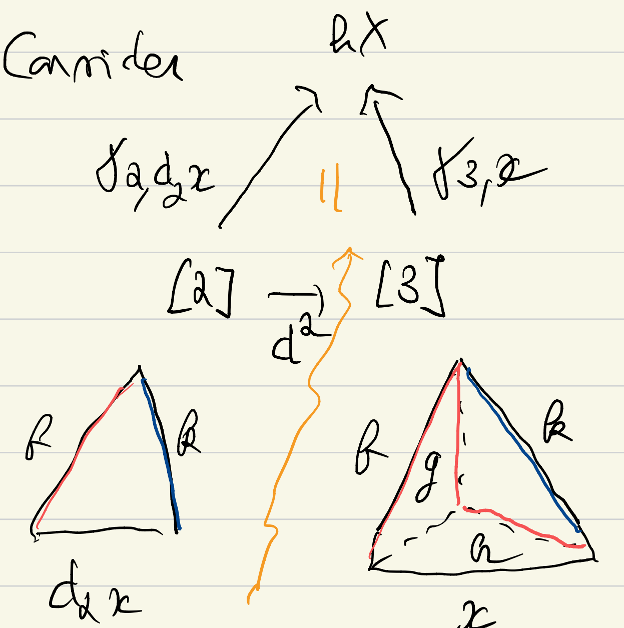
Synthesising hX, continued

- 

- candidate for the colimiting cocone:

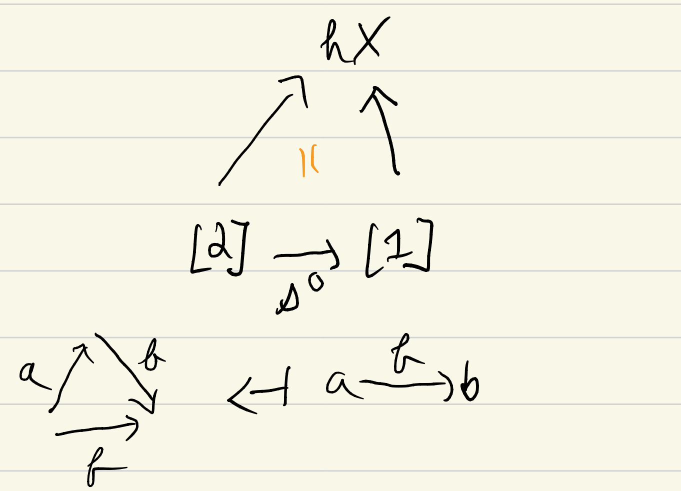
$$\gamma_{n,x} \overset{?}{=} x \restriction I^n \qquad I^n \hookrightarrow \Delta^n \xrightarrow{x} X$$

Consider



Since $d^2(1\to 2) = 1\to 2\to 3$

$k = h\circ g$ is forced in hX



Since $s^0(0\to 1\to 2) = 0\to 1$

$f\circ \mathrm{id}_a = f$ is forced in hX

- Dealing with compositions: add them (as) freely (as possible)

$$\mathrm{Mor}(hX) \overset{?}{=} \coprod_n \mathrm{Spet}(I^n, X)$$

formal compositions of composable 1-simplices

Combining • and • :

$$\mathrm{Mor}(hX) = \coprod_n \mathrm{Spet}(I^n, X) \Big/ \; \sim \xrightarrow{f} \cdot \xrightarrow{g} \sim \;\; \overset{h}{=} \;\; \sim \xrightarrow{h} \sim, \quad \forall\ f, g, d$$

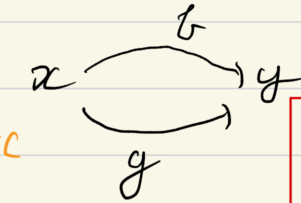
$$\sim \xrightarrow{f} b \xrightarrow{\mathrm{id}_b} b \sim \;=\; \sim \xrightarrow{f} \sim \qquad \sim a \xrightarrow{\mathrm{id}_a} a \xrightarrow{f} \sim \;=\; \sim \xrightarrow{f} \sim$$

Exercise Check that $\gamma_{n,x} = [x \restriction I^n]$ is indeed a colimiting cocone.

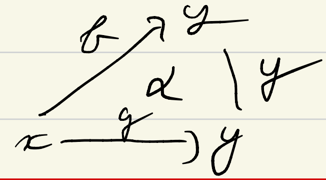
equivalence class

The Boardman-Vogt construction

If X satisfies some conditions, then there is yet another description of hX. We define a relation on parallel 1-simplices, i.e



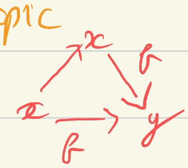
We write $f \sim g$ (right homotopic) if $\exists \alpha \in X_2$ s.t.



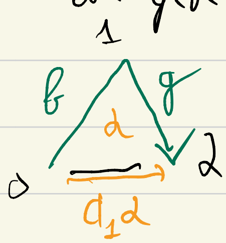
<u>Theorem</u> If X has the (non necessarily unique) lifting property for Λ^2_1 and Λ^3_1 and Λ^3_2 (= all inner 2 and 3-horns), then $\sim$ is an equivalence, and the following defines a category $\pi(X)$ which is isomorphic to hX:

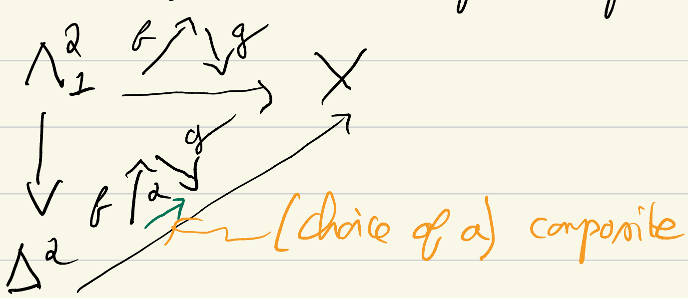
- $\operatorname{Ob} \pi(X) = X_0$
- $\pi(X)(x,y) = \{ f \in X_1 \mid d_1 f = x \; d_0 f = y \} / \sim$

- Moreover, we have $f \sim g$ (left homotopic) iff

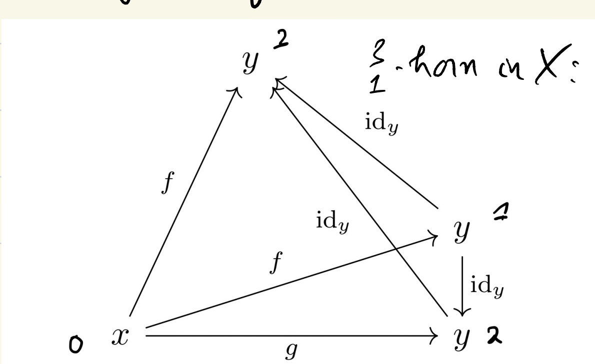


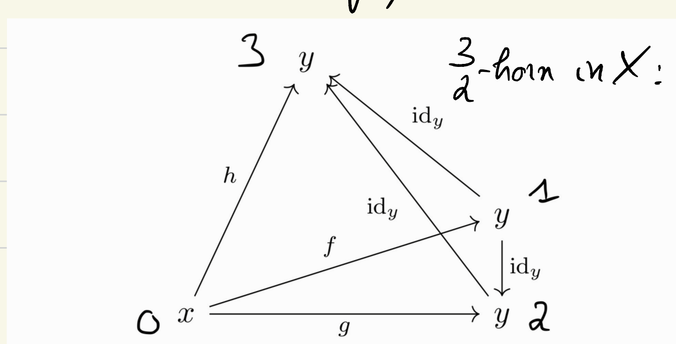
<u>Exercise</u> Prove this theorem. Hints (see Land's notes p.18-19): • candidates for composition are given by Λ^2_1 liftings:





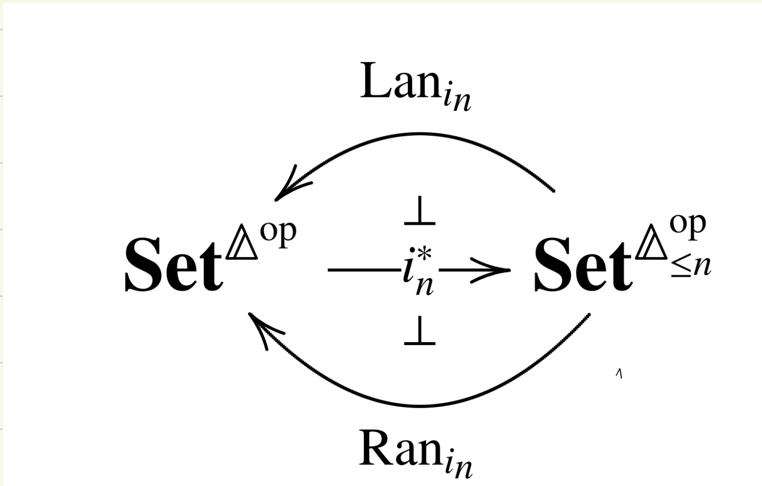
- For symmetry, build this
- For transitivity, build this

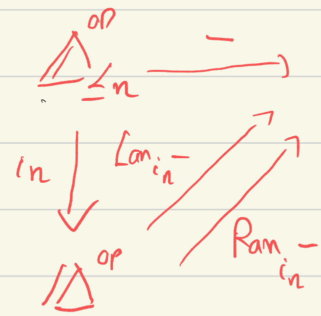




Skeleton and coskeleton

Let $\Delta_{\leq n}$ be the full subcategory of Δ on objects $[i]$ $(i \leq n)$.

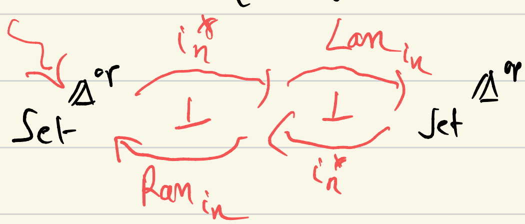




We write $sk_n = Lan_{i_n} \circ i_n^*$ comonad } on $Set^{\Delta^{op}}$

$cosk_n = Ran_{i_n} \circ i_n^*$ monad }

We have $sk_n \dashv cosk_n$ (as functors)



We say that X is n-skeletal if $X \cong sk_n X$ — more precisely via the counit of the comonad

X is n-coskeletal if $X \cong cosk_n X$ — unit iso

Skeleton in details

Explicitly, we have

$$\begin{array}{ccc} \Delta^{op}_{\leq n} & \xrightarrow{i_n} & \Delta^{op} \xrightarrow{X} Set \\ {\scriptstyle i_n}\downarrow & = & \nearrow {\scriptstyle sk_n X} \\ \Delta^{op} & & \end{array}$$

i_n full and faithful ⇝   ⇜ left Kan ext.

$$(sk_n X)_m = \int^{i \leq n} \Delta^{op}([i],[m]) \cdot X_i = \{(i, f \in \Delta([m],[i]), x \in X_i)\} / \text{gluing conditions}$$

We note that all m simplices in $sk_n X$, for $m > n$, are degenerate (cf. first exercise on p. 4!)

Hence $sk_n(X)$ has

- the same simplices as X in dimensions $\leq n$,
- and, above n, has only degeneracies of those.

<u>Exercise</u> Show that $\partial \Delta^n$ (as defined earlier) is $Sk_{n-1} \Delta^n$.

<u>Exercise</u> Show that for arbitrary X, $hX \simeq h(sk_2 X)$.

Note that $id: X i_n \to X i_n$ induces a natural transformation $Sk_n X \to X$

<u>Exercise</u> Spell out this natural transformation, and that under the identifications above, it is the inclusion $Sk_n(X) \subseteq X$.

Coskeleton in details

In contrast, $\mathrm{cosk}_n(X)$ has many m-simplices for $m > n$. We have:

$$(\mathrm{cosk}_n X)_m \simeq \hat{\Delta}(\Delta^m, \mathrm{cosk}_n X) \simeq \hat{\Delta}(\mathrm{sk}_n \Delta^m, X)$$

Yoneda — sk ⊣ cosk

In English, this gives the following description of $(\mathrm{cosk}_n X)_m$ for $m > n$: each consistent decoration of the n-faces of Δ^m by n-simplices of X gives rise to a unique simplex in $(\mathrm{cosk}_n X)_m$

(called an m n-shell) (the filler)

Exercise
- Spell out the meaning of "consistent" here.
- Prove that if all simplices in the shell are non-degenerate, then the unique filler is non-degenerate.

Exercise Synthesise the same description of cosk_n from its definition by the end formula

$$(\mathrm{cosk}_n X)_m = \int_{i \leq n} X_i^{\Delta([i],[m])}$$

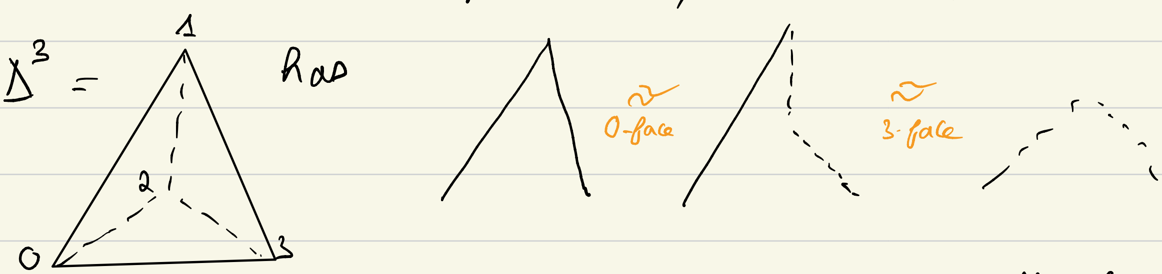
(hint: in this formula, we can replace = restrict $\Delta([i],[m])$ by the injective morphisms $[i] \to [m]$).

Preparations for the nerve theorem

Exercise We have that $sk_2(\Lambda^n_j) \to sk_2(\Delta^n)$ (→ $sk_2(\partial\Delta^n)$) is iso $\forall n \geq 4$, $0 \leq j \leq n$
(Hint all (iterated) faces of the missing face are (iterated) faces of another face in Δ^n.)

Lemma We have $h(\Lambda^3_1) = h(\partial\Delta^3) = h(\Lambda^3_2)$.

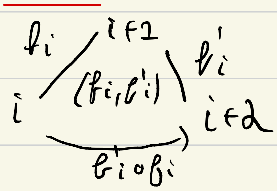
Proof $\partial\Delta^3 =$



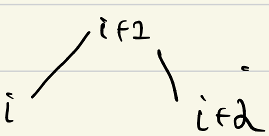
as sequences of composable arrows, and they are equivalent thanks to the presence of the 0 face and of the 3-face in $\partial\Delta^3$.
The same is true for Λ^3_1 and Λ^3_2 (but not for Λ^3_0, Λ^3_3!)

Lemma Nerves of categories are 2-coskeletal.

Proof A ∂^n-shell of NC contains in particular decorations



for all



. Compatibility forces $f_i' = f_{i+1}$.

This already determines a candidate for the filling: $(f_0, \ldots, f_{n-1}) \in NC_n$.
This candidate is not only unique, but it also works, i.e. for every $\{i<j<k\} \subseteq [n]$ we have that $d_{\{i,j,k\}}(f_0, \ldots, f_{n-1})$ is the face prescripted by the original 2-shell (shown by compatibility).

Exercise Show that the converse does not hold.
(Hint: several 2-simplices could have the same restriction to I^2.)

Nerve theorem for simplicial sets (full statement)

Theorem TFAE for a simplicial set:

(1) X has unique liftings for all inner n-horns $(n \geq 2)$

(1)$^+$ X has unique liftings for all horns except for $\Lambda^2_0, \Lambda^2_2, \Lambda^3_0, \Lambda^3_3$

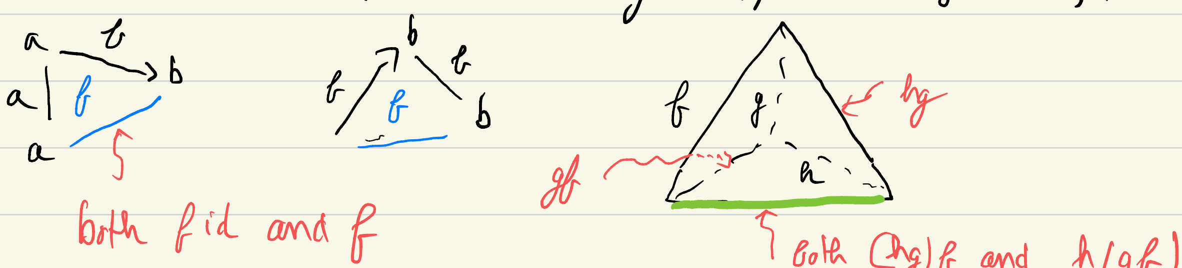
(2) X has unique liftings for all n-spines $(n \geq 2)$

(3) X is isomorphic to the nerve of a category

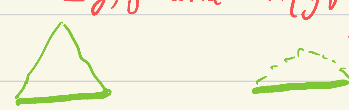
Proof We prove (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1)$^+$, and (1) $\Rightarrow$ (2)

- **(2) $\Rightarrow$ (3)** We can form a category C by setting $\mathrm{ob}\, C = X_0$, $\mathrm{mor}\, C = X_1$, $id(c) = s_0 c$, $g \circ f$ given by the unique (third side of the) extension $f \nearrow \searrow g$ of $f \nearrow \searrow g : I^2 \to X$.

The monoid laws are ensured by uniqueness of the extensions of



We then see that $X \cong NC$ by (2):



$$X_n = \hat{\Delta}(\Delta^n, X) = \hat{\Delta}(I^n, X) = NC_n$$

Yoneda; (2); by def. of C

- $(3) \Rightarrow (1)^+$

- For $n=2$, we have directly $\hat{\Delta}(\Lambda^2_1, NC) \simeq NC_2 \; (\simeq \hat{\Delta}(\Delta^2, NC))$ by definition of NC.
- For $n > 2$, we use some "categorical abstract non sense".

We know that NC is 2-coskeletal (second lemma p.23).

If Y is 2-coskeletal and X is arbitrary, we have bijections

$$\frac{\hat{\Delta}(sk_2(X), Y)}{\dfrac{\hat{\Delta}(X, cosk_2 Y)}{\hat{\Delta}(X, Y)}} \quad \begin{matrix}\text{adjunction} \\ \text{coskeletal}\end{matrix}$$

Our goal is to prove (for the specified horns):

$$\frac{\hat{\Delta}(\Lambda^n_k, NC)}{\hat{\Delta}(\Delta^n, NC)} \overset{\text{2-coskeletality}}{\Longleftrightarrow} \frac{\hat{\Delta}(Sk_2(\Lambda^n_k), NC)}{\hat{\Delta}(Sk_2(\Delta^n), NC)} \overset{h \dashv N}{\Longleftrightarrow} \frac{Cat(h(Sk_2(\Lambda^n_k)), C)}{Cat(h(Sk_2(\Delta^n)), C)}$$

and hence to prove $h(Sk_2(\Lambda^n_k)) = h(Sk_2(\Delta^n))$:

- for $n \geq 4$, we know (exercise p.23) $Sk_2(\Lambda^n_k) \simeq Sk_2(\Delta^n)$, for all horns, and we are a fortiori done.
- For $n=3$, we have $Sk_2(\Lambda^3_k) = \Lambda^3_k$, hence our goal reformulates as $h(\Lambda^3_k) = h(sk_2(\Delta^3)) = h(\partial\Delta^3)$ for $k = 1, 2$,

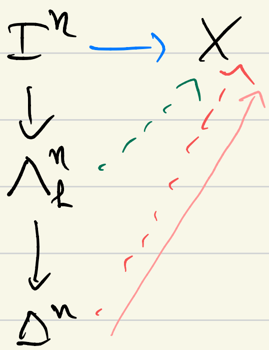
(exercise p.21)

— first lemma on p.23

- (1) ⇒ (2)

We use induction on n : • For $n=2$, this is tautological, as $\Lambda^2_1 = I^2$

• Inductive case. It is enough to prove that every n-spine in X has a unique extension to some fixed chosen inner n-horn in X, because we can then use (1) for the further lifting to Δ^n:



Overall uniqueness of ⇢ wrt to → is deduced as follows: if → is another candidate, then ↓ ↗ = ⇢ by uniqueness, (claimed)

and → = ⇢ by uniqueness (assumed)

We start with a decoration $(f_0, \cdots, f_{n-2})$ of the spine of Δ^n. By assumption for $n=2$, we have unique composites $f_{i+1} \circ f_i$ for all i, so that we can build $n+1$ $(n-1)$-spines in X :

$$\underset{x_0}{(f_1 \cdots f_{n-2})}, \underset{x_1}{(f_1 \circ f_0, f_2, \cdots f_{n-2})}, \cdots, \underset{x_{n-1}}{(f_0, \cdots, f_{n-1} \circ f_{n-2})}, \underset{x_n}{(f_0, \cdots, f_{n-2})}$$

By induction for $n-1$, we get $n+1$ simplices $x_i \in X_{n-1}$ having those spines (unique)

We are left to show to show : (cf. p. 2)

If $\{i < j\} \subseteq [n] \setminus \{i_0\}$ for some chosen $0 < i_0 < n$ $\quad (x_i)_{j-1} = (x_j)_i$

By assumption for $n-2$, it is enough to look at the spines of $(x_i)_j$ and of $(x_j)_i$, which are again obtained from $\text{spine}(x_i)$ and $\text{spine}(x_j)$ by the same process as above

Exercise Before examining general n, work out $n=3$. (next page)

(1) ⇒ (2) continued

$(f_1, \dots, f_{n-1}), (f_1 \circ f_0, f_2, \dots, f_{n-1}), \dots, (f_0, \dots, f_{n-1} \circ f_{n-2}), (f_0, \dots, f_{n-2})$

$x_0 \qquad x_1 \qquad \dots \qquad x_{n-1} \qquad x_n$

- $i = 0$, $j = n$ We have $(x_0)_{n-1} = (f_1, \dots, f_{n-2}) = (x_n)_0$
- $i = 0$, $j = 1$ $(x_0)_0 = (f_2, \dots, f_{n-1}) = (x_1)_0$
- $i = 0$, $1 < j < n$ We have $(x_0)_{j-1} = (f_1, \dots, f_j \circ f_{j-1}, \dots, f_{n-1}) = (x_j)_0$
- $0 < i < j < n$, $j - i > 1$ We have
  $(x_i)_{j-1} = (f_0, \dots, (f_i \circ f_{i-1}), \dots, (f_j \circ f_{j-1}), \dots, f_{n-1}) = (x_j)_i$
- $0 < i < n-1$ $j = i+1$ $(x_i)_i = (f_1, \dots, f_{i+1} \circ (f_i \circ f_{i-1}), \dots, f_{n-1})$
  $(x_{i+1})_i = (f_1, \dots, (f_{i+1} \circ f_i) \circ f_{i-1}, \dots, f_{n-1})$

Here we distinguish two cases:

- $n > 3$. Then we can apply induction for $n = 3$ to the 3-spine (f_{i-1}, f_i, f_{i+1}) and get a 3-simplex in X witnessing associativity, and then $(x_i)_{i+1} = (x_{i+1})_i$.
- $n = 3$. Then we simply **avoid** this case by requiring our chosen horn to be an inner horn: indeed, in an inner n-horn, x_0 and x_n have to be there. If $n=3$, there is only one left face to be chosen to form an inner horn: x_1 and x_2 cannot be both in, i.e. $0 < i < j < n$ impossible.

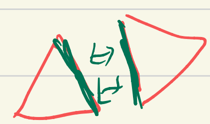
- All cases with $j = n$ are symmetric to the cases with $i = 0$.

Remark: For $n \geq 4$, it does not matter which fixed horn we choose (we could as well have chosen an outer horn, except that we need the horn to be inner for applying our assumption (1)).

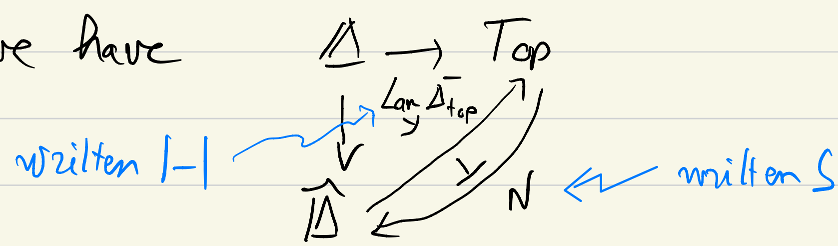
And topology in all this?

Replace Cat by Top, replace $\subseteq$ by the functor $\bar{\Delta}_{top} : \Delta \to Top$ defined by Δ^n_{top} = convex hull of $\{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}$ in $\mathbb{R}^{n+1}$.

Exercise Describe the action of $\bar{\Delta}_{top}$ on morphisms



Then we have



Instantiating the definitions, we have

$(SA)_n = Top(\Delta^n_{top}, A)$ (the singular complex of A)

$|X| = \int^n X_n \times \Delta^n_{top}$ (the geometric realisation of X)

Exercise Make sense of f^* and f_* in the following formula, and prove it:

$$|X| = \Big(\coprod_{n \in \Delta} X_n \times \Delta^n_{\mathrm{top}}\Big) / ((f^*(x), t) \sim (x, f_*(t)))$$

$C \longmapsto NC \longmapsto |NC|$

Category — simplicial set — top space