Lecture 3 Enriched categories

Motivation: Often, homsets $C(x,y)$ of a category carry some structure, and hence are not a mere set of morphisms (for C locally small).

It may even be the case that $C(x,y)$ is not a set! For example, it could be ... a simplicial set.

The framework to do this is that of enriched categories.

Formally, $C(x,y)$ may be an object of some other category $\mathcal{V}$.

And composition, which is usually a function

$$C(y,z) \times C(y,x) \to C(x,z)$$

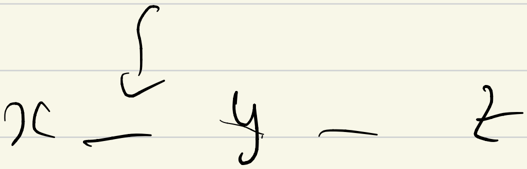
becomes a morphism in $\mathcal{V}$. It turns out that we do not need $\times$ to be a product in $\mathcal{V}$, but only a monoidal product, i.e. we need $\mathcal{V}$ to be a (symmetric) monoidal category

↓

next slide!

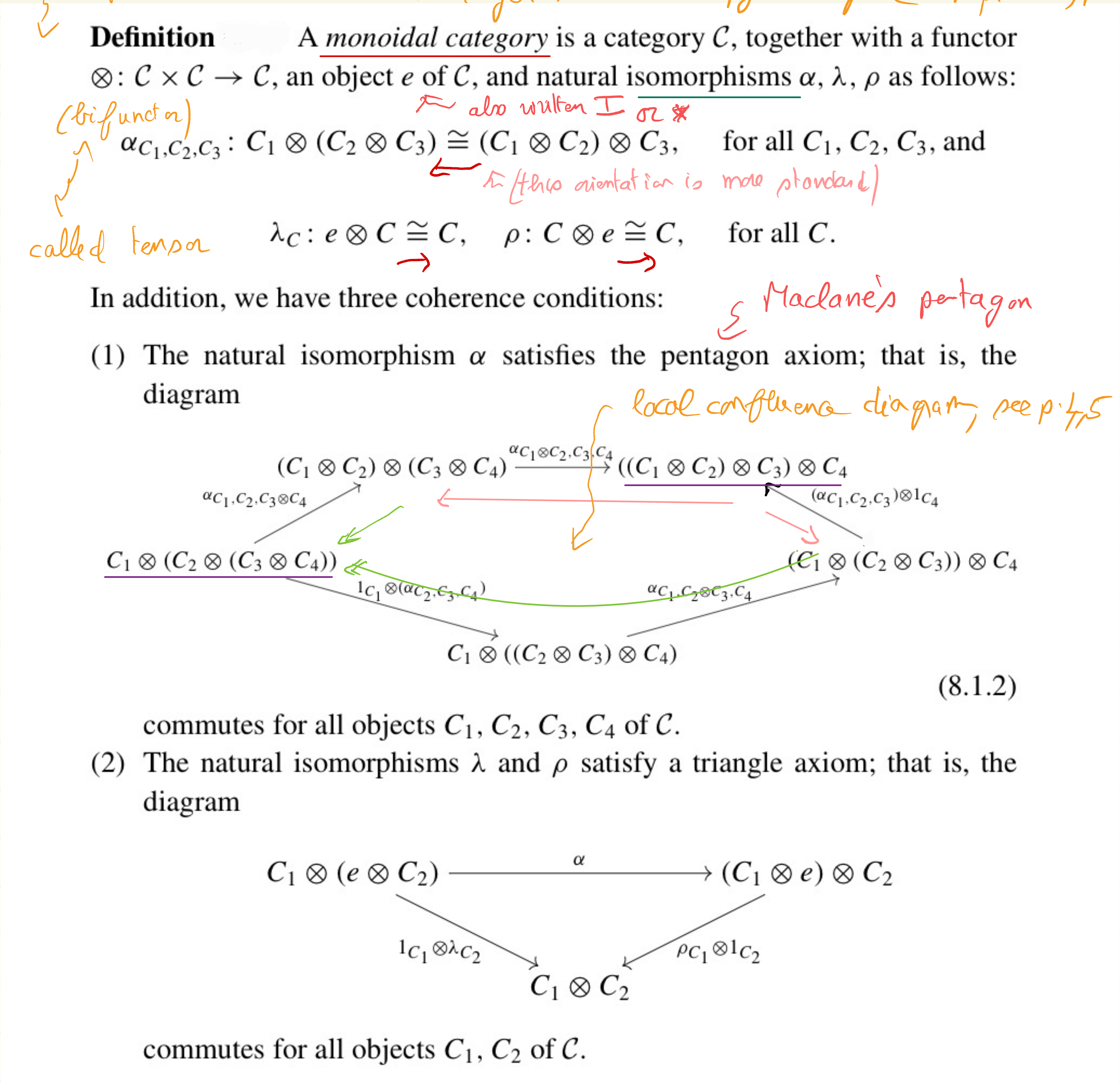
Our key example will be

$C(x,y) =$ simplicial set



Monoidal categories

Birgit Richter's book : From categories to homotopy theory (CUP 2020) p 151



Definition A *monoidal category* is a category $\mathcal{C}$, together with a functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, an object e of $\mathcal{C}$, and natural isomorphisms α, λ, ρ as follows:

(bifunctor) called tensor

also written I or *

$$\alpha_{C_1,C_2,C_3}\colon C_1\otimes(C_2\otimes C_3)\cong(C_1\otimes C_2)\otimes C_3,\quad \text{for all } C_1, C_2, C_3, \text{ and}$$

(this orientation is more standard)

$$\lambda_C\colon e\otimes C\cong C,\quad \rho\colon C\otimes e\cong C,\quad \text{for all } C.$$

In addition, we have three coherence conditions:

Maclane's pentagon

(1) The natural isomorphism α satisfies the pentagon axiom; that is, the diagram

local confluence diagram, see p.4,5

$$\begin{tikzcd}
 & (C_1\otimes C_2)\otimes(C_3\otimes C_4) \arrow[rr, "\alpha_{C_1\otimes C_2,C_3,C_4}"] & & ((C_1\otimes C_2)\otimes C_3)\otimes C_4 \arrow[ll] \arrow[dr] & \\
C_1\otimes(C_2\otimes(C_3\otimes C_4)) \arrow[ur, "\alpha_{C_1,C_2,C_3\otimes C_4}"] \arrow[drr, "1_{C_1}\otimes(\alpha_{C_2,C_3,C_4})"'] & & & & (C_1\otimes(C_2\otimes C_3))\otimes C_4 \arrow[ul, "(\alpha_{C_1,C_2,C_3})\otimes 1_{C_4}"'] \arrow[llll, bend left] \\
 & & C_1\otimes((C_2\otimes C_3)\otimes C_4) \arrow[urr, "\alpha_{C_1,C_2\otimes C_3,C_4}"'] & &
\end{tikzcd}$$

(8.1.2)

commutes for all objects C_1, C_2, C_3, C_4 of $\mathcal{C}$.

(2) The natural isomorphisms λ and ρ satisfy a triangle axiom; that is, the diagram

$$\begin{tikzcd}
C_1\otimes(e\otimes C_2) \arrow[rr, "\alpha"] \arrow[dr, "1_{C_1}\otimes\lambda_{C_2}"'] & & (C_1\otimes e)\otimes C_2 \arrow[dl, "\rho_{C_1}\otimes 1_{C_2}"] \\
 & C_1\otimes C_2 &
\end{tikzcd}$$

commutes for all objects C_1, C_2 of $\mathcal{C}$.

If α, λ, ρ are all identities, then (1), (2) hold for free and we have a strict monoidal category

Prototypical example : the category Vect of vector spaces → whence the notation ⊗!

Coherence

The diagrams of the definition in p. 2 are called coherence diagrams. More generally, we have:

Coherence theorem (stated informally)

All meaningful coherence diagrams commute.

To formalise this, we introduce some syntax:

shorthand for id_A, id_B

Object terms: $A ::= X \mid I \mid A \otimes A$

object variable

Canonical iso terms: $\varphi ::= \alpha \mid \alpha^{-1} \mid \lambda \mid \lambda^{-1} \mid \rho \mid \rho^{-1} \mid A \otimes \varphi \otimes B \mid \varphi \circ \varphi$

Typing rule (we consider linear object terms only (each object variable in A occurs only once in A)

implicit in those rules

$$\frac{}{\alpha : (A \otimes B) \otimes C \underset{\leftarrow}{\to} A \otimes (B \otimes C) : \alpha^{-1}} \qquad \frac{}{\lambda : I \otimes C \underset{\leftarrow}{\to} C : \lambda^{-1}} \qquad \frac{}{\rho : C \otimes I \underset{\leftarrow}{\to} C : \rho^{-1}}$$

$$\frac{\varphi : B_1 \to B_2}{A \otimes \varphi \otimes C : A \otimes B_1 \otimes C \to A \otimes B_2 \otimes C} \qquad \frac{\varphi : B \to C \quad \psi : A \to B}{\psi \circ \varphi : A \to C}$$

It is immediate that if $\varphi : A \to B$ is provable, then

$$X \text{ occurs in } A \iff X \text{ occurs in } B$$

and that if $\{X \mid X \text{ occurs in } A\} = \{X_1, \dots, X_n\}$, then A, B determine functors $A, B : \mathcal{C}^n \to \mathcal{C}$, and φ determines a natural transformation $\varphi : A \to B$

Coherence theorem (formal statement) For A, B linear, all canonical iso terms $\varphi : A \to B$ denote the same natural transformation.

$$\begin{tikzcd} A \arrow[r, bend left, "\varphi_1"] \arrow[r, bend right, "\varphi_2"'] \arrow[r, phantom, "\parallel"] & B \end{tikzcd}$$

Mac Lane's proof of coherence in modern dresses

(= Molière's M. Jourdain!)

Mac Lane was doing rewriting theory without knowing it

- $_1$ Remove all the canonical iso information (and the inverses) from the system of previous page:

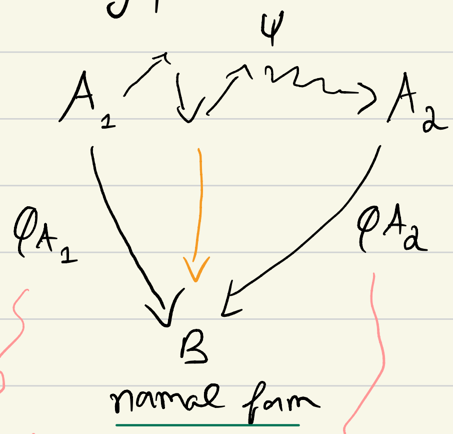
$$\frac{}{(A\otimes B)\otimes C \to A\otimes(B\otimes C)} \qquad \frac{}{I\otimes C\to C} \qquad \frac{}{C\otimes I\to C}$$

$$\frac{B_1\to B_2}{A\otimes B_1\otimes C\to A\otimes B_2\otimes C} \qquad \frac{B\to C \quad A\to B}{A\to C}$$

- $_2$ This rewriting system on (linear) object terms is
  - locally confluent (this is what the diagrams in the definition on p.2 ensure!)
  - strongly normalising (remove I's, push parentheses to the right)

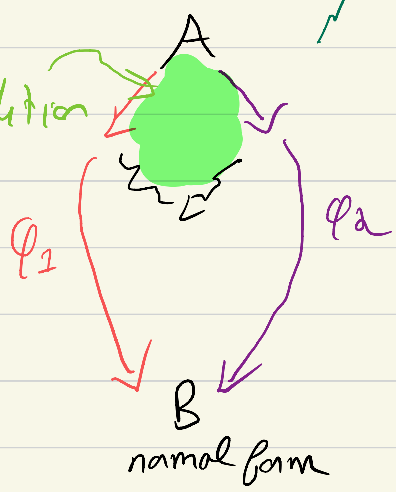
$$\boxed{\varphi_1, \varphi_2 : A \to B \quad (B \text{ in normal form}) \Rightarrow \varphi_1 = \varphi_2}$$

- $_3$ Finally, restore the inverses, and show that $\forall\psi$:



$\Rightarrow$

$$\boxed{\begin{array}{c} \forall\; A_1 \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} A_2 \\ \psi_1 = \varphi_{A_2}^{-1}\circ\varphi A_1 = \psi_2 \end{array}}$$



coherence condition

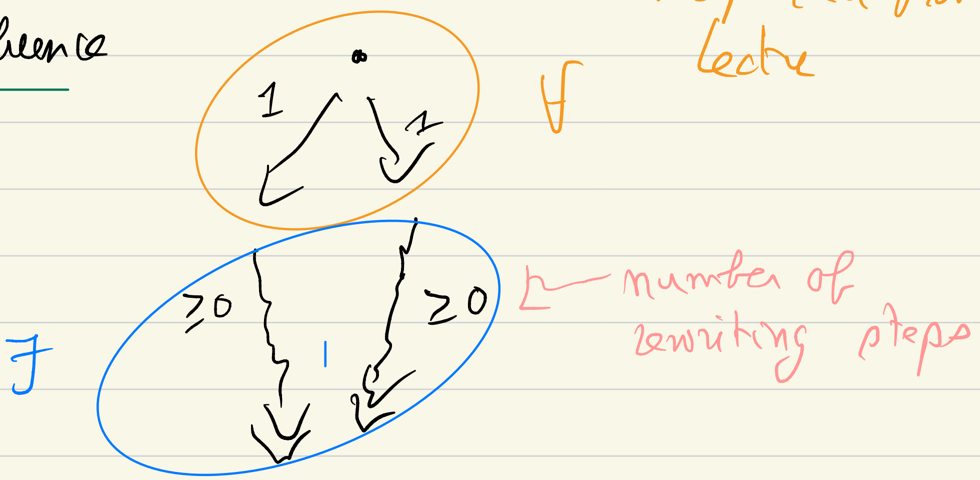
depending only on A_1, A_2 by $_2$

(see next slide for a rewriting reminder)

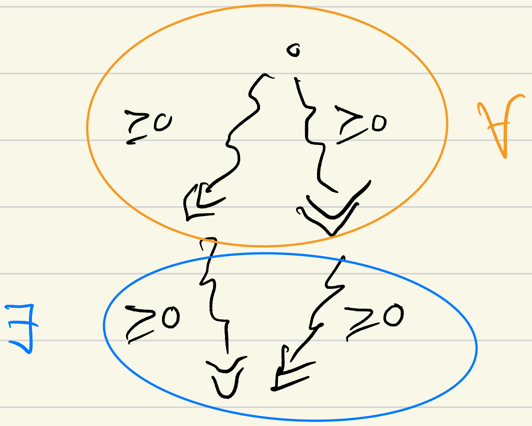
Rewriting theory background

⇑ repeated from lecture

- local confluence



- confluence :

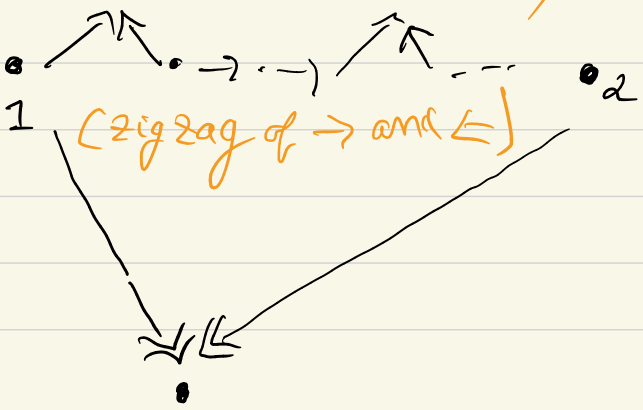


- strong normalisation $\nexists$ $\bullet \to \bullet \to \bullet \cdots \bullet \to \bullet \cdots\cdots$ (infinite)

- normal form : $\bullet \not\to$ $\nexists$

- Local confluence + strong normalisation ⇒ confluence (Newman Lemma)

- If confluence + strong norm., then



⇒ $\bullet_1$ and $\bullet_2$ have the same normal form.

Symmetric monoidal categories

Richter p.158

Definition A *symmetric monoidal category* consists of a monoidal category $(\mathcal{C}, \otimes, e; \alpha, \lambda, \rho)$, together with a natural isomorphism τ in $\mathcal{C}$ with

$$\tau_{C_1,C_2} \colon C_1 \otimes C_2 \cong C_2 \otimes C_1$$

for all objects C_1, C_2 of $\mathcal{C}$, such that τ satisfies the following conditions:

(1) For all objects C_1, C_2 of $\mathcal{C}$, $\tau_{C_2,C_1} \circ \tau_{C_1,C_2} = 1_{C_1 \otimes C_2}$.

(2) For all objects C of $\mathcal{C}$, $\rho_C = \lambda_C \circ \tau_{C,e} \colon C \otimes e \cong C$.

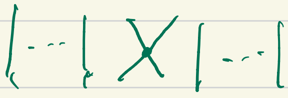
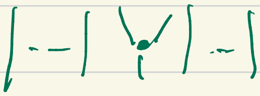
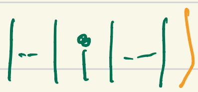
$$\begin{tikzcd} C \otimes e \arrow[r] \arrow[d] & C \\ e \otimes C \arrow[ur] & \end{tikzcd}$$

(3) The natural isomorphism τ is compatible with α in the sense that for all objects C_1, C_2, C_3 of $\mathcal{C}$, the following hexagon diagram commutes:

$$\begin{tikzcd}
 & (C_1 \otimes C_2) \otimes C_3 \arrow[r, "\tau_{C_1 \otimes C_2, C_3}"] & C_3 \otimes (C_1 \otimes C_2) \arrow[dr, "\alpha_{C_3,C_1,C_2}"] & \\
C_1 \otimes (C_2 \otimes C_3) \arrow[ur, "\alpha_{C_1,C_2,C_3}"] \arrow[dr, "1_{C_1} \otimes \tau_{C_2,C_3}"'] & & & (C_3 \otimes C_1) \otimes C_2. \\
 & C_1 \otimes (C_3 \otimes C_2) \arrow[r, "\alpha_{C_1,C_3,C_2}"] & (C_1 \otimes C_3) \otimes C_2 \arrow[ur, "\tau_{C_1,C_3} \otimes 1_{C_2}"'] &
\end{tikzcd}$$

The coherence theorem extends to sym. monoidal categories

ξ In terms of string diagrams, the hero is

 (the little brother of  and )

Exercise Show that if C has finite products (i.e. terminal object + binary products), then it is symmetric monoidal.

⇓

as a presheaf category, has all limits.

we have that $\boxed{\text{Cat, Sset, ... are symmetric monoidal}}$

Proof of coherence for symmetric monoidal categories

The original proof by Mac Lane goes in two steps:

- it "gets rid of parentheses" = monoidal
- and then it amounts to contemplating the Coxeter presentation of the symmetric group.

Exercise: If we extend the syntax of p.3 with

$\tau_{A,B} : A\otimes B \rightleftarrows B\otimes A : \tau_{B,A}$, and if we decide to orient any τ step as follows:

$$C\otimes(A\otimes B)\otimes D \longrightarrow C\otimes(B\otimes A)\otimes D$$

if $\max\{i \mid X_i \in A\otimes B\} \in \{j \mid X_j \in B\}$, and if one orients α-steps as previously, then the resulting rewriting system is convergent (i.e. terminating and confluent), and the proof of coherence can be carried out "in one step" as in p.4.

Monoidal functors

Richter pp 161-162

Definition Let $(\mathcal{C}, \otimes, e_{\mathcal{C}})$ and $(\mathcal{D}, \Box, e_{\mathcal{D}})$ be two monoidal categories.

- A functor $F\colon (\mathcal{C}, \otimes, e_{\mathcal{C}}) \to (\mathcal{D}, \Box, e_{\mathcal{D}})$ is a *lax monoidal functor* if for each pair of objects C_1, C_2 of $\mathcal{C}$, there is a morphism

$$\varphi_{C_1,C_2}\colon F(C_1)\Box F(C_2) \to F(C_1 \otimes C_2)$$

in $\mathcal{D}$, which is natural in C_1 and C_2, and there is a morphism

$$\eta\colon e_{\mathcal{D}} \to F(e_{\mathcal{C}})$$

in $\mathcal{D}$, satisfying

$$\begin{tikzcd}
F(C_1)\Box(F(C_2)\Box F(C_3)) \arrow[r, "\alpha_{\mathcal{D}}"] \arrow[d, "1_{F(C_1)}\Box\varphi_{C_2,C_3}"'] & (F(C_1)\Box F(C_2))\Box F(C_3) \arrow[d, "\varphi_{C_1,C_2}\Box 1_{F(C_3)}"] \\
F(C_1)\Box(F(C_2 \otimes C_3)) \arrow[d, "\varphi_{C_1,C_2\otimes C_3}"'] & (F(C_1 \otimes C_2))\Box F(C_3) \arrow[d, "\varphi_{C_1\otimes C_2,C_3}"] \\
F(C_1 \otimes (C_2 \otimes C_3)) \arrow[r, "F(\alpha_{\mathcal{C}})"] & F((C_1 \otimes C_2) \otimes C_3),
\end{tikzcd}$$

$$\begin{tikzcd}
F(C)\Box e' \arrow[r, "\rho^{\mathcal{D}}_{F(C)}"] \arrow[d, "1_{F(C)}\Box\eta"'] & F(C) \\
F(C)\Box F(e) \arrow[r, "\varphi_{C,e}"] & F(C \otimes e), \arrow[u, "F(\rho^{\mathcal{C}}_{C})"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
e'\Box F(C) \arrow[r, "\lambda^{\mathcal{D}}_{F(C)}"] \arrow[d, "\eta\Box 1_{F(C)}"'] & F(C) \\
F(e)\Box F(C) \arrow[r, "\varphi_{e,C}"] & F(e \otimes C). \arrow[u, "F(\lambda^{\mathcal{C}}_{C})"']
\end{tikzcd}$$

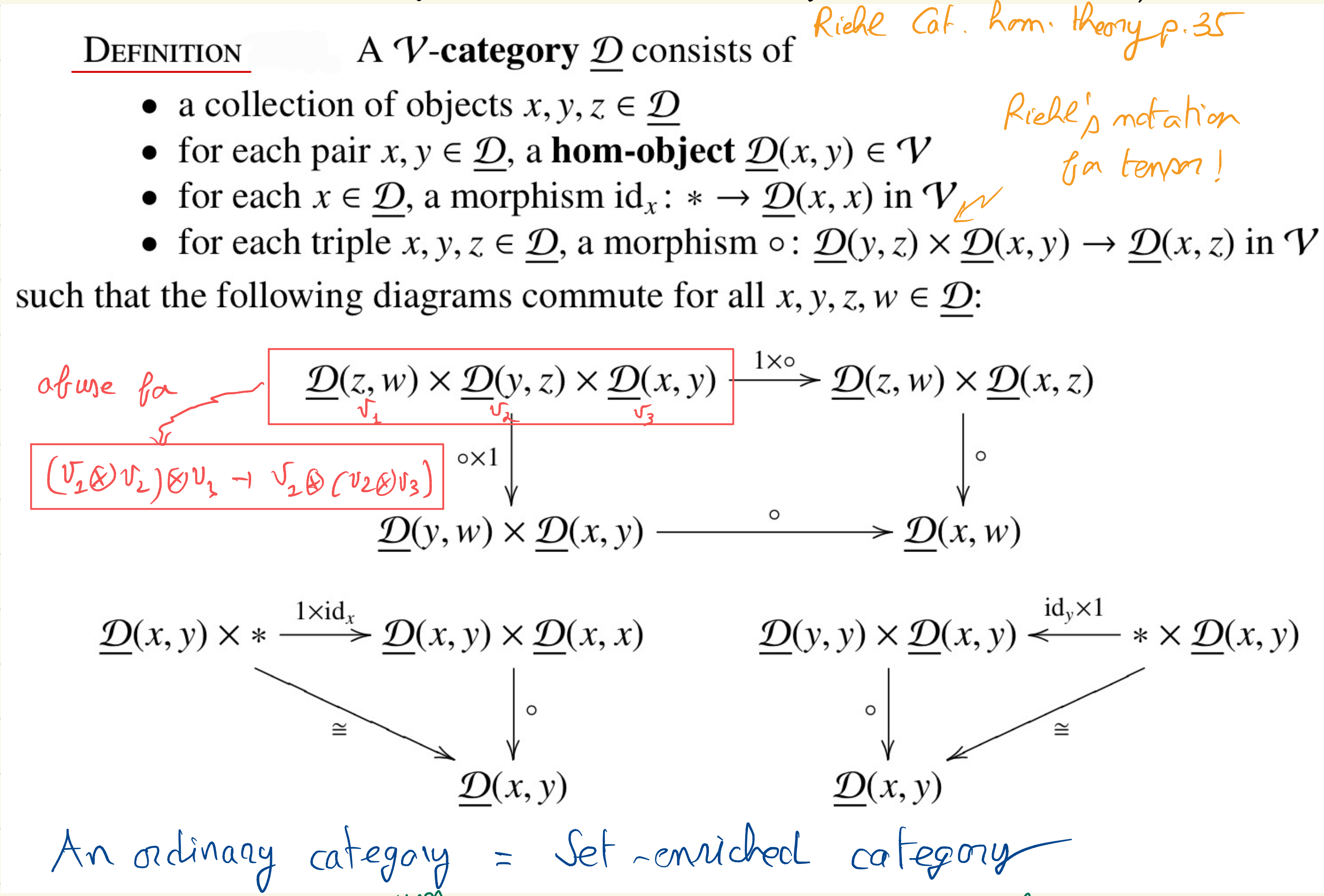
- A lax monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ is *strong monoidal* if the structure morphisms φ and η are natural isomorphisms.
- A lax monoidal functor F is *strictly monoidal* if φ and η are identities.

Example If V is monoidal, then $V(*,-)\colon V \to Set$ is monoidal (with cartesian structure of Set as mon. structure)

$$\underset{f}{V(*,v)} \times \underset{g}{V(*,w)} \underset{\mapsto}{\longrightarrow} \underset{f\otimes g}{V(*\otimes *, v\otimes w)} \longrightarrow \underset{\cong}{V(*, v\otimes w)}$$

post composition with $\lambda^{-1} = \rho^{-1}$

$\mathcal{V}$- categories ($\mathcal{V}$ sym. monoidal)



Riehl Cat. hom. theory p.35

Definition A **$\mathcal{V}$-category** $\underline{\mathcal{D}}$ consists of

- a collection of objects $x, y, z \in \underline{\mathcal{D}}$
- for each pair $x, y \in \underline{\mathcal{D}}$, a **hom-object** $\underline{\mathcal{D}}(x,y) \in \mathcal{V}$
- for each $x \in \underline{\mathcal{D}}$, a morphism $\mathrm{id}_x : * \to \underline{\mathcal{D}}(x,x)$ in $\mathcal{V}$
- for each triple $x, y, z \in \underline{\mathcal{D}}$, a morphism $\circ : \underline{\mathcal{D}}(y,z) \times \underline{\mathcal{D}}(x,y) \to \underline{\mathcal{D}}(x,z)$ in $\mathcal{V}$

Riehl's notation for tensor!

such that the following diagrams commute for all $x, y, z, w \in \underline{\mathcal{D}}$:

abuse for

$(v_1 \otimes v_2) \otimes v_3 \to v_1 \otimes (v_2 \otimes v_3)$

$$\begin{tikzcd}
\underbrace{\underline{\mathcal{D}}(z,w)}_{v_1} \times \underbrace{\underline{\mathcal{D}}(y,z)}_{v_2} \times \underbrace{\underline{\mathcal{D}}(x,y)}_{v_3} \arrow[r, "1\times\circ"] \arrow[d, "\circ\times 1"'] & \underline{\mathcal{D}}(z,w) \times \underline{\mathcal{D}}(x,z) \arrow[d, "\circ"] \\
\underline{\mathcal{D}}(y,w) \times \underline{\mathcal{D}}(x,y) \arrow[r, "\circ"] & \underline{\mathcal{D}}(x,w)
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathcal{D}}(x,y) \times * \arrow[r, "1\times \mathrm{id}_x"] \arrow[dr, "\cong"'] & \underline{\mathcal{D}}(x,y) \times \underline{\mathcal{D}}(x,x) \arrow[d, "\circ"] \\
& \underline{\mathcal{D}}(x,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathcal{D}}(y,y) \times \underline{\mathcal{D}}(x,y) \arrow[d, "\circ"'] & * \times \underline{\mathcal{D}}(x,y) \arrow[l, "\mathrm{id}_y \times 1"'] \arrow[dl, "\cong"] \\
\underline{\mathcal{D}}(x,y) &
\end{tikzcd}$$

An ordinary category = Set-enriched category

Definition A sym. monoidal closed category is a sym. monoidal category in which $-\otimes v : \mathcal{V} \to \mathcal{V}$ admits a right adjoint $\underline{\mathcal{V}}(v,-)$: $\mathcal{V}(u \otimes v, w) \simeq \mathcal{V}(u, \underline{\mathcal{V}}(v,w))$

Proposition If $\mathcal{V}$ is mon. closed, then $\mathcal{V}$ is enriched over itself (and then we write $\underline{\mathcal{V}}$).

Proof Easy: $Ob(\underline{\mathcal{V}}) = Ob\,\mathcal{V}$ $\quad \underline{\mathcal{V}}(x,y) = \underline{\mathcal{V}}(x,y)$

Cat-enriched categories (= 2-categories)

Let us spell out what a Cat-enriched category $\underline{C}$ is

- C_0 (a Ob C): a collection of objects or 0-cells
- $C_1 = \mathrm{Ob}\,\underline{C}(x,y)$: 1-morphisms
- $C_2 = \underline{C}(x,y)(f,g)$: 2-morphisms

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

- We have identities at two levels:
  - $* \to \underline{C}(x,x)$ gives $x \xrightarrow{id_x} x$
  - the category structure on $\underline{C}(x,y)$ gives $$\begin{tikzcd} x \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "f"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, "id_f"] \end{tikzcd}$$
- The object part of $\underline{C}(y,z)\times\underline{C}(x,y)\to\underline{C}(x,z)$ gives composition of 1-morphisms: $g\circ_0 f$ $x \xrightarrow{f} y \xrightarrow{g} z$

along a 0-cell

yielding an ordinary category structure $(\mathrm{Ob}=C_0,\ \mathrm{Mor}=C_2)$

- We have composition of 2-morphisms at two levels:
  - composition in $\underline{C}(x,y)$

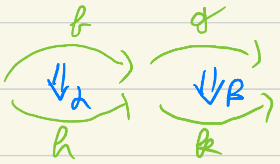
(vertical) $\beta\circ_1\alpha$ ← along a 1-morphism

$$\begin{tikzcd}[row sep=large] x \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{name=G, near start}] \arrow[r, bend right=60, "h"'{name=H}] & y \arrow[Rightarrow, from=F, to=G, "\alpha"] \arrow[Rightarrow, from=G, to=H, "\beta"] \end{tikzcd}$$

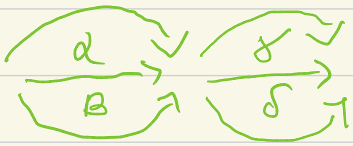
  - Functor part of $\underline{C}(y,z)\times\underline{C}(x,y)\to\underline{C}(x,z)$

(horizontal) $\beta\circ_0\alpha$

along a 0-cell



- Functoriality of $\underline{C}(y,z)\times\underline{C}(x,y)\to\underline{C}(x,z)$ ensures non-ambiguity in



Interchange law: $\boxed{(\delta\circ_1\gamma)\circ_0(\beta\circ_1\alpha) = (\delta\circ_0\beta)\circ_1(\gamma\circ_0\alpha)}$

A few words on 2-categories

The prototypical 2-category is the 2-category $\underline{\mathrm{Cat}}$ of

- categories as 0-cells
- functors as 1-morphisms
- natural transformations

(This picture extends to the enriched setting, see exercises p.11)

The notions of adjunction, equivalence of categories, monad, comonad,

all born in $\underline{\mathrm{Cat}}$, make sense in any 2-category

(See second exercise on p.13 for the instantiation to the 2-category $\underline{\mathcal{V}\text{-}\mathrm{Cat}}$ of enriched categories)

Underlying category of a V-category

or just C

If $\underline{C}$ is a V-category, we define an ordinary category C_0 by

- Ob C_0 = Ob $\underline{C}$ • $C_0(x,y) = V(*, \underline{C}(x,y))$
- id_x in C_0 = id_x in $\underline{C}$
- Composition is given by

Riehl CHT p.41

$$\begin{tikzcd}
C_0(y,z) \times C_0(x,y) \arrow[rr, dashed] \arrow[d, equal] & & C_0(x,z) \arrow[d, equal] \\
\mathcal{V}(*, \underline{C}(y,z)) \times \mathcal{V}(*, \underline{C}(x,y)) \arrow[r, "{V(*,-)\ \text{monoidal}}"'] & \mathcal{V}(*, \underline{C}(y,z) \times \underline{C}(x,y)) \arrow[r, "\text{precomposing with } \circ"'] & \mathcal{V}(*, \underline{C}(x,z))
\end{tikzcd}$$

Exercise Show that if V is mon. closed, then the underlying category of $\underline{V}$ is V.

Exercise Show that the underlying category of a 2-category is the category formed by its 0-cells and 1-morphisms. ← In particular Cat is the underlying category of $\underline{Cat}$

The underlying category functor is just one example of change of base: for any V, U, and F: V → U lax-monoidal, and for any V-category $\underline{D}_V$, we obtain a U-category $\underline{D}_U$ as follows:

- Ob $\underline{D}_U$ = Ob $\underline{D}_V$ • $\underline{D}_U(x,y) = F\underline{D}_V(x,y)$
- Composition + identities:

$$F\underline{\mathcal{D}}_{\mathcal{V}}(y,z) \otimes F\underline{\mathcal{D}}_{\mathcal{V}}(x,y) \xrightarrow{lax} F(\underline{\mathcal{D}}_{\mathcal{V}}(y,z) \times \underline{\mathcal{D}}_{\mathcal{V}}(x,y)) \xrightarrow{F(\circ)} F\underline{\mathcal{D}}_{\mathcal{V}}(x,z) = \underline{\mathcal{D}}_{\mathcal{U}}(x,z)$$

$$\mathbb{1} \xrightarrow{lax} F(*) \xrightarrow{F(\mathrm{id}_x)} F\underline{\mathcal{D}}_{\mathcal{V}}(x,x) = \underline{\mathcal{D}}_{\mathcal{U}}(x,x)$$

Exercise Prove that $\underline{D}_U$ is indeed a V-category.

Unenrichment = change of base induced by $V(*,-): V \to Set$

$\underline{\mathcal{V}}$-functors and $\underline{\mathcal{V}}$-natural transformations

Riehl CHT p.44

Definition A $\mathcal{V}$-**functor** $F\colon \underline{C} \to \underline{D}$ between $\mathcal{V}$-categories is given by an object map $\underline{C} \ni x \mapsto Fx \in \underline{D}$ together with morphisms

$$\underline{C}(x,y) \xrightarrow{F_{x,y}} \underline{D}(Fx,Fy)$$

in $\mathcal{V}$ for each $x, y \in \underline{C}$ such that the following diagrams commute for all $x, y, z \in \underline{C}$:

$$\begin{tikzcd}
\underline{C}(y,z)\times\underline{C}(x,y) \arrow[r, "\circ"] \arrow[d, "F_{y,z}\times F_{x,y}"'] & \underline{C}(x,z) \arrow[d, "F_{x,z}"] \\
\underline{D}(Fy,Fz)\times\underline{D}(Fx,Fy) \arrow[r, "\circ"] & \underline{D}(Fx,Fz)
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{Fx}"'] & \underline{C}(x,x) \arrow[d, "F_{x,x}"] \\
& \underline{D}(Fx,Fx)
\end{tikzcd}$$

p 45

Definition A $\mathcal{V}$-**natural transformation** $\alpha\colon F \Rightarrow G$ between a pair of $\mathcal{V}$-functors $F, G\colon \underline{C} \rightrightarrows \underline{D}$ consists of a morphism $\alpha_x\colon * \to \underline{D}(Fx,Gx)$ in $\mathcal{V}$ for each $x \in \underline{C}$ such that for all $x, y \in \underline{C}$ the following diagram commutes:

$$\begin{tikzcd}
\overset{f}{\underline{C}(x,y)} \arrow[r, "F_{x,y}"] \arrow[d, "G_{x,y}"'] & \overset{Ff}{\underline{D}(Fx,Fy)} \arrow[d, "(\alpha_y)_*"] \\
\underset{Gf}{\underline{D}(Gx,Gy)} \arrow[r, "(\alpha_x)^*"'] & \underline{D}(Fx,Gy)
\end{tikzcd}$$

α o Ff
Gf o α

where $-_*$ is defined by

$(g: * \to \underline{D}(y,z))$

$$g_*\colon \underline{D}(x,y) \cong * \times \underline{D}(x,y) \xrightarrow{g\times 1} \underline{D}(y,z)\times\underline{D}(x,y) \xrightarrow{\circ} \underline{D}(x,z).$$

Exercise Show that $\underline{\mathcal{V}}$-categories, $\underline{\mathcal{V}}$-functors, and $\underline{\mathcal{V}}$-natural transformations form a 2-category $\underline{\mathcal{V}\text{-Cat}}$.

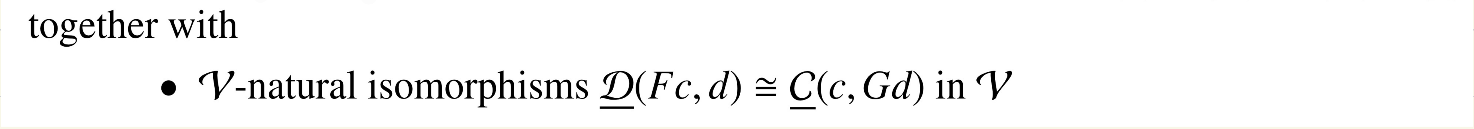
Exercise Show that change of base extends to a 2-functor (= Cat-enriched functor). Instantiate this to unenrichment.

Exercise Show that when $\underline{D} = \underline{V}$, then a $\underline{V}$-natural transformation $\alpha : F \to G : \underline{C} \to \underline{V}$ consists of morphisms $\alpha_x \in V(Fx, Gx)$ s.t.

$$\begin{tikzcd}
\underline{C}(x,y) \otimes Fx \arrow[r, "\text{F enriched functor}"'] \arrow[d, "1 \otimes \alpha_x"'] & \underline{V}(Fx, Fy) \otimes Fx \arrow[r, "\text{counit (or eval)}"'] & Fy \arrow[d, "\alpha_y"] \\
\underline{C}(x,y) \otimes Gx \arrow[r, "\text{G enriched functor}"'] & \underline{V}(Gx, Gy) \otimes Gx \arrow[r, "\text{eval}"'] & Gy
\end{tikzcd}$$

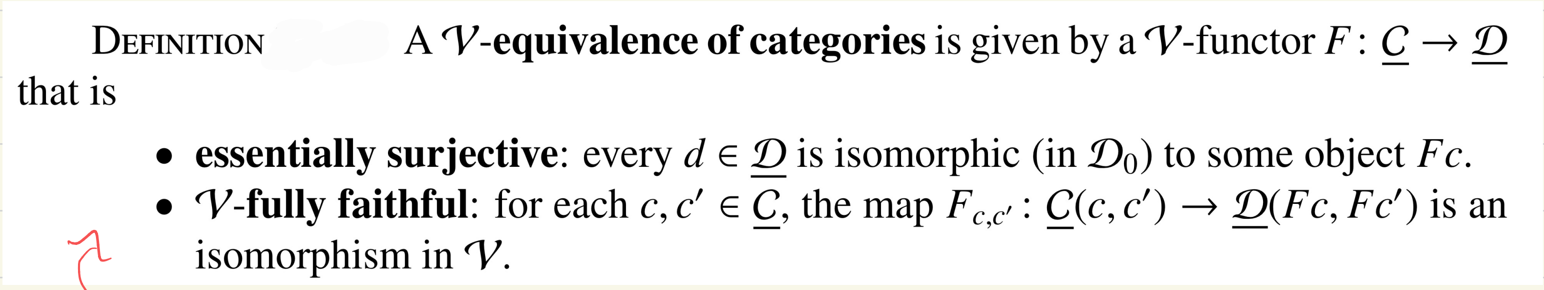
Exercise Show that V-adjunctions and V-equivalences, i.e. adjunctions and equivalences in $\underline{V\text{-Cat}}$, cf. p. 10 instantiate as expected:

CHT p. 47



DEFINITION A $\mathcal{V}$-**adjunction** consists of $\mathcal{V}$-functors $F : \underline{C} \to \underline{D}$, $G : \underline{D} \to \underline{C}$ together with

- $\mathcal{V}$-natural isomorphisms $\underline{D}(Fc, d) \cong \underline{C}(c, Gd)$ in $\mathcal{V}$



DEFINITION A $\mathcal{V}$-**equivalence of categories** is given by a $\mathcal{V}$-functor $F : \underline{C} \to \underline{D}$ that is

- **essentially surjective**: every $d \in \underline{D}$ is isomorphic (in $\mathcal{D}_0$) to some object Fc.
- $\mathcal{V}$-**fully faithful**: for each $c, c' \in \underline{C}$, the map $F_{c,c'} : \underline{C}(c,c') \to \underline{D}(Fc, Fc')$ is an isomorphism in $\mathcal{V}$.

Hint. This involves proving an enriched version of the proposition on p. 4 of lecture 0.

Exercises for the road

← used in lecture 6

Exercise^1 Show that if V has copowers $X.I = \overbrace{I+I\cdots+I}^{\#(X)\text{ times}}$ (cf. lecture 1 p.5) that are preserved by $\otimes$, i.e. $(X.I)\otimes v \simeq X.(I\otimes v)$ $[\simeq X.v]$, then the unenrichment functor $V\text{-cat} \to \mathrm{Cat}$ has a left adjoint (free $\underline{V}$-category functor).

Hints
- For C, take $\mathrm{Ob}(FC) = \mathrm{ob}\, C$, $FC(x,y) = \underline{C(x,y).I}$
- composition by reindexing copies of I.

$$\begin{tikzcd}
\mathrm{Free}_{\underline{V}}(C) & C \arrow[l, mapsto] \\
V\text{-cat} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend right] \\
\underline{m} \arrow[r, mapsto] & M
\end{tikzcd}$$

Exercise^2 Consider a V-adjunction
$$\begin{tikzcd}
\underline{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \underline{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$
Show that G preserves arbitrary limits of ordinary functors $E \xrightarrow{H} D$ in the strong sense of a certain canonical map (in V) being iso.

Variations on Hom-functors

Given a $\mathcal{V}$ enriched category $\underline{\mathsf{M}}$, we have three hom functors

↖ mon. closed

- An enriched functor $\boxed{\underline{\mathsf{M}}[m,-] : \underline{\mathsf{M}} \to \underline{\mathcal{V}},}$ defined by transposition of composition:

$$\underline{\mathsf{M}}[m_1, m_2] \to \underline{\mathcal{V}}[\underline{\mathsf{M}}[m, m_1], \underline{\mathsf{M}}[m, m_2]]$$

- The unenrichment $\boxed{\underline{\mathsf{M}}[m,-] : \mathsf{M} \to \mathcal{V}}$ of $\underline{\mathsf{M}}[m,-]$, mapping $f : I \to \underline{\mathsf{M}}[m_1, m_2]$ to

$$\underline{\mathsf{M}}[m, m_1] \to I \otimes \underline{\mathsf{M}}[m, m_1] \to \underline{\mathsf{M}}[m_1, m_2] \otimes \underline{\mathsf{M}}[m, m_1] \to \underline{\mathsf{M}}[m, m_2]$$

- The representable functor $\boxed{\mathsf{M}[m,-] : \mathsf{M} \to \textbf{\textit{Set}}}$ in the underlying category M of $\underline{\mathsf{M}}$. We have $\mathsf{M}[m,-] = \mathcal{V}[I,-] \circ \underline{\mathsf{M}}[m,-]$.

Lemma (CHT 3.5.12) Let $\underline{C}$ be a $\underline{V}$-category. TFAE

(i) $x, y \in \underline{C}$ are isomorphic (i.e. isomorphic in the underlying cat. C)

(ii) the representable functors $C(x,-), C(y,-) : C \rightrightarrows \mathbf{Set}$ are naturally isomorphic

(iii) the unenriched representable functors $\underline{C}(x,-), \underline{C}(y,-) : C \rightrightarrows \mathcal{V}$ are naturally isomorphic

(iv) the representable $\mathcal{V}$-functors $\underline{C}(x,-), \underline{C}(y,-) : \underline{C} \rightrightarrows \underline{\mathcal{V}}$ are $\mathcal{V}$-isomorphic

Proof (i)⇒(iv) If $g : I \to \underline{C}(y,x)$ is iso, then ↙ cf. p.10 $g^* : \underline{C}(x,z) \to \underline{C}(y,z)$ is iso, yielding (cf. exercise p.17) a $\underline{V}$-iso $\underline{C}(x,-) \to \underline{C}(y,-)$.

(iv) ⇒ (iii) is by unenrichment

(iii) ⇒ (ii) uses the factorisation $C(x,-) = V(I,-) \circ \underline{C}(x,-)$

(ii) ⇒ (i) follows from the usual Yoneda lemma

(This lemma is a heating up for enriched Yoneda lemma, coming...)

Tensored and cotensored $\underline{V}$-categories

CHT 3.7.2

closed $\mathcal{V}$

DEFINITION A $\mathcal{V}$-category $\underline{\mathcal{M}}$ is **tensored** if for each $v \in \mathcal{V}$ and $m \in \mathcal{M}$ there is an object $v \otimes m \in \mathcal{M}$ together with isomorphisms

or copowered

$$\underline{\mathcal{M}}(v \otimes m, n) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{M}}(m,n)) \qquad \forall \quad v \in \mathcal{V} \quad m, n \in \mathcal{M}.$$

i.e. $\forall m$

$$\boxed{\begin{tikzcd} (-\otimes m) : \underline{V} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \underline{M} : \underline{M}(m,-) \arrow[l, bend left] \end{tikzcd}}$$

$\underline{V}$-adjunction

DEFINITION 3.7.3. A $\mathcal{V}$-category $\underline{\mathcal{M}}$ is **cotensored** if for each $v \in \mathcal{V}$ and $n \in \mathcal{M}$ there is an object $n^v \in \mathcal{M}$ together with isomorphisms

or powered

$$\underline{\mathcal{M}}(m, n^v) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{M}}(m,n)) \qquad \forall \quad v \in \mathcal{V} \quad m, n \in \mathcal{M}.$$

i.e. $\forall n$

$$\boxed{\begin{tikzcd} \underline{M}(-,n) : \underline{M}^{op} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \underline{V} : n^{-} \arrow[l, bend left] \end{tikzcd}}$$

$\underline{V}$-adjunction

If $\underline{M}$ is tensored and cotensored, then

other notation $\{v,n\}$

$$\underline{\mathcal{M}}(v \otimes m, n) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{M}}(m,n)) \cong \underline{\mathcal{M}}(m, n^v).$$

and we get a third adjunction

$\forall v$

$$\boxed{\begin{tikzcd} (v\otimes -) : \underline{M} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \underline{M} : -^{v} \arrow[l, bend left] \end{tikzcd}}$$

Hence $\otimes$ preserves colimits in both variables

When $V = Set$, tensor (resp. cotensor) = (#v)-fold coproduct (resp. product)

Exercise Show that

$$\boxed{(-\otimes=) : \underline{V} \otimes \underline{M} \to \underline{M}, \; (=^{-}) : \underline{V}^{op} \otimes \underline{M} \to \underline{M}, \; \underline{M}(-,=) : \underline{M}^{op} \otimes \underline{M} \to \underline{V}}$$

are $\underline{V}$-bifunctors. (On the way, define $\underline{C} \otimes \underline{D}$, and $\underline{C}^{op}$)

Here, the symmetric structure of V is used!

Exercise Show that if $\underline{M}$ is tensored, then

$I \otimes m \cong m \qquad v \otimes (w \otimes m) \cong (v \otimes w) \otimes m$ (Hint: use ex. on p. 14)

More on this p. 18!

Two-variable adjunctions

Shulman Homotopy limits and colimits and enriched homotopy theory 14.8

Definition A $\mathscr{V}$*-adjunction of two variables* $(\circledast, \hom_\ell, \hom_r) : \underline{\mathscr{M}} \otimes \underline{\mathscr{N}} \to \underline{\mathscr{P}}$ between $\mathscr{V}$-categories consists of $\mathscr{V}$-bifunctors

$$\begin{aligned} \circledast &: \underline{\mathscr{M}} \otimes \underline{\mathscr{N}} \to \underline{\mathscr{P}} \\ \hom_\ell &: \underline{\mathscr{M}}^{op} \otimes \underline{\mathscr{P}} \to \underline{\mathscr{N}} \\ \hom_r &: \underline{\mathscr{N}}^{op} \otimes \underline{\mathscr{P}} \to \underline{\mathscr{M}} \end{aligned}$$

together with $\mathscr{V}$-natural isomorphisms between hom-objects in $\mathscr{V}$:

$$\underline{\mathscr{P}}(M \circledast N, P) \cong \underline{\mathscr{M}}(N, \hom_\ell(M, P)) \cong \underline{\mathscr{N}}(M, \hom_r(N, P)).$$

By the first excercise on p. 16,

a tensored an cotensored $\mathcal{V}$-category exhibits a two-variable adjunction:

$$\underline{M}(v \otimes m, n) \cong \underline{V}(v, \underline{M}(m, n)) \cong \underline{M}(m, n^v).$$

$$\left(\text{dictionary} \quad \begin{array}{llll} v \leftrightarrow M & \underline{V} \leftrightarrow \underline{\mathscr{M}} & \underline{\mathscr{M}}(m,n) & = \hom_r(m,n) \\ m \leftrightarrow N & \underline{\mathscr{M}} \leftrightarrow \underline{\mathscr{N}} & n^v & = \hom_\ell(v,n) \\ n \leftrightarrow P & \underline{\mathscr{M}} \leftrightarrow \underline{P} & v \otimes m & = v \circledast m \end{array}\right)$$

Categorical actions

We arrived to tensored $\mathcal{V}$-categories through ($\mathcal{V}$ fixed)

- a $\mathcal{V}$-category $\underline{\mathcal{M}}$
- a left adjoint $(-\otimes m) \dashv \underline{\mathcal{M}}(m,-)$

We can instead start from

categorical action

- an ordinary category $\mathcal{M}$
- a bifunctor $\otimes : \mathcal{V} \times \mathcal{M} \to \mathcal{M}$ satisfying

$$I \otimes m \xrightarrow{\hat{\lambda}} m \qquad (v \otimes w) \otimes m \xrightarrow{\hat{\alpha}} v \otimes (w \otimes m)$$

via isomorphisms compatible with α, λ, ρ

- a right adjoint for $-\otimes m$

$\mathcal{C}$ tensored is then defined by requiring $v \otimes -$ to have a right adjoint

Exercise Show that action + right adjoint $\Leftrightarrow$ enriched + left adjoint. On the way, make the required compatibilities explicit.

(Hints • Prolongate the second exercise p.16 by checking the compatibilities

• Use the right adjoint G_m of $-\otimes m$ to define $\underline{\mathcal{M}}(m,n)$ in such a way that its unenrichment is $\mathcal{M}$.)

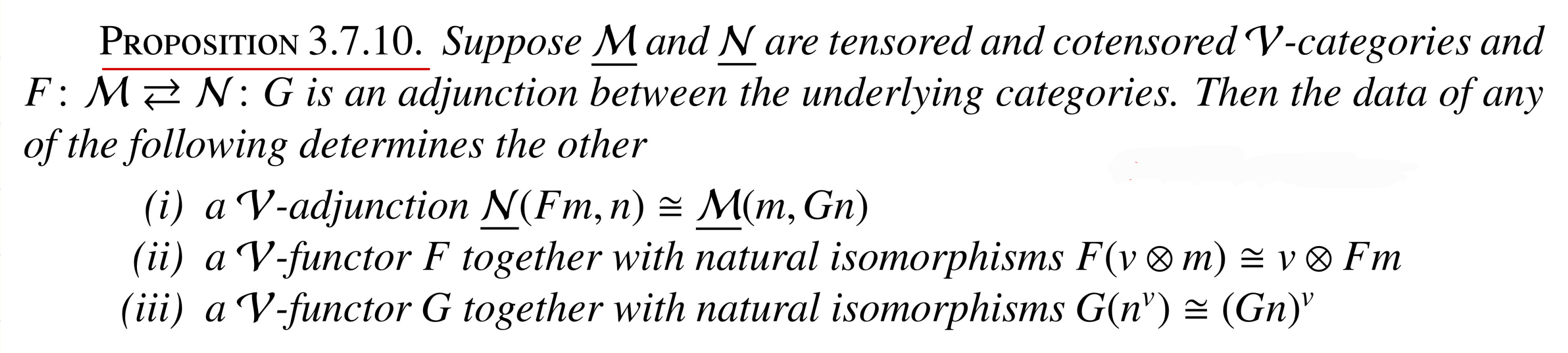
$$\begin{tikzcd}
(v \otimes I) \otimes m \arrow[r, "\hat{\alpha}"] \arrow[d, "\lambda \otimes m"'] & v \otimes (I \otimes m) \arrow[dl, "v \otimes \hat{\lambda}"] \\
v \otimes m &
\end{tikzcd}$$

(Similarly $(I \otimes v) \otimes m \cdots$)

$$\begin{tikzcd}
((v_1 \otimes v_2) \otimes v_3) \otimes m \arrow[r, "\hat{\alpha}"] \arrow[d, "\alpha \otimes m"'] & (v_1 \otimes v_2) \otimes (v_3 \otimes m) \arrow[dr, "\hat{\alpha}"] & \\
 & & v_2 \otimes (v_a \otimes (v_3 \otimes m)) \\
(v_1 \otimes (v_2 \otimes v_3)) \otimes m \arrow[r, "\hat{\alpha}"] & v_1 \otimes ((v_2 \otimes v_3) \otimes m) \arrow[ur, "v_1 \otimes \hat{\alpha}"'] &
\end{tikzcd}$$

$\mathcal{V}$-adjunctions between tensored (or cotensored) categories

CHT



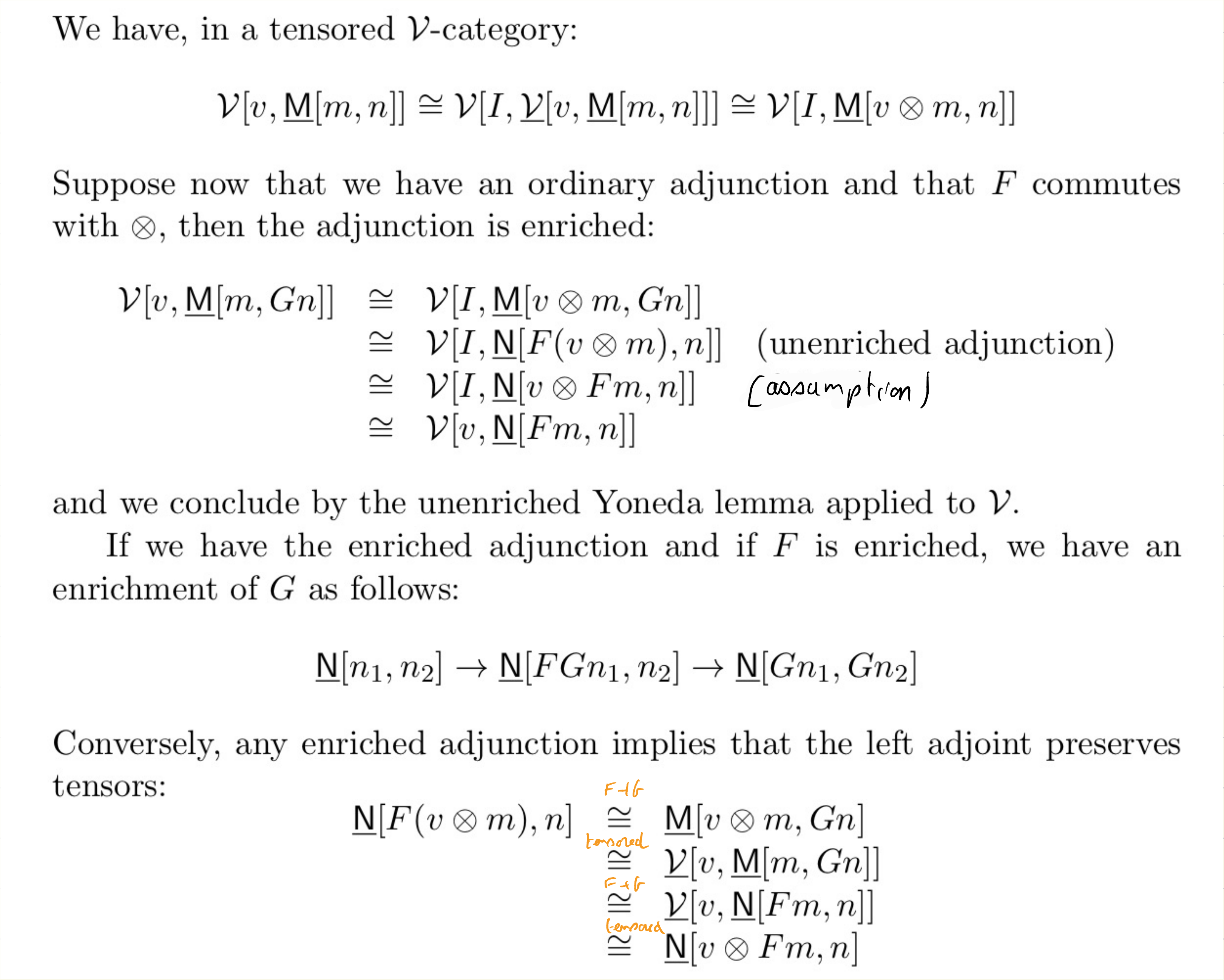
Proposition 3.7.10. *Suppose $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are tensored and cotensored $\mathcal{V}$-categories and $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon G$ is an adjunction between the underlying categories. Then the data of any of the following determines the other*

(i) *a $\mathcal{V}$-adjunction $\underline{\mathcal{N}}(Fm,n) \cong \underline{\mathcal{M}}(m,Gn)$*

(ii) *a $\mathcal{V}$-functor F together with natural isomorphisms $F(v\otimes m) \cong v\otimes Fm$*

(iii) *a $\mathcal{V}$-functor G together with natural isomorphisms $G(n^v) \cong (Gn)^v$*

Proof (We make use twice of the lemma p.15)



We have, in a tensored $\mathcal{V}$-category:

$$\mathcal{V}[v,\underline{\mathsf{M}}[m,n]] \cong \mathcal{V}[I,\underline{\mathcal{V}}[v,\underline{\mathsf{M}}[m,n]]] \cong \mathcal{V}[I,\underline{\mathsf{M}}[v\otimes m,n]]$$

Suppose now that we have an ordinary adjunction and that F commutes with $\otimes$, then the adjunction is enriched:

$$\begin{aligned}
\mathcal{V}[v,\underline{\mathsf{M}}[m,Gn]] &\cong \mathcal{V}[I,\underline{\mathsf{M}}[v\otimes m,Gn]] \\
&\cong \mathcal{V}[I,\underline{\mathsf{N}}[F(v\otimes m),n]] \quad \text{(unenriched adjunction)} \\
&\cong \mathcal{V}[I,\underline{\mathsf{N}}[v\otimes Fm,n]] \quad \text{(assumption)} \\
&\cong \mathcal{V}[v,\underline{\mathsf{N}}[Fm,n]]
\end{aligned}$$

and we conclude by the unenriched Yoneda lemma applied to $\mathcal{V}$.

If we have the enriched adjunction and if F is enriched, we have an enrichment of G as follows:

$$\underline{\mathsf{N}}[n_1,n_2] \to \underline{\mathsf{N}}[FGn_1,n_2] \to \underline{\mathsf{N}}[Gn_1,Gn_2]$$

Conversely, any enriched adjunction implies that the left adjoint preserves tensors:

$$\begin{aligned}
\underline{\mathsf{N}}[F(v\otimes m),n] &\overset{F\dashv G}{\cong} \underline{\mathsf{M}}[v\otimes m,Gn] \\
&\overset{\text{tensored}}{\cong} \underline{\mathcal{V}}[v,\underline{\mathsf{M}}[m,Gn]] \\
&\overset{F\dashv G}{\cong} \underline{\mathcal{V}}[v,\underline{\mathsf{N}}[Fm,n]] \\
&\overset{\text{tensored}}{\cong} \underline{\mathsf{N}}[v\otimes Fm,n]
\end{aligned}$$

Change of base for (co)tensored categories

CHT **Theorem** *Suppose we have an adjunction $F\colon \mathcal{V} \rightleftarrows \mathcal{U}\colon G$ between closed symmetric monoidal categories such that the left adjoint F is strong monoidal. Then any tensored, cotensored, and enriched $\mathcal{U}$-category becomes canonically enriched, tensored, and cotensored over $\mathcal{V}$.*

Proof hint We set

$$v \star m := Fv \otimes m, \quad \{v, m\} := m^{Fv}, \quad \underline{M}_{\mathcal{V}}(m,n) := G\underline{M}_{\mathcal{U}}(m,n).$$

And then, by a sequence of natural bijections, we have

$$\mathcal{M}(v \star m, n) = \begin{array}[t]{c} \mathcal{M}(Fv \otimes m, n) \\ \hline\hline \mathcal{V}(Fv, \underline{\mathcal{M}}_{\mathcal{U}}(m,n)) \\ \hline\hline \mathcal{V}(v, G\underline{\mathcal{M}}_{\mathcal{U}}(m,n)) = \mathcal{V}(v, \underline{\mathcal{M}}_{\mathcal{V}}(m,n)) \end{array}$$

lecture 0, p.6

Object of V-natural transformations

- Recall the definition of (ordinary) end as limit, for $H: C^{op}\times C \to D$

$$\boxed{\int_c H(c,c) = \lim\Big(\prod_c H(c,c) \rightrightarrows \prod_{c,c'} \prod_{f: c\to c'} H(c,c')\Big)}$$

"zone to be enriched"

where

$$\xrightarrow[c,c',f]{} \;=\; \prod_c H(c,c) \xrightarrow{\text{projection}} H(c,c) \xrightarrow{Fcf} H(c,c')$$

$$\xrightarrow[c,c',f]{} \;=\; \prod_c H(c,c) \xrightarrow{\text{projection}} H(c',c') \xrightarrow{Ffc'} H(c,c')$$

- Recall also (lecture 0, p.5) the end formula for the set of nat. transf.

$$\boxed{D^C[F,G] = \int_c D[Fc,Gc]}$$

for $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & D \end{tikzcd}$, $D(F-,G-): C^{op}\times C \to Set$

We seek an enriched version: $\begin{tikzcd} \underline{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \underline{D} \end{tikzcd}$, $\underline{D}(F-,G-)$ $\underline{C}^{op}\otimes\underline{C} \to \underline{V}$

Towards this goal, we slightly rephrase the formula for $H = D(F-,G-)$:

$$\int_c D[Fc,Gc] = \lim\Big(\prod_c D(Fc,Gc) \rightrightarrows \prod_{c,d} D(Fc,Gd)^{C(c,d)}\Big)$$

in Set

and we define

$$\boxed{\underline{V}^{\underline{D}}(F,G) = \lim\Big(\prod_c \underline{D}(Fc,Gc) \rightrightarrows \prod_{c,d} \underline{V}\big(\underline{C}(c,d), \underline{D}(Fc,Gd)\big)\Big)}$$

in V

CHT

Lemma 7.3.5 ($\mathcal{V}$-Yoneda lemma). *Given a small $\mathcal{V}$-category $\underline{\mathcal{D}}$, and object $d \in \underline{\mathcal{D}}$, and a $\mathcal{V}$-functor $F: \underline{\mathcal{D}} \to \underline{\mathcal{V}}$, the canonical map is a $\mathcal{V}$-natural isomorphism*

$$Fd \xrightarrow{\cong} \underline{\mathcal{V}}^{\underline{\mathcal{D}}}(\underline{\mathcal{D}}(d,-), F).$$

Enriched ends

The enrichment of page 21 works in fact for ordinary ends in a cotensored V-category:

for $\underline{H} : \underline{C}^{op} \otimes \underline{C} \to \underline{D}$ (enriched functor) we set

enriched end

$$\boxed{\int_{\underline{C}} \underline{H}(c,c) = \lim \left(\prod_{c} \underline{H}(c,c) \rightrightarrows \prod_{c,c'} \underbrace{\underbrace{\underline{H}(c,c')}_{D}{}^{\overbrace{\underline{C}(c,c')}^{V}}}_{D} \right)}$$

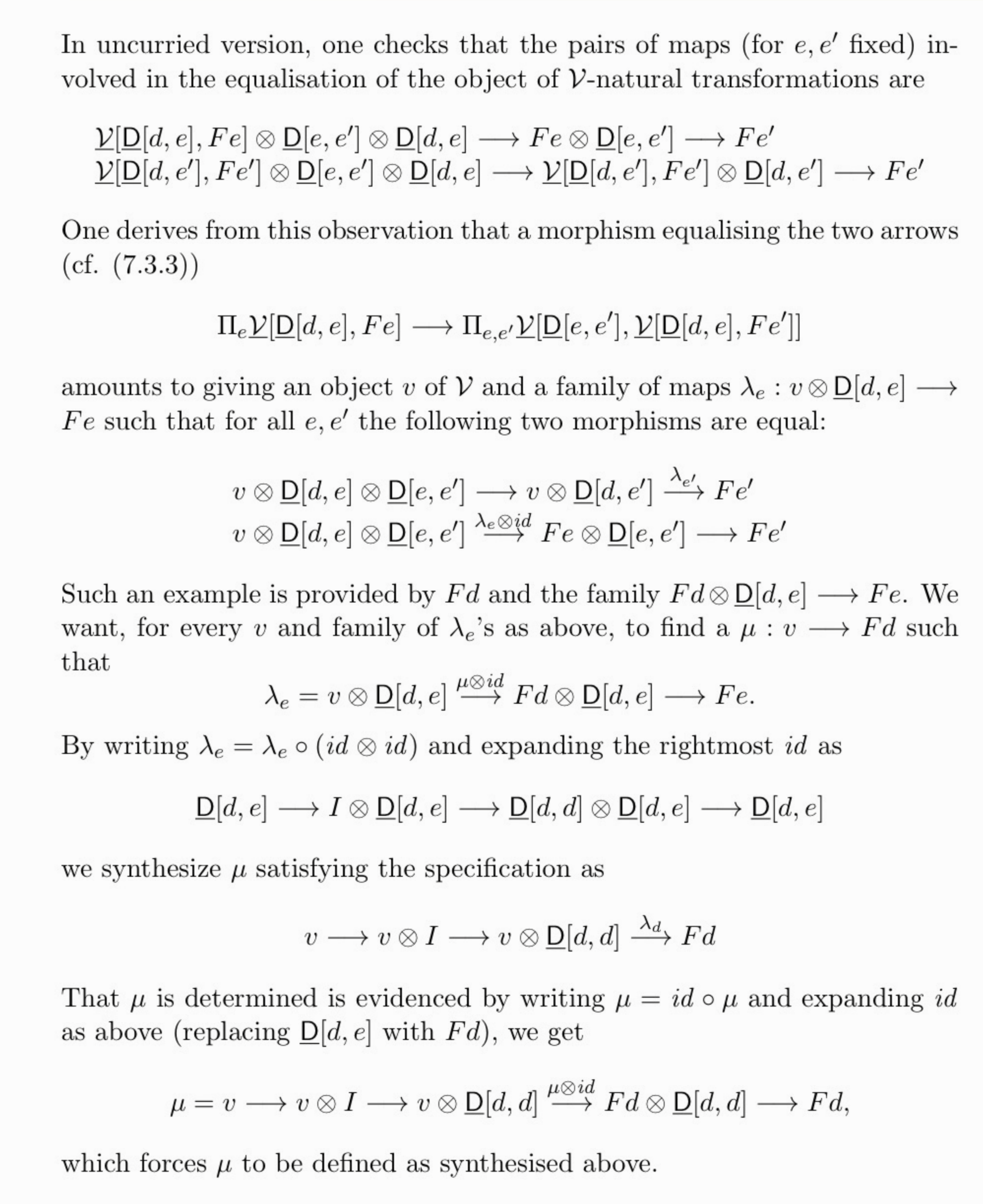
(Contrast with $\int_{c} H(c,c) = \lim\left(\prod_{c} H(c,c) \rightrightarrows \prod_{c,c'} \prod_{f:c\to c'} H(c,c') \right)$ "gone to be enriched")

Thus we can reformulate the object of natural transf.

$$\boxed{\underline{V}^{\underline{D}}(F,G) = \int_{\underline{C}} \underline{D}(Fc, Gc)}$$

One defines enriched coends dually.

Proof of enriched Yoneda lemma



In uncurried version, one checks that the pairs of maps (for e, e' fixed) involved in the equalisation of the object of $\mathcal{V}$-natural transformations are

$$\begin{array}{l} \underline{\mathcal{V}}[\underline{\mathsf{D}}[d,e], Fe] \otimes \underline{\mathsf{D}}[e,e'] \otimes \underline{\mathsf{D}}[d,e] \longrightarrow Fe \otimes \underline{\mathsf{D}}[e,e'] \longrightarrow Fe' \\ \underline{\mathcal{V}}[\underline{\mathsf{D}}[d,e'], Fe'] \otimes \underline{\mathsf{D}}[e,e'] \otimes \underline{\mathsf{D}}[d,e] \longrightarrow \underline{\mathcal{V}}[\underline{\mathsf{D}}[d,e'], Fe'] \otimes \underline{\mathsf{D}}[d,e'] \longrightarrow Fe' \end{array}$$

One derives from this observation that a morphism equalising the two arrows (cf. (7.3.3))

$$\Pi_e \underline{\mathcal{V}}[\underline{\mathsf{D}}[d,e], Fe] \longrightarrow \Pi_{e,e'} \underline{\mathcal{V}}[\underline{\mathsf{D}}[e,e'], \underline{\mathcal{V}}[\underline{\mathsf{D}}[d,e], Fe']]$$

amounts to giving an object v of $\mathcal{V}$ and a family of maps $\lambda_e : v \otimes \underline{\mathsf{D}}[d,e] \longrightarrow Fe$ such that for all e, e' the following two morphisms are equal:

$$\begin{array}{c} v \otimes \underline{\mathsf{D}}[d,e] \otimes \underline{\mathsf{D}}[e,e'] \longrightarrow v \otimes \underline{\mathsf{D}}[d,e'] \xrightarrow{\lambda_{e'}} Fe' \\ v \otimes \underline{\mathsf{D}}[d,e] \otimes \underline{\mathsf{D}}[e,e'] \xrightarrow{\lambda_e \otimes id} Fe \otimes \underline{\mathsf{D}}[e,e'] \longrightarrow Fe' \end{array}$$

Such an example is provided by Fd and the family $Fd \otimes \underline{\mathsf{D}}[d,e] \longrightarrow Fe$. We want, for every v and family of λ_e's as above, to find a $\mu : v \longrightarrow Fd$ such that

$$\lambda_e = v \otimes \underline{\mathsf{D}}[d,e] \xrightarrow{\mu \otimes id} Fd \otimes \underline{\mathsf{D}}[d,e] \longrightarrow Fe.$$

By writing $\lambda_e = \lambda_e \circ (id \otimes id)$ and expanding the rightmost id as

$$\underline{\mathsf{D}}[d,e] \longrightarrow I \otimes \underline{\mathsf{D}}[d,e] \longrightarrow \underline{\mathsf{D}}[d,d] \otimes \underline{\mathsf{D}}[d,e] \longrightarrow \underline{\mathsf{D}}[d,e]$$

we synthesize μ satisfying the specification as

$$v \longrightarrow v \otimes I \longrightarrow v \otimes \underline{\mathsf{D}}[d,d] \xrightarrow{\lambda_d} Fd$$

That μ is determined is evidenced by writing $\mu = id \circ \mu$ and expanding id as above (replacing $\underline{\mathsf{D}}[d,e]$ with Fd), we get

$$\mu = v \longrightarrow v \otimes I \longrightarrow v \otimes \underline{\mathsf{D}}[d,d] \xrightarrow{\mu \otimes id} Fd \otimes \underline{\mathsf{D}}[d,d] \longrightarrow Fd,$$

which forces μ to be defined as synthesised above.

Simplicial categories

Simplicial categories are by definition $\widehat{\Delta}$-enriched categories. From lecture 5 on, we shall consider tensored and cotensored simplicial categories. Let $\underline{\mathcal{M}}$ be a simplicial category. Then

- The underlying category $\mathcal{M}$ of $\underline{\mathcal{M}}$ is given by

$$\boxed{\mathcal{M}(x,y) = \underline{\mathcal{M}}(x,y)_0}$$

$\Delta^0_n = \Delta([n],[0]) = \{*\}$

Indeed, in $\widehat{\Delta}$, $*$ is the constant $*_n = \{*\} = \underline{\Delta^0}$, and we have thus $\mathcal{M}(x,y) = \mathcal{M}(\Delta^0 \otimes x, y) \simeq \widehat{\Delta}(\Delta^0, \underline{\mathcal{M}}(x,y)) \simeq \underline{\mathcal{M}}(x,y)_0$

- Writing $\bar{A}$ for the constant simplicial set $A_n = A$ ($A \in Set$), we have

$$\boxed{\bar{A} \otimes x = (\# A)\text{-fold coproduct of } x} = A \cdot x$$

written A for that

This follows from

- the following exercise
- the meaning of tensor and cotensor when $V = Set$ (cf. p 17)
- the change of base theorem p. 20

Exercise Show that

$$\boxed{(A \mapsto \bar{A}) : Set \underset{\perp}{\rightleftarrows} \widehat{\Delta} : (X \mapsto X_0)}$$

(Hint: contemplate naturality for the maps $! : [n] \to [0]$).

Exercise Show that the data of a simplicial category $\underline{C}$ can be arranged as that of a simplicial object $C_\bullet : \Delta^{op} \to Cat$ s.t.

- $Ob(C_i) = Ob(C_j)$ $\forall i, j$
- Cf is identity-on-objects $\forall f \in Mor\Delta$.

Simplicial homotopy

A simplicial category structure gives rise to a natural notion of homotopy between parallel maps $f,g : m \to n$ of the underlying category:

$$\begin{tikzcd}
\Delta^0 \arrow[d, "d^1"'] \arrow[rd, "f"] & \text{in } \mathbf{sSet} \\
\Delta^1 \arrow[r, "H"] & \underline{\mathcal{M}}(m,n) \\
\Delta^0 \arrow[u, "d^0"] \arrow[ru, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^0 \otimes m \cong m \arrow[d, "d^1 \otimes 1"'] \arrow[rd, "f"] & \text{in } \mathcal{M} \\
\Delta^1 \otimes m \arrow[r, "H"] & n \\
\Delta^0 \otimes m \cong m \arrow[u, "d^0 \otimes 1"] \arrow[ru, "g"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{in } \mathcal{M} & n \cong n^{\Delta^0} \\
m \arrow[ru, "f"] \arrow[r, "H"] \arrow[rd, "g"'] & n^{\Delta^1} \arrow[u, "n^{d^1}"'] \arrow[d, "n^{d^0}"] \\
& n \cong n^{\Delta^0}
\end{tikzcd}$$

Simplicial structure on $\mathcal{M}^{\Delta^{op}}$ ← category of simplicial objects

We suppose that $\mathcal{M}$ is a cocomplete category. We consider the following structure on $\mathcal{M}^{\Delta^{op}}$.

- We define $\otimes : \hat{\Delta} \otimes \mathcal{M}^{\Delta^{op}} \to \mathcal{M}^{\Delta^{op}}$ as follows:

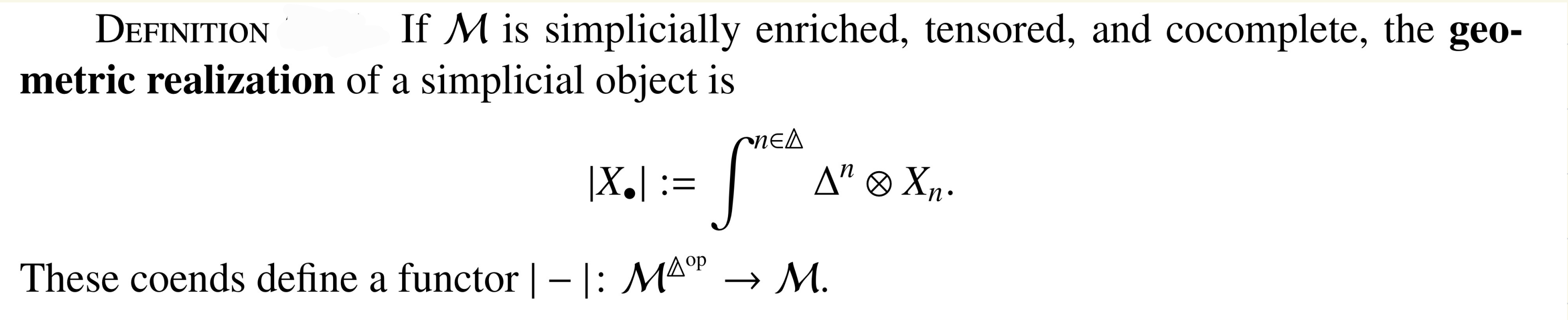
$$\boxed{(K\otimes X)_n = K_n \cdot X_n} \quad \Big(= \coprod_{x\in K_n} X_n\Big)$$

Exercise Show that this provide, an action and a $\hat{\Delta}$-enrichment (cf. p. 18)

Hint: for the adjunction, imitate the synthesis of the internal hom in $\hat{\Delta}$:

$$\underline{\mathcal{M}^{\Delta^{op}}}(X,Y)_n \underset{\text{Yoneda}}{\simeq} \hat{\Delta}(\Delta^n, \underline{\mathcal{M}}^{\Delta^{op}}(X,Y)) \underset{\text{claimed adj.}}{\simeq} \mathcal{M}^{\Delta^{op}}(\Delta^n \otimes X, Y)$$

Geometric realization



DEFINITION If $\mathcal{M}$ is simplicially enriched, tensored, and cocomplete, the **geometric realization** of a simplicial object is

$$|X_\bullet| := \int^{n\in\Delta} \Delta^n \otimes X_n.$$

These coends define a functor $|-| \colon \mathcal{M}^{\Delta^{op}} \to \mathcal{M}$.

Exercise Show that for a constant simplicial object $\bar{m}$, we have $|\bar{m}| \simeq m$

Hint Use exercise of lecture 0 p. 1

Summary of enriched structures encountered

- Set-categories = ordinary categories
- Cat-categories = 2-categories
- $\hat{\Delta}$-categories = simplicial categories
- Change of base $\mathcal{U}$-categories → $\mathcal{V}$-categories

(in particular underlying category of an enriched category)

Enrichment demystified

Tensored $\mathcal{V}$-categories as ordinary categories with some structure (action + adjoint)

(The situation is a bit similar to cartesian closed categories vs closed categories, which have a rather complicated definition)

⇐ internal homsets without cart. product