(We follow closely Chapter 2 of CHT)

Lecture 4 : Deformations and derived functors

We want to consider categories with a class of morphisms that are morally isos, in the sense that we would like to make them isos. Typically homotopy equivalences in topology, i.e.

$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that

$$\begin{tikzcd} X \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\simeq"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, bend left, "f\circ g"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\simeq"] & Y \end{tikzcd},$$

homotopy $H : I\times X \to X$

$H(0,-) = g\circ f$
$H(1,-) = id$

or in fact the wider class of topological weak equivalences, i.e. maps $f : X \to Y$ inducing isos on all homotopy groups.

There are then two basic issues:

- How to formally invert a given class W of arrows in a category $\mathcal{M}$, i.e. solve the following universal problem: construct a category $\mathcal{M}[W^{-1}]$ and a functor $\gamma : \mathcal{M} \to \mathcal{M}[W^{-1}]$ ← called localisation functor
  also called $\mathrm{Ho}\,\mathcal{M}$ s.t. $\gamma(W) \subseteq \mathrm{Iso}(\mathcal{M}[W^{-1}])$ –

s.t.

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[dr, "\forall F"'] & \mathcal{M}[W^{-1}] \arrow[d, "\exists ! G"] \\ & \mathcal{N} \end{tikzcd}$$

s.t. $F(W) \subseteq \mathrm{Iso}(\mathcal{N})$

We write $G = \tilde{F}$

- What axioms should we ask W to satisfy for developing a meaningful theory?
  - The most popular answer (backed by a long preexisting tradition that started with the thesis of Jean-Pierre Serre) is to introduce two other classes Fib an Cof (fibrations and cofibrations), and to give axioms governing Fib, Cof and W
    ⇝ model categories (covered later in the course)
  - For the time being, we follow the more lightweight approach of homotopical categories.

Constructing $\mathcal{M}[W^{-1}]$ (also called $\mathrm{Ho}\,\mathcal{M}$)

Construction due to Gabriel and Zisman

(Here W may be an arbitrary class of morphisms.)

The category Ho$\mathcal{M}$ has the same objects as $\mathcal{M}$. Its morphisms are equivalences classes of finite zig-zags of morphisms in $\mathcal{M}$, with only those arrows in $\mathcal{W}$ permitted to go backwards, subject to the following relations:

- adjacent arrows pointing in the same direction may be composed
- adjacent pairs $\xleftarrow{w}\xrightarrow{w}$ or $\xrightarrow{w}\xleftarrow{w}$ with $w \in \mathcal{W}$ may be removed
- identities pointing either forwards or backwards may be removed

Main drawback: $\mathrm{Ho}\mathcal{M}[a,b]$ might not be a set (even in a locally small category), hence the construction may take place in a "huge" universe.

One of the points of model categories is to give a description of a (dense full subcategory of) $\mathrm{Ho}\mathcal{M}$ that is simpler: quotient morphisms of $\mathcal{M}$ (no need to take formal zig zags) by a notion of homotopy defined thanks to the apparatus of model categories.

<u>Remark</u>: The situation is a bit similar to what we encountered in lecture 2 (up to the size issues): the general description of hX, and its "optimized version" via a notion of homotopy between 1-simplices.

<u>Exercise</u> Show that the relations above are redundant, in the sense that if $w_1, w_2, (w_2 w_1) \in W$, then $\xleftarrow[w_2]{}\xleftarrow[w_1]{} = \xleftarrow[w_2 w_1]{}$ is provable

(hint: consider $\xleftarrow[w_2 w_1]{}\xrightarrow[w_1]{}\xrightarrow[w_2]{}\xleftarrow[w_2]{}\xleftarrow[w_1]{}$)

Homotopical categories

Definition A homotopical category is a category $\mathcal{M}$ together with a collection $W_{\mathcal{M}}$ (W for short) of morphisms called weak equivalences (w.e.)

- containing all identity morphisms
- satisfying the 2-for-6 property: for all composable (f, g, h)

$$(2.1.2)\qquad \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "W\ni gf"'] \arrow[drr, "hgf"] & \cdot \arrow[d, "g"] \arrow[dr, "hg\in W"] & \\ & \cdot \arrow[r, "h"'] & \cdot \end{tikzcd} \qquad \Rightarrow \qquad f, g, h, hgf \in W$$

Clearly, if $\mathcal{N}$ is a subcategory of $\mathcal{M}$, then $(\mathcal{N}, W_{\mathcal{M}} \cap \mathrm{Mor}\,\mathcal{N})$ is homotopical.

Exercise Show that in a homotopical category, $\mathrm{Iso}(\mathcal{M}) \subseteq W$.
Show that $(\mathcal{M}, \mathrm{Iso}(\mathcal{M}))$ is a homotopical category (the smallest!)

Exercise Show that the 2-for-6 property implies the 2-for-3 property:

> for all composable
> $$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "gf"'] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd}$$
> if any two of f, g, gf are in W, then so is the third.

(In particular, in a homotopical category W is closed under composition.)

Exercise Show that if W is a collection of morphisms such that $f \in W$ iff $\gamma(f)$ is iso (recall $\gamma: \mathcal{M} \to \mathcal{M}[W^{-1}]$ from p.2), then W satisfies the 2-for-6 property.

This property is called saturation.

lecture 7 p.15

We shall see later that every model category is saturated.
Thus model categories are homotopical.

Diagram categories of homotopical categories.

Let $(\mathcal{M}, W_{\mathcal{M}})$ be a homotopical category.

Then, by immediate pointwise reasoning, we get that for any (small) category D, $(\mathcal{M}^D, \mathcal{W})$ is homotopical, where

$$\boxed{\mathcal{W} = \{\alpha \mid \forall d\ \alpha_d \in W_{\mathcal{M}}\}}$$

Homotopical functors

Definition A homotopical functor is a functor between hom. categories preserving the weak equivalences. Such a functor induces a unique functor between the homotopy categories:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[d, "\gamma"] \\
\mathrm{Ho}\,\mathcal{M} \arrow[r, "\exists!\, H"] & \mathrm{Ho}\,\mathcal{N}
\end{tikzcd}$$

Notation $\bar{F}$

Indeed, $\bar{F} = \widetilde{\gamma F}$ (cf. p. 1)

Useful easy lemmas

Lemma For any category $\mathcal{C}$ and any homotopical homotopical category $\mathcal{M}$, for any functors $(\mathcal{M} \xrightarrow{\gamma})\ Ho(\mathcal{M}) \underset{K}{\overset{H}{\rightrightarrows}} \mathcal{C}$, we have

strict equality

$$Cat(H,K) \equiv Cat(H\gamma, K\gamma)$$

Proof: Note that $\mathcal{M}$ and $Ho\,\mathcal{M}$ have the same objects, and that γ is identity-on-objects. Hence, for $\mu : H \to K$, $\mu\gamma = \mu$.
Hence we have, immediately, $\mu = \mu\gamma : H\gamma \to K\gamma$.
Conversely, let $\nu : H\gamma \to K\gamma$, and suppose, say, we want to check naturality for $[g w^{-1} f] = [g][w]^{-1}[f]$. Then we get the commutation of the outer rectangle from the commutation of the 3 inner squares (the point is that $H[w]$ is iso since $[w]$ is iso).

$$\begin{tikzcd}
\cdot \arrow[r, "\nu"] \arrow[d, "{H[f]}"'] & \cdot \arrow[d, "{K[f]}"] \\
\cdot \arrow[r, "\nu"] \arrow[d, shift right, "{H[w^{-2}] = (H[w])^{-1}}"'] & \cdot \arrow[d, shift right] \\
\cdot \arrow[r, "\nu"] \arrow[u, shift right, "{H[w]}"'] \arrow[d, "{H[g]}"'] & \cdot \arrow[u, shift right, "{K[w]}"'] \arrow[d, "{K[g]}"] \\
\cdot \arrow[r, "\nu"] & \cdot
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \\
\cdot \arrow[r] \arrow[u] & \cdot \arrow[u]
\end{tikzcd}$$
commutes iff

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$
commutes

Lemma If $\alpha : F \to G$ is s.t. all its components are weak equivalences, (natural weak equivalence)
then F is homotopical iff G is homotopical

Proof Apply the 2-for-3 property to

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}$$

Derived functors

Definition A **total left derived functor** $\mathbf{L}F$ of a functor F between homotopical categories $\mathcal{M}$ and $\mathcal{N}$ is a *right* Kan extension $\mathrm{Ran}_\gamma \delta F$

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow\varepsilon"] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}\mathcal{M} \arrow[r, dashed, "\mathbf{L}F"'] & \mathrm{Ho}\mathcal{N} \end{tikzcd}$$

$= \mathrm{Ran}_\gamma \delta F$

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\delta F"] \arrow[d, "\gamma"'] & \mathrm{Ho}\mathcal{N} \\ \mathrm{Ho}\mathcal{M} \end{tikzcd}$$

where γ and δ are the localization functors for $\mathcal{M}$ and $\mathcal{N}$.

Proposition $\mathbf{L}F, \varepsilon$ are characterised by the following property:

$\forall G: \mathcal{M} \to \mathrm{Ho}\mathcal{N}$ mapping w.e. to isos
$\forall \alpha: G \to \delta F$, α factors uniquely through ε

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[dr, bend right, "G"'] \arrow[dr, phantom, "\alpha \Nearrow"] & \mathcal{N} \arrow[d, "\delta"] \\ & \mathrm{Ho}\mathcal{N} \end{tikzcd} = \begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow\varepsilon"] \arrow[d, bend right=60, "G"'] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}\mathcal{M} \arrow[r, dashed, "\mathbf{L}F"'] \arrow[r, phantom, bend right, "!\beta \Nearrow"] & \mathrm{Ho}\mathcal{N} \end{tikzcd}$$

Proof We observe that the data of G is equivalent to the data of $\tilde{G}: \mathrm{Ho}\mathcal{M} \to \mathrm{Ho}\mathcal{N}$. Then, applying the definition of $\mathrm{Ran}_\gamma \delta F$ to $G = \tilde{G}\gamma$, we get a unique $\beta: \tilde{G} \to \mathbb{L}F$, which by the first lemma on p.5 can be written $\beta: G = \tilde{G}\gamma \to \mathbf{L}F\gamma$

Definition A **left derived functor** of $F: \mathcal{M} \to \mathcal{N}$ is a homotopical functor $\mathbb{L}F: \mathcal{M} \to \mathcal{N}$ equipped with a natural transformation $\lambda: \mathbb{L}F \Rightarrow F$ such that $\delta\lambda: \delta \cdot \mathbb{L}F \Rightarrow \delta \cdot F$ is a total left derived functor of F.

$\overline{\mathbb{L}F}$ and

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, bend right, "\mathbb{L}F"'] \arrow[r, phantom, "\Uparrow\lambda"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}\mathcal{M} \arrow[r, "\overline{\mathbb{L}F}"'] & \mathrm{Ho}\mathcal{N} \end{tikzcd}$$

$\mathbb{L}F = \overline{\mathbb{L}F}$

Corollary $\mathbb{L}F, \lambda$ are characterised by:

$$\boxed{\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[dr, bend right, "\forall G"'] \arrow[dr, phantom, "\forall\alpha \Nearrow"] & \mathcal{N} \arrow[d, "\delta"] \\ & \mathrm{Ho}\mathcal{N} \end{tikzcd} = \begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, bend right, "\mathbb{L}F"'] \arrow[r, phantom, "\Uparrow\lambda"] \arrow[dr, bend right=60, "G"'] & \mathcal{N} \arrow[d, "\delta"] \arrow[d, phantom, bend right, "\exists!\beta \Nearrow"] \\ & \mathrm{Ho}\mathcal{N} \end{tikzcd} \quad G(\mathcal{W}) \subseteq \mathrm{isos}(\mathrm{Ho}\mathcal{N})}$$

← This is all you need to remember!

Remark While total left derived functors are warranted by the existence of Kan extensions, left derived functors may not exist. But here comes a recipe!

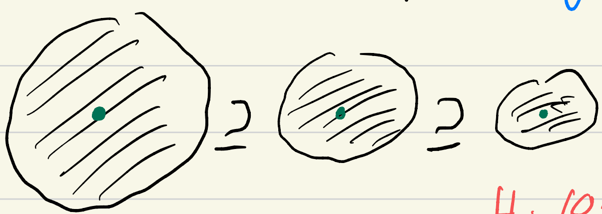
Our main use of derived functors

We want functors that respect homotopy equivalences

Ordinary colimits do not preserve them $\rightsquigarrow$ REPLACE colimits by their left derived functors $=$ **HOMOTOPY COLIMITS**

- A disk can be contracted to a point

D^2



$$H : [0,1] \times D^2 \to D^2 \qquad H(0,-) = id \qquad H(1,-)x = \bullet$$

- A sphere cannot

S^1

$\not\simeq$ (but $S^1 - \{x\}$ can)

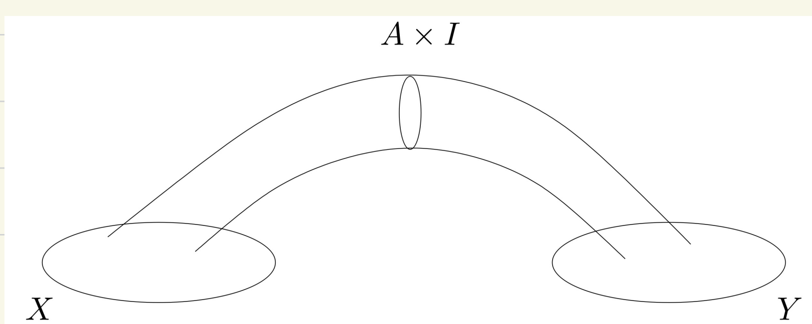
Problem: pushouts do not preserve homotopy equivalences

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & \{*\} \arrow[d] \\
D^2 \arrow[r] \arrow[ur, phantom, "\text{p.o.}"] & S^2
\end{tikzcd}$$

$D^2 \simeq \{x\}$, $S^2 \not\simeq \{*\}$

What we need is a homotopy pushout

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] \arrow[ur, phantom, "\text{h.p.o.}"] & \,
\end{tikzcd}$$



Deformations

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "Q"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow q"] & \mathcal{M} \end{tikzcd}$$

DEFINITION A **left deformation** on a homotopical category $\mathcal{M}$ consists of an endofunctor Q together with a natural weak equivalence $q\colon Q \overset{\sim}{\Rightarrow} 1$.

By the second lemma on p.5, Q is then homotopical.

Definition A functor $F: \mathcal{M} \to \mathcal{N}$ between homotopical categories is called left deformable (relative to given Q, q) if $\boxed{\forall m, n,\ f: Qm \to Qn \quad f \in W \Rightarrow Ff \in W}$

Proposition F is left deformable iff

- FQ is homotopical
- $FqQ\colon FQ^2 \Rightarrow FQ$ is a natural weak equivalence

- "Only if"
  - FQ homotopical: $\overbrace{FQ\underbrace{w}_{\in W \cap Im(Q)}}^{\in W_{\mathcal{N}}}$
  - FqQ natural w.e.: $\overbrace{F\underbrace{q Q}_{\in \mathcal{M}_Q}}^{\in W_{\mathcal{N}}}$
- "If" let $w: Qm \to Qn$ w.e. We have

$$\begin{tikzcd} QQm \arrow[r, "Qw"] \arrow[d, "qQm"'] & QQn \arrow[d, "qQn"] \\ Qm \arrow[r, "w"'] & Qn \end{tikzcd}$$

We observe

$\left.\begin{array}{l} FQw \\ FqQm \\ FqQn \end{array}\right\}$ w.e. (assumptions)

In most example, there exists a natural full subcategory $\mathcal{M}_Q$ of $\mathcal{M}$ containing all objects Qm, and we have

objects of $\mathcal{M}_Q$ are called cofibrant

$$\boxed{F(W \cap \mathcal{M}_Q) \subseteq W_{\mathcal{N}}.}$$

(a stronger property than in the definition, but the definition above is all what we need!)

Deformations arising from model categories

Among the properties required from Fib, Cof and W in a model category, is the following: every morphism $f : m \to n$ has a factorisation

$$\begin{tikzcd} m \arrow[r, "\in Cof"] \arrow[rr, bend right, "f"'] & m' \arrow[r, "\in Fib \cap W"] & n \end{tikzcd}$$

initial object

In particular, for $m = 0$, for every object n, there exists an object φn and

$$0 \xrightarrow{\in Cof} \varphi_n \xrightarrow[q_n]{\in Fib \cap W} n$$

An object m s.t. $0 \to m$ is a cofibration is called cofibrant.

We say that φn above is a cofibrant replacement of n.

When the model category structure arises in a certain way (cofibrantly generated), φ can be made into a functor and q_n to a natural transformation $\varphi \xrightarrow[q]{} 1$

Since $q_n \in W$, we have a left deformation.

see lecture 7 p. 12

Derived functors via deformations

Proposition If F is left deformable, then (FQ, Fq) is a left derived functor of F.

Proof • FQ is homotopical cf. p.8

• We have to prove

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[dr, bend right, "G"'] & \mathcal{N} \arrow[d, "\delta"] \arrow[dl, phantom, "\alpha \Nearrow"] \\ & \mathrm{Ho}\,\mathcal{N} \end{tikzcd} = \begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[r, bend right, "FQ"'] \arrow[r, phantom, bend right=20, "\Uparrow Fq"] \arrow[dr, bend right, "G"'] & \mathcal{N} \arrow[d, "\delta"] \arrow[dl, phantom, "!? \Nearrow"] \\ & \mathrm{Ho}\,\mathcal{N} \end{tikzcd}$$

valid candidate for ?

• From $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "Q"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow q"] & \mathcal{M} \arrow[r, bend left, "G"] \arrow[r, bend right, "\delta F"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathrm{Ho}\,\mathcal{N} \end{tikzcd}$, we derive $\alpha \circ \underset{\text{invertible}}{Gq} = \delta Fq \circ \alpha Q$

$\Downarrow$ $\alpha = \delta Fq \circ \alpha Q \circ (Gq)^{-1}$

and δF by δFQ

• We are left to show uniqueness. Suppose $\delta Fq \circ \beta' = \alpha$.

• Then $\underbrace{\overbrace{\delta FqQ}^{\in W_{\mathcal{N}} \text{ cf. p.8}}}_{\text{iso}} \circ \beta' Q = \alpha Q$

$\Downarrow$

$\beta' Q$ determined

• Replacing α by β', we get

$\beta' \circ Gq = \delta FQq \circ \beta' Q$

$\Downarrow$ $\beta' = \delta FQq \circ \underbrace{\beta' Q}_{\text{determined}} \circ (Gq)^{-1}$

Deformable adjunctions

There are obvious dual definitions of

- (total) right derived functor (left Kan extensions)
- right deformation

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "1"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\Downarrow r"] & \mathcal{M} \end{tikzcd}$$

with a specified full subcategory $\mathcal{M}_R$ of $\mathcal{M}$ of fibrant objects

- right deformable functor (same definition, replacing $\mathcal{M}_Q$ by $\mathcal{M}_R$)

leading to the dual proposition that (FR, Fr) is a right derived functor.

Definition An adjoint pair $F \dashv G$ with F left deformable and G right deformable is called a deformable adjunction.

Exercise Show that the total left derived functor of a left deformable functor is an absolute right Kan extension.
cf. lecture 1, p. 8
(Hint: replace everywhere δ by $H\delta$ in the proof of the proposition p. 6)

The following is a consequence of this exercise and exercise p. 8 lecture 1

THEOREM *If* $F\colon \mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{N} \colon G$ *is a deformable adjunction, then the total derived functors form an adjunction*

$$\mathbf{L}F\colon \mathrm{Ho}\mathcal{M} \underset{\perp}{\rightleftarrows} \mathrm{Ho}\mathcal{N}\colon \mathbf{R}G$$

Furthermore, the total derived adjunction $\mathbf{L}F \dashv \mathbf{R}G$ *is the unique adjunction compatible with the localizations in the sense that the diagram of hom-sets*

$$\begin{tikzcd} \mathcal{N}(Fm,n) \arrow[r, phantom, "\cong"] \arrow[d, "\delta"'] & \mathcal{M}(m,Gn) \arrow[d, "\gamma"] \\ \mathrm{Ho}\mathcal{N}(Fm,n) \arrow[d, "Fq^*"'] & \mathrm{Ho}\mathcal{M}(m,Gn) \arrow[d, "Gr_*"] \\ \mathrm{Ho}\mathcal{N}(\mathbf{L}Fm,n) \arrow[r, phantom, "\cong"] & \mathrm{Ho}\mathcal{M}(m,\mathbf{R}Gn) \end{tikzcd}$$

commutes for each pair $m \in \mathcal{M}$, $n \in \mathcal{N}$.

Derived functors in homological algebra

(This page can be skipped if you have zero familiarity with basic concepts of homological algebra.)

- Chain complexes of R-modules form a homotopical category $Ch_{\geq 0}(R)$, with quasi-isomorphisms as weak equivalences.

  morphisms inducing isos in homology

- Every R-module M, viewed as a chain-complex concentrated in degree 0, is quasi-isomorphic to an acyclic chain-complex QM of projective modules. More generally, the construction can be done starting from an arbitrary chain-complex.

  $\rightsquigarrow (Q, q)$ left deformation

- Take as M_Q the full subcategory of chain-complexes of projective modules.

- A quasi-isomorphism between objects of M_Q iff it is a chain homotopy equivalence, an equational notion that is thus preserved by "any" functor F. It follows that "any" F is left deformable.

  (for a suitable sense of "any")

This matches the essence of the traditional derived functors L_iF of the literature:

$$
\begin{tikzcd}
Mod_R \arrow[r, hook] \arrow[rrrr, bend right, "L_iF"'] & Ch_{\geq 0}(R) \arrow[r, "Q"] \arrow[rr, bend right, "\mathbb{L}F"'] & Ch_{\geq 0}(R) \arrow[r, "F"] & Ch_{\geq 0}(S) \arrow[r, "H_i"] & Mod_S
\end{tikzcd}
$$