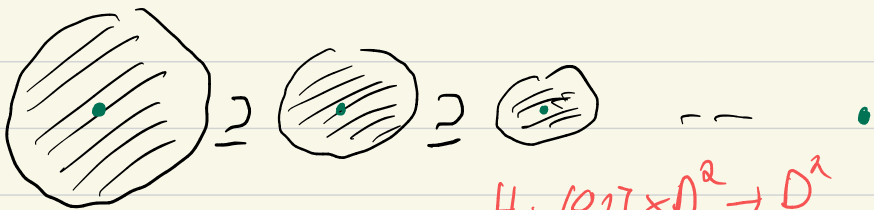
Lecture 5 : Bar construction and homotopy colimits

Recall from Lecture 4 p. 7 : the main goal of homotopical algebra is to capture and study spaces under deformations by homotopy.

- A disk can be contracted to a point:

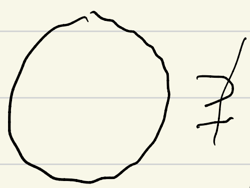
D^2



$H: [0,1] \times D^2 \to D^2 \quad H(0,-) = id$
$H(1,-)x = \bullet$

- A sphere cannot:

S^1



(but $S^1 - \{x\}$ can)

Problem : pushouts do not preserve homotopy equivalences

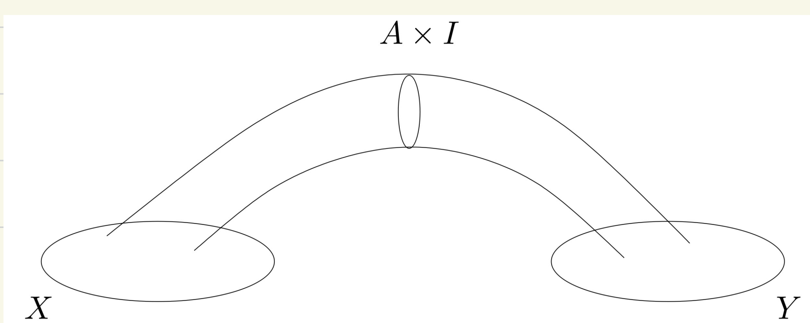
$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ D^2 \arrow[r] \arrow[ur, phantom, "\text{p.o.}"] & S^2 \end{tikzcd}$$

$D^2 \simeq \{*\}$, $\quad S^2 \cancel{\simeq} \{*\}$

What we need is a **homotopy pushout**

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\text{h.p.o.}"] & \end{tikzcd}$$

In this lecture, we shall construct homotopy colimits using abstract categorical tools.



What we need to recall from lecture 3

- In a tensored and cotensored $\mathcal{V}$-category $\mathcal{M}$
  $\otimes : \mathcal{V} \otimes \mathcal{M} \to \mathcal{M}$ preserves colimits in both variables
  (by adjunction)

- In a simplicial (i.e. $\hat{\Delta}$-) category, we have, for every set A and object m:
  $$\boxed{\bar{A} \otimes m = \coprod_{a \in A} m}$$

- ## Simplicial structure on $\mathcal{M}^{\Delta^{op}}$ ← category of simplicial objects

We suppose that $\mathcal{M}$ is a cocomplete category. We consider the following structure on $\mathcal{M}^{\Delta^{op}}$:

We define $\otimes : \hat{\Delta} \otimes \mathcal{M}^{\Delta^{op}} \to \mathcal{M}^{\Delta^{op}}$ as follows:
$$\boxed{(K \otimes X)_n = K_n \cdot X_n} \ \left(= \coprod_{x \in K_n} X_n\right)$$

+ cotensored!

Exercise Show that this provides an action and a $\hat{\Delta}$-enrichment (cf. p. 18)

Hint: for the adjunction, imitate the synthesis of the internal hom in $\hat{\Delta}$:
$$\underline{\mathcal{M}^{\Delta^{op}}}(X,Y)_n \underset{\text{Yoneda}}{\simeq} \hat{\Delta}(\Delta^n, \underline{\mathcal{M}^{\Delta^{op}}}(X,Y)) \underset{\text{claimed adj.}}{\simeq} \mathcal{M}^{\Delta^{op}}(\Delta^n \otimes X, Y)$$

- ## Geometric realisation

DEFINITION If M is simplicially enriched, tensored, and cocomplete, the **geometric realization** of a simplicial object is
$$|X_\bullet|_{\mathcal{M}} := \int^{n \in \Delta} \Delta^n \otimes X_n.$$
These coends define a functor $|-|_{\mathcal{M}} : M^{\Delta^{op}} \to M.$

Exercise Show that for a constant simplicial object $\bar{m}$, we have $|\bar{m}| \simeq m$

Functor tensor product

Definition In the presence of a bifunctor $-\otimes -\colon \mathcal{V}\times\mathcal{M}\to\mathcal{M}$, the **functor tensor product** of $F\colon\mathcal{D}\to\mathcal{M}$ with $G\colon\mathcal{D}^{\mathrm{op}}\to\mathcal{V}$ is the coend

$$(4.1.1)\qquad G\otimes_{\mathcal{D}} F := \int^{d\in\mathcal{D}} \underset{Gd\otimes Fd}{G\otimes F} = \mathrm{coeq}\left(\coprod_{f\colon d\to d'} Gd'\otimes Fd \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_d Gd\otimes Fd \right)$$

This construction applies in particular to a tensored $\mathcal{V}$-category $\mathcal{M}$.

The following exercise justifies the terminology.

Exercise Let Ab be the category of abelian (= commutative) groups. Show that a ring "is" a one-object Ab-enriched category. Show that a right (resp. left) module is an Ab-functor $R^{op}\to Ab$ (resp. $R\to Ab$). Show that, instantiating above $\mathcal{D},\mathcal{M},\mathcal{V}$ as R, Ab, Ab, we get that $A\otimes_R B$ is the usual tensor product of R-modules (i.e. $A\times B/(ra,b)\sim(a,rb)$).

Exercise Show that the functor tensor product is functorial in G, F, and that it commutes with colimits in both variables. à

Hint: cf. Lecture 2 p. 16 for $\mathcal{M}$ tensored and cotensored

We shall need the following associativity property, for

$F\colon \mathcal{D}\to\mathcal{M}$, $L\colon \mathcal{D}^{op}\times\mathcal{D}\to\mathcal{V}$, $G\colon\mathcal{D}^{op}\to\mathcal{V}$

Proposition For G, L, F as above, we have

$$\boxed{G\otimes_{\mathcal{D}}\big(d\mapsto (L-d)\otimes_{\mathcal{D}} F\big) \simeq \big(d'\mapsto G\otimes_{\mathcal{D}}(Ld'-)\big)\otimes_{\mathcal{D}} F}$$

Proof $G\otimes_{\mathcal{D}}(d\mapsto (L-d)\otimes F) = \int^d Gd\otimes \int^{d'}(Ld'd\otimes Fd')$ Fubini

(tensor-colimit commutation) $\simeq \int^d\int^{d'} Gd\otimes(Ld'd\otimes Fd') \simeq \int^{d'}\int^{d} Gd\otimes(Ld'd\otimes Fd')$

$\wr\wr$ $\mathcal{M}$ tensored

$(d'\mapsto G\otimes(Ld'-))\otimes F = \int^{d'}\left(\int^d Gd\otimes Ld'd\right)\otimes Fd' \simeq \int^{d'}\int^d (Gd\otimes Ld'd)\otimes Fd'$

(tensor-colimit commutation)

Instances of the functor tensor product

- $\boxed{* \otimes_D F =' \operatorname{colim} F}$ (cf. lecture 0, dummy contra/co variant!)

  ↑ constant $*(d) = *$ (unit object of V)

- $\boxed{\underline{D}(-,d) \otimes_D F \simeq Fd \qquad G \otimes_D \underline{D}(d,-) \simeq Gd}$ (enriched version of) co-Yoneda cf. lecture 1 p. 6 and 11

- The left Kan extension formula gets reformulated as:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ D & \end{tikzcd} \qquad \boxed{\operatorname{Lan}_K F(d) = D(K-,d) \otimes_C F}$$

and, for $K = y$, $\underline{\operatorname{Lan}_y F X = X \otimes F}$

in part. for $K = y$ and $F = \Delta^\bullet_{top} : \Delta \to Top$

$$\begin{tikzcd} \Delta \arrow[r, "\Delta_{top}"] \arrow[d, "y"'] & Top \\ \hat{\Delta} \arrow[ur, "|-|_{top}"'] & \end{tikzcd}$$

$$\operatorname{Lan}_y \Delta^\bullet_{top} X = \boxed{|X|_{top} = X \otimes_\Delta \Delta^\bullet_{top}}$$

- The geometric realisation of a simplicial object $X_\bullet$ in a (tensored) simplicial category $\mathcal{M}$ gets reformulated as $\boxed{|X_\bullet|_{\mathcal{M}} = \Delta^\bullet \otimes_{\Delta^{op}} X_\bullet}$ cf. lecture 2 p. 27

<u>Exercise</u> Show that for $\mathcal{M} = \hat{\Delta}$ and $X \in \hat{\hat{\Delta}}$ considered as a bisimplicial set, (i.e. $X : \Delta^{op} \to (\Delta^{op} \to Set)$, with $X_{nm} = X_n$), then $|X| = X$ (hint: co-Yoneda!).

Relating the functor tensor product and Kan extensions

Consider

$$\begin{tikzcd} \mathcal{V} \\ & D^{op} \arrow[ul, "H"'] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\ D \arrow[ur, "\mathrm{Lan}_K F"'] & \end{tikzcd}$$

and $\underline{\mathcal{E}}$ tensored and cotensored $\mathcal{V}$-category

Recall from lecture 3 p.16 that $\otimes$ preserves colimits in each variable (since $-\otimes e$ and $v\otimes -$ are left adjoints).

Proposition We have $\boxed{H \otimes_D \mathrm{Lan}_K F \cong (HK) \otimes_C F}$

Proof We have

$$\begin{aligned}
H \otimes_D \mathrm{Lan}_K F &= \int^d Hd \otimes (\mathrm{Lan}_K F d) \\
&= \int^d Hd \otimes \int^c D(Kc,d) \cdot Fc \\
\text{(tensor-colimit commutation)} &\cong \int^d \int^c D(Kc,d) \cdot (Hd \otimes Fc) \\
\text{(commutation of coends) a.k.a. Fubini} &\cong \int^c \int^d D(Kc,d) \cdot (Hd \otimes Fc) \\
\text{(tensor-colimit commutation)} &\cong \int^c \Big(\underbrace{\int^d D(Kc,d) \cdot Hd}\Big) \otimes Fc \\
\text{(co-Yoneda)} &\cong \int^c HKc \otimes Fc \\
&= (HK) \otimes_C F
\end{aligned}$$

PRACTICAL!

used p.23-24

As an example:

other notation for $sk_n X$

Corollary If X is n-skeletal, then $\boxed{|X| \cong \Delta^\bullet_{\le n} \otimes_{\Delta^{op}_{\le n}} X_{\le n}}$

Proof $|X| = \Delta^\bullet \otimes_{\Delta^{op}} X_\bullet \cong \Delta^\bullet \otimes \mathrm{Lan}_{i_n} \underbrace{X i_n}_{X_{\le n}} \cong \underbrace{\Delta^\bullet i_n}_{\Delta_{\le n}} \otimes_{\Delta^{op}_{\le n}} X_{\le n} = \Delta^\bullet_{\le n} \otimes_{\Delta^{op}_{\le n}} X_{\le n}$

The simplicial bar construction

The bar construction is a variation on the functor tensor product, that yields a simplicial object in $\mathcal{M}$, rather than an object of $\mathcal{M}$.

With, again, $F: \mathcal{D} \to \mathcal{M}$ and $G: \mathcal{D}^{op} \to \mathcal{V}$, we set

$$\boxed{B_n(G,\mathcal{D},F) = \coprod_{\vec{d}\colon [n]\to\mathcal{D}} Gd_n \otimes Fd_0,} \quad \longleftarrow \text{ i.e. } \in (N\mathcal{D})_n$$

writing $\vec{d}$ as shorthand for a sequence $d_0 \to d_1 \to \cdots \to d_n$ of n composable arrows in $\mathcal{D}$.

We call $B_\bullet(G,\mathcal{D},F)$ the two-sided simplicial bar construction.

The simplicial structure stems from the simplicial structure of the nerve.

Exercise Spell this simplicial structure out, either directly, or through the following abstraction. Let C be a category and $X \in \hat{C}$. Let $H: el\,X \to \mathcal{M}$ into a category with all coproducts. Then show that $\hat{H}c = \coprod_{x\in X_c} H(c,x)$ extends to a functor $\hat{H}: C^{op} \to \mathcal{M}$. Apply this to $X = N\mathcal{D}$ and $H(n,\vec{d}) = Gd_n \otimes Fd_0$

The following lemma establishes a relation between $G \otimes_{\mathcal{D}} F$ and $B_\bullet(G,\mathcal{D},F)$

Lemma We have $\underset{n}{\mathrm{colim}}(B_\bullet(G,\mathcal{D},F)) \cong G \otimes_{\mathcal{D}} F$

Proof hint A cocone for $B_\bullet(G,\mathcal{D},F)$ consists of morphisms $\lambda_{n,\vec{d}}$ indexed over $el(N\mathcal{D})$, which are entirely determined by the components at 0:

$$(\vec{d} = d_0 \xrightarrow{f_0} d_1 \cdots \xrightarrow{f_{n-1}} d_n) \rightsquigarrow$$

$$\begin{tikzcd}
Gd_n \otimes Fd_0 \arrow[r, "\lambda_{n,\vec{d}}"] \arrow[d, "G(f_{n-1}\circ\cdots\circ f_0)\otimes Fd_0"'] & m \\
Gd_0 \otimes Fd_0 \arrow[ur, "\lambda_{0,d_0}"'] &
\end{tikzcd}$$

The bar construction

We take $V = \hat{\Delta}$ and $\mathcal{M}$ a (tensored) simplicial category.

Definition The **bar construction** is the geometric realization of the simplicial bar construction, i.e.,

$$\boxed{B(G,\mathcal{D},F) = |B_\bullet(G,\mathcal{D},F)|_{\mathcal{M}} = \Delta^\bullet \otimes_{\Delta^{op}} B_\bullet(G,\mathcal{D},F).}$$

The relation between $G \otimes_D F$ and $B(G,D,F)$ is as follows.

Lemma The natural transformation $! : \Delta^\bullet \to * : \Delta \to \hat{\Delta}$ induces a map $B(G,D,F) \to G \otimes_D F$.

Proof $B(G,D,F) \to * \otimes_{\Delta^{op}} B_\bullet(G,D,F) \simeq \operatorname{colim}(B_\bullet(G,D,F)) \simeq G \otimes_D F$

exercise p.3 p.4 Lemma p.6

$$\boxed{\text{We are in particular interested in the case where } G = D(-,d)}$$

(Considered as a functor $D^{op} \to Set \to \hat{\Delta}$
Then we write

$$\boxed{B(\mathcal{D},\mathcal{D},F)\colon \mathcal{D} \to \mathcal{M} \qquad \text{for the functor} \qquad d \mapsto B(\mathcal{D}(-,d),\mathcal{D},F).}$$

Remembering from p.2 that $\bar{A} \otimes x = \coprod_{a \in A} x$,

we can write $$\boxed{B_n(D(-,d),D,F) = \coprod_{a_0 \to \cdots \to a_n \to d} Fa_0}$$ (coproduct of coproducts)

$D(a_n,d)$

and, dually $\boxed{B(G,D,D)\, d' = B(G,D,D(d',-))}$

A fundamental simplicial homotopy equivalence

There are two natural transformations $\in \mathrm{Mor}\, \mathcal{M}^{\Delta^{op}}$

$$B_\bullet(D(-,d),D,F) \underset{\iota}{\overset{\varepsilon}{\rightleftarrows}} \overline{Fd}$$

see next slide — constant simplicial object

- The component at $a_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} a_n \xrightarrow{f_n} d$ of $\varepsilon_n : \coprod_{a_0 \to \cdots a_n \to d} Fa_0 \to Fd$ is $F(f_n \circ \cdots \circ f_0) : Fa_0 \to Fd$

- ι_n is the coproduct injection Fd at component $\underbrace{d \xrightarrow{id} \cdots \to d \xrightarrow{id} d}_{n+1 \text{ identities}}$

We have $\varepsilon_n \circ \iota_n = id_{Fd}$ $\quad (F(id \circ \cdots \circ id) = id)$

Remember from p. 2 that there is a tensored simplicial structure on $\mathcal{M}^{\Delta^{op}}$.

Theorem There exists a simplicial homotopy

$$\boxed{H : \Delta^1 \otimes B_\bullet(D(-,d),D,F) \to B_\bullet(D(-,d),D,F)}$$

between $(\iota_n \circ \varepsilon_n)$ and id. (see next 2 pages for what it boils down to)

Proof hint We seek $H_n : \coprod_{a_0 \to \cdots \to a_n \to d,\ g:[n]\to[1]} Fa_0 \longrightarrow \coprod_{e_0 \to \cdots \to e_n \to d} Fe_0$, i.e.,

for each $a_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} a_n \xrightarrow{f_n} d$ and $g : [n] \to [1]$ • a sequence $e_0 \to \cdots \to e_n \to d$ • and a map $Fa_0 \to Fe_0$.

We can represent g as a separator $\quad | \quad | \quad \downarrow \quad | \quad |$

(e.g. for $n=2$: $000 = |||\downarrow$, $001 = ||\downarrow|$, $011 = |\downarrow||$, $111 = \downarrow|||$)

- With $a_0 \to \cdots a_i \xrightarrow{f_i} a_{i+1} \to \cdots \xrightarrow{f_n} d$ we associate $\underset{e_0}{a_0} \xrightarrow{f_0} \cdots \to \underset{e_i}{a_i} \xrightarrow{f_n \circ \cdots \circ f_i} \underset{e_{i+1}}{d} \xrightarrow{id} d \cdots \to d$ and $id : Fa_0 \to Fa_0$

- With $a_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} a_n \xrightarrow{f_n} d$ we associate $\underbrace{\underset{e_0}{d} \xrightarrow{id} \underset{e_1}{d} \cdots \xrightarrow{id} d \text{ and } F(f_n \circ \cdots \circ f_0) : Fa_0 \to Fd}_{\iota_n \circ \varepsilon_n\,!}$

We sketch naturality of H (on generators d^i, s^i) on p. 9bis

Simplicial homotopy

(recalled from lecture 3, p. 24)

A simplicial category structure gives rise to a natural notion of homotopy between parallel maps $f, g : m \to n$ of the underlying category:

$$\begin{tikzcd} \Delta^0 \arrow[d, "d^1"'] \arrow[dr, "f"] & \\ \Delta^1 \arrow[r, "H"] & \underline{\mathcal{M}}(m,n) \\ \Delta^0 \arrow[u, "d^0"] \arrow[ur, "g"'] & \end{tikzcd} \quad \text{in } \mathbf{sSet}$$

$$\boxed{\begin{tikzcd} \Delta^0 \otimes m \cong m \arrow[d, "d^1 \otimes 1"'] \arrow[dr, "f"] & \\ \Delta^1 \otimes m \arrow[r, "H"] & n \\ \Delta^0 \otimes m \cong m \arrow[u, "d^0 \otimes 1"] \arrow[ur, "g"'] & \end{tikzcd} \quad \text{in } \mathcal{M}}$$

$$\text{in } \mathcal{M} \quad \begin{tikzcd} & n \cong n^{\Delta^0} \\ m \arrow[ur, "f"] \arrow[r, "H"] \arrow[dr, "g"'] & n^{\Delta^1} \arrow[u, "n^{d^1}"'] \arrow[d, "n^{d^0}"] \\ & n \cong n^{\Delta^0} \end{tikzcd}$$

Simplicial homotopy in $\mathcal{M}^{\Delta^{op}}$

$H : \Delta^1 \otimes X \to Y$ amounts to

$$H_n : \underbrace{(\Delta^1 \otimes X)_n}_{\Delta([n],[1]) \cdot X_n \,\cong\, \coprod_{\Delta([n],[1])} X_n} \to Y_n$$

$$H_{n,\, g:[n]\to[1]} : X_n \to Y_n$$

s.t.

$$\begin{tikzcd} X_n \arrow[r, "H_{n,g}"] \arrow[d, "d_i"'] & Y_n \arrow[d, "d_i"] \\ X_{n-1} \arrow[r, "H_{n-1,\, g \circ d^i}"'] & Y_{n-1} \end{tikzcd}$$

9 bis We have

$$\underbrace{|\cdots}_{k}\overset{\overset{(0\leq k\leq n+1)}{\downarrow}}{|}\underbrace{\cdots|}_{n+1-k} \circ d^i = \begin{cases} \underbrace{|\cdots}_{k-1}\overset{\downarrow}{|}\underbrace{\cdots|}_{n+1-k} & (i<k) \\[2ex] \underbrace{|\cdots}_{k}|\underbrace{\cdots|}_{n-k} & (i\geq k) \end{cases}$$

$$\underbrace{|\cdots}_{k}\overset{\overset{(0\leq k\leq n+1)}{\downarrow}}{|}\underbrace{\cdots|}_{n+1-k} \circ s^i = \begin{cases} \underbrace{|\cdots}_{k+1}\overset{\downarrow}{|}\underbrace{\cdots|}_{n+1-k} & (i<k) \\[2ex] \underbrace{|\cdots}_{k}|\underbrace{\cdots|}_{n+2-k} & (i\geq k) \end{cases}$$

The only interesting case is $|\overset{\downarrow}{|}\cdots| \circ d^0 = \overset{\downarrow}{|}\cdots|$:

$$\begin{tikzcd}
\overset{\downarrow}{a_0} \xrightarrow{f_0} a_1 \to \cdots \to a_n \xrightarrow{f_n} d \arrow[d, mapsto, "d_0"] & Fa_0 \arrow[d, "Ff_0"] \arrow[r, mapsto, "H_{n,g}"] \arrow[rr, bend left, "id"] & a_0 \xrightarrow{f_n\circ\cdots\circ f_0} d \xrightarrow{id} \cdots \xrightarrow{id} d \arrow[d, mapsto] & Fa_0 \arrow[d, "F(f_n\circ\cdots\circ f_0)"] \\
\overset{\downarrow}{a_1} \xrightarrow{f_1} \cdots \to a_n \xrightarrow{f_n} d & Fa_1 \arrow[r, mapsto, "H_{n-1,\, g\circ d^0}"'] \arrow[rr, bend right, "F(f_n\circ\cdots\circ f_1)"'] & d \xrightarrow{id} \cdots \xrightarrow{id} d & Fd
\end{tikzcd}$$

A recipe for defining a map from a coproduct to a coproduct

Specifying a map $f : \coprod_{i \in I} c_i \to \coprod_{j \in J} d_j$

amounts to specify maps $f_i : c_i \to \coprod_{j \in J} d_j$ $(\forall i \in I)$

and a tempting way to do it is to specify a map $\alpha : I \to J$ and maps $h_i : c_i \to d_{\alpha(i)}$ and then to define $f_i = c_i \xrightarrow{h_i} d_{\alpha(i)} \xrightarrow[\text{injection}]{\text{coproduct}} \coprod_{j \in J} d_j$.

(Of course, not all f arise in this way)

We speak of f_i (and even h_i) as the components of f.

Realising this homotopy equivalence

Exercise Consider

$$\begin{tikzcd} {[n]} \arrow[d, mapsto] & \underline{\Delta} \arrow[r, "Y = \Delta^\bullet"] \arrow[d, "\delta"'] & \widehat{\Delta} \\ ({[m]},{[n]}) & \underline{\Delta}\times\underline{\Delta} \end{tikzcd}$$

Show that

$$\boxed{\operatorname{Lan}_\delta Y([m],[n]) = \Delta^m \times \Delta^n.}$$

Hint A cocone to some K is a collection $\alpha_{p,f,g} : \Delta^p \to K$ indexed by $f \in \Delta^m_p$, $g \in \Delta^n_p$, which can be instantiated at p and $\mathrm{id} \in \Delta^p_p$.

Lemma *Suppose M is simplicially enriched, tensored, and cotensored, and admits geometric realizations of simplicial objects. Then geometric realization preserves tensors.*

Proof We have

$$K \otimes |X| = \Big(\overbrace{\int^m K_m \cdot \Delta^m}^{\text{density}}\Big) \otimes \Big(\int^n \Delta^n \otimes X_n\Big)$$

(tensor-colimit commutation) $\cong \int^{m,n} (K_m \cdot \Delta^m) \otimes (\Delta^n \otimes X_n)$

$$\cong \int^{m,n} \Delta^m \otimes (\Delta^n \otimes (K_m \cdot X_n))$$

(action) $\cong \int^{m,n} (\Delta^m \times \Delta^n) \otimes (K_m \cdot X_n)$

$$= (\Delta^- \times \Delta^=) \otimes_{\Delta^{op} \times \Delta^{op}} (K_- \cdot X_=)$$

(exercise) $\cong \operatorname{Lan}_\delta(\Delta^\bullet) \otimes_{\Delta^{op} \times \Delta^{op}} (K_- \cdot X_=)$ $\quad K \boxtimes X$

(similar to Proposition p.5) $\cong \Delta^\bullet \otimes_{\Delta^{op}} ((K_- \cdot X_=)\delta) = \Delta^\bullet \otimes_{\Delta^{op}} \overbrace{(K_\bullet \cdot X_\bullet)}^{''} = |K \otimes X|$

Corollary *If M is simplicially enriched, tensored, and cotensored, then geometric realization preserves simplicial homotopy equivalences.*

(homotopies and simplicial)

Proof We have

$$\begin{tikzcd} {|K\otimes X|} \arrow[r, phantom, "\cong"] \arrow[d, "|f\otimes X|"] & K\otimes|X| \arrow[d, "f\otimes|X|"] \\ {|L\otimes X|} \arrow[r, phantom, "\cong"] & L\otimes|X| \end{tikzcd}$$

, and from $H : \Delta^1 \otimes X \to X$, we get

$$\Delta^1 \otimes |X| \cong |\Delta^1 \otimes X| \xrightarrow{|H|} |X|.$$

Simplicial model categories (a preview)

A simplicial model category is

- a simplicial tensored and cotensored category $\underline{\mathcal{M}}$ +
- a model category structure on $\mathcal{M}$ +
- some compatibility.

Simplicial model categories enjoy the foll. properties (that we assume for the time being)

(A1) W contains all simplicial homotopy equivalences
(A2) Cofibrant objects are preserved by tensoring with any simplicial set.
(A3) Cofibrations and trivial cofibrations are preserved by tensoring with any simplicial set.

We shall also use

$\hat{\Delta}$ itself is a simplicial model category

See Lecture 7 for the proof of these properties

A left deformation

see also p. 17 for the global scene

Putting • Theorem p.8 and • corollary p.11 together, and using exercise p.2 we get that

$$\boxed{\begin{tikzcd} B(D(-,d),D,F) \arrow[r, bend left, "|\varepsilon|"] & Fd \arrow[l, bend left, "|\iota|"] \end{tikzcd}}$$

written $\varepsilon_{F,d}$ for short

is a simplicial homotopy equivalence

We have that

- $\varepsilon_{F,d}$ is a weak equivalence in $\mathcal{M}$ (property (A1) of p.12)
- ε_F is a weak equivalence in $\mathcal{M}^D$ (cf lecture 4 p.4)

Thus, we have shown

LEMMA. *The natural weak equivalence $\epsilon\colon B(\mathcal{D},\mathcal{D},F) \Rightarrow F$ makes $B(\mathcal{D},\mathcal{D},-)$ a left deformation* on *$\mathcal{M}^{\mathcal{D}}$.*

not always good enough!

We shall combine this left deformation • $\varepsilon : B(D,D,-) \to 1$ on $\mathcal{M}^D$ with the left deformation • $\varphi \xrightarrow{q} 1$ induced by the cofibrant replacement on $\mathcal{M}$: cf. lecture 4 p.9

$B(D,D,\varphi -) \to 1$ defined by

$$B(D,D,\varphi F) \xrightarrow{\varepsilon_{\varphi F}} \varphi F \xrightarrow{q^F} F$$

$$\boxed{\begin{tikzcd} \mathcal{M}^D \arrow[r, bend left, "\varphi -"{name=U}] \arrow[r, bend right, "id"'{name=D}] \arrow[from=U, to=D, phantom, "q-"] & \mathcal{M}^D \arrow[r, bend left, "{B(D,D,-)}"{name=U2}] \arrow[r, bend right, "id"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\varepsilon"] & \mathcal{M}^D \end{tikzcd}}$$

Homotopy colimits

Main theorem!

already shown!

TO BE SHOWN

Theorem *Let M be a simplicial model category with cofibrant replacement Q and fibrant replacement R. The pair*

$$B(\mathcal{D},\mathcal{D},Q-)\colon M^{\mathcal{D}} \to M^{\mathcal{D}} \qquad B(\mathcal{D},\mathcal{D},Q-) \overset{\epsilon_Q}{\Rightarrow} Q \overset{q}{\Rightarrow} 1$$

is a left deformation for colim: $M^{\mathcal{D}} \to M$. (is a left deformation and colim is left deformable)

Definition We define $\operatorname{hcolim} = \mathbb{L}\operatorname{colim} = \operatorname{colim} \circ B(\mathcal{D},\mathcal{D},Q-)$

i.e. $\operatorname{hcolim} F = \operatorname{colim}\big(B(\mathcal{D},\mathcal{D},QF)\big)$

Before addressing the proof of the theorem (quite involved), we give a simpler formula for $\operatorname{hcolim} F$.

Proposition We have, for all $H\colon \mathcal{D}\to M$, $K\colon \mathcal{D}^{op}\to \hat{\Delta}$ ($\mathcal{D}^{op}\to\hat{\Delta}$)

$$\boxed{K \otimes_{\mathcal{D}} \underbrace{B(\mathcal{D},\mathcal{D},H)}_{\mathcal{D}\to M} \cong B(K,\mathcal{D},H) \cong B(K,\mathcal{D},\mathcal{D})\otimes_{\mathcal{D}} H}$$ (natural)

Proof We prove the first isomorphism (second isom. similar)

We have $K\otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},H) = \int^{d} Kd \otimes \int^{n} \Delta^n \otimes B_n(\mathcal{D}(-,d),\mathcal{D},H)$

$$= \int^{n,d} (Kd\otimes\Delta^n)\otimes B_n(\mathcal{D}(-,d),\mathcal{D},H)$$

$$B(K,\mathcal{D},H) = \int^{n}\Delta^n \otimes \underbrace{B_n(K,\mathcal{D},H)} \qquad = \int^{n}\Delta^n\otimes\Big(\underbrace{\int^{d} Kd\otimes B_n(\mathcal{D}(-,d),\mathcal{D},H)}\Big)$$

$$K\otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},H) = \qquad\qquad = \int^{d} Kd\otimes \coprod_{\vec{d}} (\mathcal{D}(d_n,d)\otimes Hd_0)$$

(tensor-colimit commutation) $\cong \int^{d}\coprod_{\vec{d}}\big(Kd\otimes(\mathcal{D}(d_n,d))\otimes Hd_0\big) \cong \int^{d}\coprod_{\vec{d}}\big[(Kd\times\mathcal{D}(d_n,d))\otimes Hd_0\big]$ § Fubini

$$\underbrace{B_n(K,\mathcal{D},H)} \cong \coprod_{\vec{d}}\Big(\underbrace{\int^{d} Kd\times\mathcal{D}(d_n,d)}_{\cong Kd_n \text{ (co-Yoneda)}}\Big)\otimes Hd_0 \cong \coprod_{\vec{d}}\int^{d}(Kd\times\mathcal{D}(d_n,d))\otimes Hd_0$$ (tensor-colimit commutation)

Hence we have $\boxed{\operatorname{hcolim} F = B(*,\mathcal{D},QF)}$

$= * \otimes_{\mathcal{D}} B(\mathcal{D},\mathcal{D},QF)$

Ken Brown's lemma

and $\twoheadrightarrow$ for a fibration but not needed here

We use $\rightarrowtail$ for a cofibration and $\xrightarrow{\sim}$ for a w.e.

cf. lecture 4 p. 8

All we need to know is that
- [1] every map factors as $\rightarrowtail \xrightarrow{\sim}$
- [2] cofibrations are stable under composition
- [3] cofibrations are stable under pushouts

Lemma cof n W (Ken Brown's lemma). *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and suppose $F\colon \mathcal{M} \to \mathcal{N}$ sends trivial cofibrations between cofibrant objects to weak equivalences. Then F is homotopical on the full subcategory of cofibrant objects.*

Proof Let A, B cofibrant and $w : A \xrightarrow{\sim} B$. Consider

$$\begin{tikzcd}
\emptyset \arrow[r, tail] \arrow[d, tail] & B \arrow[dl, tail] \arrow[d, dashed, "j"] \arrow[ddr, bend left, "1"] & \\
A \sqcup B \arrow[dr] & C \arrow[dr, "q", "\sim"'] & \\
A \arrow[u, tail] \arrow[ur, dashed, "i"'] \arrow[rr, bend right, "w"'] & & B
\end{tikzcd}$$

p.o.

We have $B \rightarrowtail A \sqcup B$ and $A \rightarrowtail A \sqcup B$ by [3]
$B \rightarrowtail C$ and $A \rightarrowtail C$ by [2]
$B \xrightarrow{\sim} C$ and $A \xrightarrow{\sim} C$ by 2-of-3
C cofibrant by [2]

$\Rightarrow$ $Fj : FB \xrightarrow{\sim} FC$ and $Fi : FA \xrightarrow{\sim} FC$

$\Downarrow$ by 2-of-3

$Fq : FC \xrightarrow{\sim} FB$

$\Downarrow$ by 2-of-3

$$\boxed{Fw : FA \xrightarrow{\sim} FB}$$

(yellow) = factorization of $[w, 1]$

Quillen functors (preview)

If $f \in W \cap C$, we say that f is a trivial cofibration

Let $\mathcal{M}$, $\mathcal{N}$ be model categories.

Definition A left Quillen functor $F: \mathcal{M} \to \mathcal{N}$ is a functor that preserves cofibrations, trivial cofibrations, and colimits.

We note that by Ken Brown's lemma, a left Quillen functor preserves weak equivalences between cofibrant objects.

16bis

The setting of homotopical categories is not enough

The setting of homotopical categories was sufficient to formulate the notions of left deformation and left deformable functor.

- We needed an implication simplicial homotopy equivalence $\Downarrow$ weak equivalence

to prove that $B(\mathcal{D}, D, -)$ (and hence $\underbrace{B(D, D, \varphi -)}_{\varphi}$) is a left deformation

For this we need to "upgrade" our homotopical category $\mathcal{M}$ to a simplicial model category

- We need also to prove that colim is left deformable, and in particular that

$$\operatorname{colim} \varphi = \operatorname{hcolim} = B(*, D, \varphi -)$$

is homotopical, and for this we need to consider the Reedy model structure on the intermediate category $\mathcal{M}^{\Delta^{op}}$ (p. 17 and Cours 7 p. 20-29)

[The final part, that colim q φ is a natural equivalence (with $q = \overset{\curvearrowright}{q_\sim} \; \overset{\curvearrowright}{\varepsilon_\sim}$, cf. p. 13) is technical but does not require conceptual additions to the picture (p. 20-21)

Homotopical sense of the bar construction

The functor $B(G,D,-): \mathcal{M}^D \to \mathcal{M}$ factors as

without interaction!

Reedy model structure + $\hat{\Delta}$-tensored-cotensored simplicial model cat.

$$\mathcal{M}^D \xrightarrow[B_\cdot(G,D,-)]{} \mathcal{M}^{\Delta^{op}} \xrightarrow[|-|]{} \mathcal{M}$$

- In $\mathcal{M}^D$ we consider
  - pointwise cofibrant functors
  - pointwise weak equivalences

defined as the n-th component of the counit (remember from lecture 2 p. 19 that sk_{n-1} is a comonad)

- In $\mathcal{M}^{\Delta^{op}}$ we consider
  - Reedy cofibrant simplicial objects, defined as follows
    - The functor sk_n of Lecture 2 exists also for simplicial objects (replace Set by $\mathcal{M}$)
    - We define $L_n X = (sk_{n-1} X)_n$ ($L_n X$ = n-th latching object)

    (for $\mathcal{M}$ = Set, $L_n X$ is the subset of X_n consisting of its degenerate simplices)
    - We say that X is $\boxed{\text{Reedy cofibrant if all maps } L_n X \to X_n \text{ are cofibrant}}$

    called latching maps
  - pointwise weak equivalence
  - We assume the following results (for the time being)

Reedy model structure

For any model category, there is a model structure on $\mathcal{M}^{\Delta^{op}}$ whose cofibrant objects and weak equivalences are as above.

THEOREM [Hirschhorn 18.4.11] *If $\mathbf{M}$ is a simplicial model category, then*

$$|-|: \mathbf{M}^{\Delta^{op}} \to \mathbf{M}$$

is left Quillen with respect to the Reedy model structure. In particular, $|-|$ sends Reedy cofibrant simplicial objects to cofibrant objects and preserves pointwise weak equivalences between them.

consequence of Ken Brown's lemma cf. p. 15

Contrast with corollary p. 11: less weak equivalences between more objects!

What about $B_\cdot(G,D,-)$? next page!

Homotopical properties of the simplicial bar construction

Proposition Let D be a small category and $\mathcal{M}$ be a simplicial model category. Then $B_\bullet(G, D, -)$ sends pointwise cofibrant functors to Reedy cofibrant simplicial objects and preserves pointwise weak equivalences between them.

Exercise Show that $L_n(B_\bullet(G,D,F))$ is the coproduct of the collection $Gd_n \otimes Fd_0$ indexed over the degenerate simplices of $(ND)_n$ and that $L_n(B_\bullet(G,D,F)) \to B_n(G,D,F)$ is the coproduct inclusion.

We need to know:
- •4 cofibrations are stable under coproduct
- •5 trivial cofibrations are stable under coproducts

Proof of proposition By the assumption and by property (A2) (p. 12) we have that all the components $Gd_n \otimes Fd_0$ in $B_n(G,D,F)$ (and a fortiori, of exercise above, in $L_n(B_\bullet(G,D,F))$) are cofibrant. By the exercise again, we can write a push out

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, tail, "\bullet 4"] & L_n(B_\bullet(G,D,F)) \arrow[d, tail, "\bullet 3 \text{ cf. p. 25}"] \\
\bullet \arrow[r] & B_n(G,D,F)
\end{tikzcd}$$

coproduct of the other copies

Let now $\alpha : F_1 \to F_2$ (F_1, F_2 pointwise cofibrant, α pointwise w.e.

We have that $B_\bullet(G,D,\alpha)$ is a coproduct of maps $Gd_n \otimes \alpha_{d_0}$.

Thanks to (A2) and (A3) (p. 12), we can apply Ken Brown's Lemma (p. 15) to $Gd_n \otimes -$ and conclude that $Gd_n \otimes \alpha_{d_0}$ is a w.e.

Finally, we observe that by •5 we can apply Ken Brown's lemma again to the functor $m \sqcup -$ (m arbitrary) and we conclude that $\coprod_{I} Gd_n \otimes \alpha_{d_0}$ is a weak equivalence, as a composite of maps of the form $m_1 \sqcup (Gd_n \otimes \alpha_{d_0}) \sqcup m_2$

$(f \sqcup g = (f \sqcup id) \circ (id \sqcup g))$

Most of the proof of the main theorem

Proposition The functor $B(G,D,-)$ maps pointwise cofibrant functors to cofibrations and preserves pointwise weak equivalences between pointwise cofibrant objects.

Proof By the proposition p.18 and the (assumed) theorem on p.15

Exercise Show that when F is pointwise cofibrant, $B(-,D,F)$ preserves weak equivalences. (Hint: as for proposition p.18, mutatis mutandis.)

We write $\mathcal{Q}$ for $B(D,D,Q-)$. We have to prove (cf. lecture 4, p.8)

- *FQ is homotopical* with $F \rightsquigarrow \operatorname{colim}$, $Q \rightsquigarrow \mathcal{Q}$, and $q \rightsquigarrow q-,\ \varepsilon$
- *$FqQ\colon FQ^2 \Rightarrow FQ$ is a natural weak equivalence*

- $\operatorname{colim} \mathcal{Q}$ is homotopical

Proof: We have proved p.14 that $(\operatorname{colim}\mathcal{Q})F = B(*,D,QF)$, and we conclude by the proposition above, noting that QF is pointwise cofibrant

- $$\begin{tikzcd}
\mathcal{M}^D \arrow[r, "\mathcal{Q}"] & \mathcal{M}^D \arrow[r, bend left, "Q-"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "q-"] & \mathcal{M}^D \arrow[r, bend left, "{B(D,D,-)}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\varepsilon"] \arrow[rr, bend left=60, "{B(*,D,-)}"] & \mathcal{M}^D \arrow[r, "\operatorname{colim}"] & \mathcal{M}
\end{tikzcd}$$
is a natural w.e.

$\cong$ p.12

We can write this (following) as $B(*,D,-)(q-)\mathcal{Q}$ followed by $\operatorname{colim}\varepsilon\mathcal{Q}$

- We observe that $(q-)\mathcal{Q}_F\colon Q\,B(D,D,QF) \to \underbrace{B(D,D,QF)}_{\text{proposition}}$ is between pointwise cofibrant functors

Hence by the proposition again $B(*,D,-)(q-)\mathcal{Q}$ is a natural w.e.

- We need some preparations to prove that $\operatorname{colim}\varepsilon\mathcal{Q}$ is a natural w.e.

(Remark: we need the proposition above for $G = D(-,d)$ and $G = *$)

recall $B(G,D,D) = (d' \mapsto B(G,D,D(d',-)))$

Some calculations

Exercise[1] Show that there exists a natural weak equivalence $\boxed{\varepsilon^G : B(G,D,D) \to G}$. (Hint: mimick the treatment of $B(D,D,F)$.)

Exercise[2] Show that the following diagram commutes

$$\begin{tikzcd}
B(D(-,d),D,H) \arrow[r, "(\varepsilon_H)_d"] \arrow[d, phantom, "\cong"' , "\text{proposition p. 14}"] & Hd \arrow[d, phantom, "\cong", "\text{co-Yoneda, cf. p.4}"'] \\
B(D(-,d),D,D)\otimes_D H \arrow[r, "\varepsilon^{D(-,d)}\otimes H"] & D(-,d)\otimes_D H
\end{tikzcd}$$

r ℓ

We have considered $B(D,D,F) : M^D \to M^D$ and $B(G,D,D) : M^{D^{op}} \to M^{D^{op}}$.
We have also $B(D,D,D) : D^{op} \times D \to Set$ defined by

$$B(D,D,D)(d',d) = B(D(-,d),D,D(d',-))$$

Exercise[3] Show that $\boxed{(\varepsilon_{D(d',-)})_d = (\varepsilon^{D(-,d)})_{d'}} : B(D(-,d),D,D(d',-)) \to D(d',d)$.

r ℓ

Exercise[4] Show that the following diagram commutes, for arbitrary H

$$\begin{tikzcd}
(d' \mapsto H \otimes_D B(D,D,D(d',-))) \arrow[r, "(d' \mapsto H\otimes_D \varepsilon_{D(d',-)})"] \arrow[d, phantom, "\cong"', "\text{proposition p.14}"] & (d' \mapsto H \otimes_D D(d',-)) \arrow[d, phantom, "\cong", "\text{co-Yoneda cf. p.4}"'] \\
B(H,D,D) \arrow[r, "\varepsilon^H"] & H
\end{tikzcd}$$

r ℓ

Note the wiper movements $\boxed{\begin{matrix} \ell \longleftrightarrow r \\ \varepsilon^- \quad \varepsilon_- \end{matrix}}$

Ending the proof of the main theorem

We are left to prove that $\operatorname{colim} \varepsilon \varphi$ is a natural w.e.

$$\underbrace{\operatorname{colim}\big(B(D,D,\underbrace{B(D,D,\varphi F)}_{H})\big)}^{*\otimes_D B(D,D,H)} \xrightarrow[\operatorname{colim}\varepsilon_{\underbrace{B(D,D,\varphi F)}_{H}}]{*\otimes_D \varepsilon_H} \overbrace{\operatorname{colim}\big(\underbrace{B(D,D,\varphi F)}_{H}\big)}^{*\otimes_D H}$$

We set $H = B(D,D,\varphi F)$.

r From exercise 2 p.20, we have

$$\begin{tikzcd} B(D,D,H) \arrow[r, "\varepsilon_H"] \arrow[d, phantom, "\wr"] & H \arrow[d, phantom, "\wr"] \\ \big(d \mapsto B(D(-,d),D,D)\otimes_D H\big) \arrow[r, "\alpha"] & \big(d \mapsto D(-,d)\otimes_D H\big) \end{tikzcd} \quad \text{where } \alpha_d = \varepsilon^{D(-,d)} \otimes H$$

applied with $K = B(D,D,D)$

l By applying $*\otimes_D -$ to this diagram and the proposition p.3, we obtain

$$\begin{tikzcd} *\otimes_D B(D,D,H) \arrow[r, "*\otimes_D \varepsilon_H"] \arrow[d, phantom, "\wr"] & *\otimes_D H \arrow[d, phantom, "\wr"] \\ \big(d' \mapsto *\otimes_D B(D,D,D(d',-))\big)\otimes_D H \arrow[r, "\beta\otimes_D H"] & \big(d' \mapsto \underbrace{*\otimes D(d',-)}_{=*\ \text{(lecture 0, p.1)}}\big)\otimes_D H \end{tikzcd}$$

where $\beta_{d'} = *\otimes_D \underbrace{\big(d \mapsto (\varepsilon^{D(-,d)})_{d'}\big)}_{=\ \varepsilon_{D(d',-)}\ \text{by exercise 3 p.20}}$

r By applying $-\otimes_D H$ to the diagram of exercise 4 p.20 we arrive at

$$\begin{tikzcd} *\otimes_D B(D,D,H) \arrow[r, "*\otimes_D \varepsilon_H"] \arrow[d, phantom, "\wr"] & *\otimes_D H \arrow[d, phantom, "\wr"] \\ B(*,D,D)\otimes H \arrow[r, "\varepsilon^*\otimes_D H"] & *\otimes_D H \end{tikzcd}$$

w.e. by 2-of-3!

l Finally, we expand $H = B(D,D,\varphi F)$

$$\begin{tikzcd}[column sep=large] *\otimes_D B(D,D,B(D,D,\varphi F)) \arrow[r, "*\otimes_D \varepsilon_{B(D,D,\varphi F)}", "= \operatorname{colim}\varepsilon\varphi"'] \arrow[d, phantom, "\wr"] & *\otimes_D B(D,D,\varphi F) \arrow[d, phantom, "\wr"] \\ B(*,D,D)\otimes B(D,D,\varphi F) \arrow[r, "\varepsilon^*\otimes_D B(D,D,\varphi F)", "\text{proposition p.14}"'] \arrow[d, phantom, "\wr"] & *\otimes B(D,D,\varphi F) \arrow[d, phantom, "\wr"] \\ B(B(*,D,D),D,\varphi F) \arrow[r, "B(\varepsilon^*,D,\varphi F)"'] & B(*,D,\varphi F) \end{tikzcd}$$

$\underbrace{B(\varepsilon^*,D,\varphi F)}_{\text{w.e. by exercise p.19}}$ ← pointwise cofibrant

l

Hence $\operatorname{colim}\varepsilon\varphi$ is a w.e. by 2-for-3.

Reflecting on the proof of the theorem

What's the point of this "wiper movement" $\underbrace{2 \to \ell \to 2 \to e}$?

It allowed us to "massage" our initial problem of proving that

$$* \otimes_D B(D,D,B(D,D,\varphi F)) \xrightarrow{* \otimes_D \varepsilon_{B(D,D,\varphi F)}} * \otimes_D B(D,D,\varphi F) \text{ is a w.e.}$$

by "moving the ϵ_- and ε^- around" until we "reduced" our problem to

$$B(B(*,D,D),D,\varphi F) \xrightarrow[B(\varepsilon^*,D,\varphi F)]{} B(*,D,\varphi F)$$

where "ε" entered under B on the left, enabling us to use exercise p. 19 to conclude!

TIME for EXAMPLES!

But bear for a few more exercises

More calculations

Recall the configuration $F: D \to \mathcal{M}$, $L: D^{op} \times D \to V$, $G: D^{op} \to V$ and the proposition p.3.

We write $B(L, D, F)$ for the functor $d \mapsto B(L(-,d), D, F)$ and likewise for $B(G, D, L)$.

Exercise[1] Show that there is a natural isomorphism (associativity)

$$\boxed{B(G, D, B(L, D, F)) \cong B(B(G, D, L), D, F)}$$

Exercise Show that

$$\begin{tikzcd} B(G, D, B(D, D, F)) \arrow[r, "B(G, D, \varepsilon_F)"] \arrow[d, "\cong"] \arrow[dr, phantom, "="] & B(G, D, F) \\ B(B(G, D, D), D, F) \arrow[ur, "B(\varepsilon^G, D, F)"'] & \end{tikzcd}$$

We can further extend the notation $B(L, D, F)$ for $L': D^{op} \times C \to V$ by setting $B(L', D, F)c = B(L'(-, c), D, F): C \to \mathcal{M}$.

Exercise[2] Show that for $K: C \to D$, we have (with $L: D^{op} \times D \to V$ and $F: D \to \mathcal{M}$)

$$\boxed{B(L, D, F)K \cong B(L(-, K=), D, F)}$$

Exercise[3] Let $F: D \to \mathcal{M}$, $G: D^{op} \to V$, and $K: C \to D$. Show that there is a map $B_\bullet(GK, C, FK) \xrightarrow{\alpha_{G,F}} B_\bullet(G, D, F)$ of simplicial objects inducing $B(GK, C, FK) \xrightarrow{\alpha_{G,F}} B(G, D, F)$ natural in F, G.

Exercise[4] Let $F: D \to \mathcal{M}$, $G: D^{op} \to V$, $L: D^{op} \times D \to V$ and $K: C \to D$. Then prove that the foll. diagram commute:

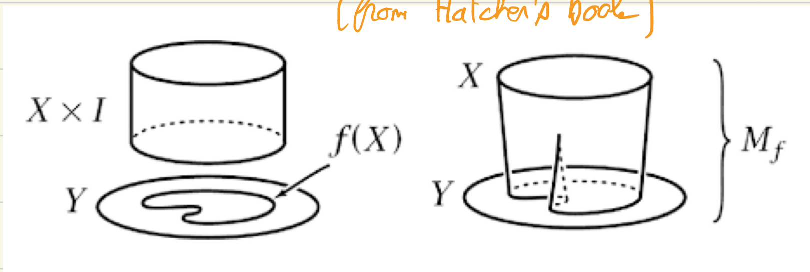
$B_d = \alpha_{G, L(d,-)}$

$$\begin{tikzcd} B(B(GK, C, \underbrace{L(-, K=)}_{d \mapsto L(d,-)K}), D, F) \arrow[r, "B(B, D, F)"] \arrow[d, "\text{associativity}"'] & B(B(G, D, L), D, F) \arrow[d, "\text{associativity}"] \\ B(GK, C, \underbrace{B(L(-, K=), D, F)}_{\text{(exercise 2)} \rightsquigarrow B(L, D, F)K}) \arrow[r, "\alpha_{G, B(L, D, F)}"'] & B(G, D, B(L, D, F)) \end{tikzcd}$$

Mapping cylinder as homotopy colimit

Given $f: X \to Y$ in Top, the mapping cylinder M_f is $\boxed{(I \times X) \sqcup Y / (1,x) \sim f(x)}$

(from Hatcher's book)



$0 \to 1$

We shall synthesise M_f as $\mathrm{hcolim}(\bar{f})$, where $\bar{f}: \mathbf{2} \to \mathrm{Top}$ $(\bar{f}(\to) = f)$

By Top, we mean a suitable full subcategory of Top that has the property of being a simplicial model category where (assumed)

$$K \otimes X = |K| \times X$$

↖ topological realisation

Moreover, one can show that for Top, the main theorem works without resorting to the cofibrant replacement Q, i.e., we have

$$\boxed{\mathrm{hcolim}\, F = B(*, D, F), \text{ for } F: D \to \mathrm{Top}}$$

(see p. 24 bis)

Exercise: Show that the nerve $N\mathbf{2}$ of $\mathbf{2}$ is 1-skeletal (hint: no space for non-degenerate 2-simplices!)

- By the exercise, and by construction, we thus have that $B_\cdot(*, \mathbf{2}, \bar{f})$ is 1-skeletal
- By the corollary p. 5, we have: $B(*, \mathbf{2}, \bar{f}) = \Delta^\cdot_{\leq 2} \otimes_{\Delta^{op}_{\leq 2}} \underbrace{B_\cdot(*, \mathbf{2}, \bar{f})_{\leq 2}}$ ← abbreviated as $B_\cdot$

$$\mathrm{colim} \left(\begin{tikzcd} \Delta^1 \otimes B_0 \arrow[r, "s_0"] \arrow[dr, "s^0"'] & \boxed{\Delta^1 \otimes B_1} \\ \Delta^0 \otimes B_1 \arrow[ur, shift left, "d^0"] \arrow[ur, shift right, "d^1"'] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \boxed{\Delta^0 \otimes B_0} \end{tikzcd} \right)$$

where $B_0 = X^0 \sqcup Y^1$

$B_1 = X^0 \sqcup X^{\to} \sqcup Y^1$

(superscripts refer to the coproduct components indexed by the 0 and 1 simplices of $N\mathbf{2}$)

The colimit induces the following identifications on $(\Delta^1 \otimes B_1) \sqcup (\Delta^0 \otimes B^0) =$

$\begin{matrix} x^0 & \Delta_0 \\ y^1 & s^0 \end{matrix}$: $(t, x^0) \simeq x^0$, $(t, y^1) \simeq y^1$

$x^{\to} \xrightarrow{d^1} \ldots$, d_1: $x^0 \simeq (0, x^{\to})$

$x^{\to} \xrightarrow{d^0} \ldots$, d_0: $f(x)^1 \simeq (1, x^{\to})$

$\cancel{I \times X^0} \sqcup I \times X^{\to} \sqcup \cancel{I \times Y^1}$

$\sqcup / \simeq$

$\cancel{X^0} \sqcup Y^1$

24bis

The case of Top (sketch)

In the case of Top, we can shortcut the need for a cofibrant replacement in $\mathcal{M}$, and the need of using a (Reedy) model structure on $\mathcal{M}^{\Delta^{op}}$.
We actually only need the homotopical category structure on Top
(w.e. = weak homotopy equivalences, i.e. maps $f: X \to Y$ inducing isomorphisms $\pi_n(X, x) \to \pi_n(Y, f(x))$ for all x and n).

- One can show (CHT Section 14.5) that for $\mathcal{M} = \mathrm{Top}$ geometric realisation maps pointwise weak equivalences between so-called split simplicial spaces in $\mathrm{Top}^{\Delta^{op}}$ to weak equivalences.
- Hence, to prove that hocolim $= B(*, D, -)$ (cf. p. 24) is homotopical, it is enough to prove that $●_1$ $B_\bullet(*, D, F)$ is split (for all F) (see CHT Example 14.5.3) and $●_2$ $B(*, D, -)$ maps pointwise w.e. to pointwise w.e., which follows from the following

($●_1$ and $●_2$ replace the proposition p. 18)

Exercise Show that coproducts of weak hom. equiv. are weak hom. equiv. (hint: connected components of a coproduct of spaces are connected components of the components of the coproduct).

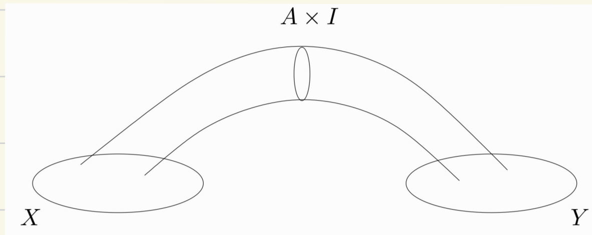
- Finally, we note that ● the proposition on p. 19 was only needed to establish that $B(*, D, -)(q_-)\varphi$ was a w.e. ← does not apply here

● the exercise on p. 19 (used in the final argument showing that $\mathrm{colim}\, \varepsilon\varphi$ is a natural w.e.) now needs to hold for all F. This follows if we prove that $●_1$ and $●_2$ above hold for arbitrary G instead of $*$. As for $●_1$ the argument given in CHT 14.5.3 works the same (even if spelled out only for $*$ in the book).
For $●_2$, this follows from the following

Exercise Let $X, Y, Z \in \mathrm{Top}$, $f: X \to Y$ w.h.e. Show that $Z \times f: Z \times X \to Z \times Y$ is a w.h.e.
(hint: universal property of products!)

Homotopy pushout

Exercise Consider a situation $\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \\ X & & \end{array}$ in Top as a functor F from $\begin{array}{ccc} \cdot & \to & \cdot \\ \downarrow & & \\ \cdot & & \end{array}$ to Top. Show that hcolim F is what this picture predicted:



(Hint, follow the same steps as in p. 24, including 1-skeletality!)

We note that the homotopy push-out

$$Z = (X \sqcup Y \sqcup I\times A) \,/\, \underbrace{(0,a)\sim f(a),\ (1,a)\sim g(a)}$$

has associated with it a homotopy $H: I\times A \to Z$ that "fills the square"

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "H"] & Y \arrow[d, "\delta"] \\ X \arrow[r, "\gamma"'] & Z \end{tikzcd}$$

i.e. H is a homotopy from δf to γg

(clearly, that $H(0,-) = \gamma f$ and $H(1,-) = \delta g$ follows exactly from the identifications)

quotient inclusions

Exercise Show that Z, δ, γ, H satisfy the following universality property:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\delta"] \arrow[ddr, bend left, "k"] & \\ X \arrow[r, "\gamma"'] \arrow[drr, bend right, "h"'] & Z \arrow[dr, dashed, "\ell"] \arrow[dr, phantom, "H'"'] & \\ & & Z' \end{tikzcd}$$

$\forall Z', h, k, H'$ (homotopy from hf to kg)

$\exists!\, \ell: Z \to Z'$ s.t. $\boxed{\begin{cases} \ell\gamma = h \\ \ell\delta = k \\ H' = \ell H \end{cases}}$

A simple functor tensor product formula for hocolim F

similar to, but not identical with, exercise p.4

Lemma Let $X \in \hat{\Delta}$, considered as a simplicial object $\tilde{X}$ in $\hat{\Delta}$, setting $\tilde{X}_n = \overline{X_n}$, hence $\tilde{X}_{nm} = X_n$. Then $|\tilde{X}| = X$.

Proof Colimits of simplicial sets are pointwise. Therefore

$$|\tilde{X}|_m = \int^n \Delta^n_m \cdot \underbrace{\tilde{X}_{nm}}_{= X_n} \simeq X_m \qquad \text{co-Yoneda}$$

Recall $B(*, D, D)$ from p.7:

$$B_n(*, D, D(d', -)) = \coprod_{d_0 \to \cdots \to d_n} \{*\} \cdot D(d', d_0) \simeq \overbrace{\coprod_{d_0 \to \cdots \to d_n} D(d', d_0)}^{\in Set,\ \text{considered as simplicial set}}$$

Lemma We have $\boxed{B(*, D, D) \simeq N(-/D)}$ $: D^{op} \to \hat{\Delta}$.

$$d'/D \qquad \begin{tikzcd} & d' \arrow[dl] \arrow[dr] & \\ \cdot \arrow[rr] & & \cdot \end{tikzcd}$$

Proof We continue the computation: $B_n(*, D, D(d', -)) \simeq \coprod_{d_0 \to \cdots \to d_n} D(d', d_0) = \coprod_{d' \to d_0 \to \cdots \to d} \{*\} \simeq \underbrace{N(d'/D)_n}_{\text{considered as a (constant) simplicial set}}$

$$\begin{tikzcd} & d' \arrow[dl] \arrow[d] \arrow[dr] & \\ d_0 \arrow[r] & \cdots \arrow[r] & d_n \end{tikzcd}$$

Thus $B_\cdot(*, D, D(d', -)) \simeq \widetilde{N(d'/D)}$

$\Downarrow$ lemma above

$$B(*, D, D(d', -)) \simeq N(d'/D)$$

Exercise Show that, for $K : C \to D$, $B(*, C, D(-, K=)) \simeq N(-/K)$.

I call this the nerve formula

Corollary We have $\boxed{\operatorname{hocolim} F \simeq N(-/D) \otimes_D QF}$ $\longleftarrow$ original definition of Bousfield-Kan

Proof We have

$$\underbrace{\operatorname{hocolim} F \simeq}_{\text{cf. p.14}} B(*, D, QF) \underbrace{\simeq}_{\text{proposition p.14}} B(*, D, D) \otimes_D QF \underbrace{\simeq}_{\text{lemma above}} N(-/D) \otimes_D QF$$

ILLUSTRATION on p. 26 bis

26bis

Computing the homotopy pushout using the naive formula

Let $D = $
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2 & \end{tikzcd}$$
and $F : D \to \mathcal{M} = $
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

We observe

$N(1/D) = *$ (only one 0-simplex $1 \xrightarrow{id} 1$, etc)

$N(2/D) = *$

$N(0/D)_0 = \{\bar{0}, \to, \downarrow\}$, $\bar{0} = 0 \xrightarrow{id} 0$

only two non-degenerate 1-simplices $0 \to 0 \to 1$, $0 \to 0 \to 2$

all ≥ 2-simplices degenerate

$$N(0/D) = \begin{tikzcd} \to & \bar{0} \arrow[l, dash] \arrow[d, dash] \\ & \downarrow \end{tikzcd}$$

Let $Z = N(-/D) \otimes_D F$. We have

$R : \overbrace{N(1/D)}^{*} \otimes \overbrace{F1}^{B} \to Z$, $\underbrace{N(1/D) \otimes F1}_{B}$

$R : \underbrace{N(2/D) \otimes F2}_{C} \to Z$

$H : \underbrace{N(0/D) \otimes F0}_{(I \sqcup_{1\,0} I) \times A \,\simeq\, I} \to Z$

We have

$N(\to/D)(1 \xrightarrow{id} 1)$

$= id_0 \to\ = \to$

$$\begin{tikzcd} \to & \bar{0} \arrow[l, dash] \arrow[d, dash] \\ & \downarrow \end{tikzcd}$$

$*$

$$\begin{tikzcd}[column sep=small] & & Z & \\ (0,x) \quad I \times A \simeq N(0/D) \otimes F0 \arrow[urr, "H"] & & & N(1/D) \otimes F1 \simeq B \arrow[ul, "R"'] \\ & N(\to/D) \otimes F0 & & \\ & & N(1/D) \otimes F0 \simeq A \arrow[uull] \arrow[uur, "f"'] & \end{tikzcd}$$

x

cf. p. 25 !

Contractible simplicial sets

Recall that $\hat{\Delta}$, as a $\hat{\Delta}$-category, allows for the notion of simplicial homotopy and homotopy equivalence. (lecture 3, p.25)

Definition A simplicial set K is contractible if the unique map $K \to \Delta^0$ is a homotopy equivalence.

terminal in $\hat{\Delta}$

Exercise[1] Recall $\subseteq : \Delta \to Cat$ from lecture 2 p.13. Prove that

$N(\subseteq[n]) \cong \Delta^n$

(Hint: say, the sequence $\circlearrowleft \longrightarrow \circlearrowleft\circlearrowleft \longrightarrow \circlearrowleft \longrightarrow$ is encoded as 0111223)

Exercise[2] Show that for categories C, D, functors $F, G : C \to D$, a natural transformation $\alpha : F \to G$ can be described equivalently as a functor $\mathcal{I} : C \times \mathbf{2} \to D$.

Exercise[3] Show that N takes natural transformations to simplicial homotopies (hints: $\mathbf{2} = \subseteq[1]$, N is a right adjoint)

Lemma If D is a category with an initial object, ND is contractible.

Proof Consider the two functors

$\mathbb{1} \xrightarrow{\bar{0}} D$ (pick initial object) and $D \xrightarrow{!} \mathbf{1}$ ($\mathbf{1}$ terminal category $\circlearrowleft$).

Since $\mathbf{1} = \subseteq[0]$, by the exercise[1], we have $N\bar{0} : \Delta^0 \to ND$, $N! : ND \to \Delta^0$

- We have $!\,\bar{0} = id$, hence $N!\, N\bar{0} = id$
- We have a nat. transformation $\alpha : \bar{0}\,! \to id$ (initiality), which by exercise[3] yields a homotopy from $N\bar{0}\, N!$ to $N id = id$.

As a consequence of the lemma, we have

$$\boxed{N(d/\Delta) \text{ is contractible}}$$

objects $\rightsquigarrow$ $\begin{array}{c} d \\ \downarrow \\ c \end{array}$ $\quad$ $\begin{array}{c} d \\ {\scriptstyle id}\downarrow \\ d \end{array}$ initial!

Homotopy finality

Definition A functor $K: C \to D$ is called homotopy final if $N(d/K)$ is contractible for all d.

THEOREM (homotopy finality). *Let $F\colon \mathcal{D} \to \mathcal{M}$ be any diagram in a simplicial model category. If $K\colon C \to \mathcal{D}$ is homotopy final, then $\operatorname{hocolim}_C FK \to \operatorname{hocolim}_{\mathcal{D}} F$ is a weak equivalence.*

Proof By the lemma p.27, the assumption, and by assumption (A1) p.12, we get that $N(d/K) \to N(d/D)$ is a weak equivalence

$$\begin{tikzcd} N(d/K) \arrow[rr] \arrow[dr] & & N(d/D) \arrow[dl] \\ & \Delta^0 & \end{tikzcd}$$

2-for-3

$$(d,\ c_0 \xrightarrow{f_1} c_1 \cdots \xrightarrow{f_n} c_n,\ g: d \to Fc_0)$$
$$\downarrow$$
$$(d,\ Fc_0 \xrightarrow{Ff_1} Fc_1 \cdots \xrightarrow{Ff_n} Fc_n,\ g: d \to Fc_0)$$

By the second lemma + exercise p.26, the map $\to$ is the map

$$\alpha_{*,D(d,-)} : B(\underset{\substack{W\\=*K}}{*}, C, D(d,-)K) \to B(*, D, D(d,-)) \text{ of exercise 3, p.23}$$

w.e. by exercise p.19

$$\begin{tikzcd}
B(B(*, C, \mathcal{D}(-, K-)), \mathcal{D}, QF) \arrow[r, "\sim"] \arrow[d, "\cong"'] & B(B(*, \mathcal{D}, \mathcal{D}), \mathcal{D}, QF) \arrow[d, "\cong"] \\
B(*, C, B(\mathcal{D}(-, K-), \mathcal{D}, QF)) \arrow[r, "\alpha"] \arrow[d] & B(*, \mathcal{D}, B(\mathcal{D}, \mathcal{D}, QF)) \arrow[d] \\
B(*, C, QFK) \arrow[r, "\alpha_{*,QF}"'] \arrow[d, "\cong"'] & B(*, \mathcal{D}, QF) \arrow[d, "\cong"] \\
\operatorname{hocolim}_C FK \arrow[r] & \operatorname{hocolim}_{\mathcal{D}} F
\end{tikzcd}$$

exercise 4 p.23

naturality

The two maps $\downarrow$ are weak equivalences by Lemma p.13 + proposition p.19

Hence $\longrightarrow$ is a weak equivalence by 2-for-3.

Quillen's Theorem A

Exercise Show that for any category C and any n we have $\int^{c} N(c/C)_n \simeq (NC)_n$

(Hint: given a cone $(\lambda_c : N(c/C)_n \to A)_{c\in C}$ and $c_0 \to \cdots \to c_n \in (NC)_n$, map it to

$$\lambda_{c_0}\left(\begin{smallmatrix} & c_0 & \\ {}_{id}\swarrow & & \searrow \\ c_0 & \dashrightarrow & c_n\end{smallmatrix}\right)\Big)$$

Top is ~~tensored~~ a simplicial category by setting $\underset{\text{simpl. set}}{L} \otimes \underset{\text{space}}{X} = |L| \times X$ — topological realisation

Lemma We have, for $\begin{array}{l} * : D \to Top \\ \quad d \mapsto \{*\} \end{array}$: $\boxed{N(-/D)\otimes_D * = |ND|}$

Proof We have $N(-/D)\otimes_D * = \int^d |N(d/D)| = \int^d \int^n N(d/D)_n \cdot \Delta^n_{top}$

$\cong$ Fubini

$$|ND| = \int^n (ND)_n \cdot \Delta^n_{top} \underset{\text{exercise}}{\simeq} \int^n \left(\int^d N(d/D)_n\right) \cdot \Delta^n_{top}$$

In the model structure on $\hat{\Delta}$ (due to Quillen), weak equivalences are defined by $f : X \to Y$ is a w.e. iff $|f| : |X| \to |Y|$ is a w.e. in Top (itself defined by the property of inducing isomorphisms of homotopy groups = black box for us here).

Corollary (Quillen's Theorem A). *If $K : C \to \mathcal{D}$ is homotopy final, then $NC \to N\mathcal{D}$ is a weak equivalence.*

Proof Remember from p.24 that cofibrant replacement in Top is not needed.

We apply the homotopy finality theorem of p.28 to $* : D \to Top$ and $* = *K : C \to Top$:

$$|NC| \underset{\text{lemma}}{\simeq} N(-/C)\otimes_C (*K) \underset{\text{corollary p.26}}{\simeq} \mathrm{hcolim}(*K) \underset{\text{w.e.!}}{\xrightarrow{\sim}} \mathrm{hcolim}(*) \simeq N(-/D)\otimes_D * \simeq |ND|$$