Lecture 6 Weighted limits and colimits

Limits are traditionally defined by an adjunction.

$$\mathcal{M}^{\mathcal{C}} \underset{\Delta}{\overset{\lim}{\underset{\top}{\rightleftarrows}}} \mathcal{M} \qquad \mathcal{M}(m, \lim F) = \mathcal{M}^{\mathcal{C}}(\Delta m, F)$$

constant functor

We can repackage $\underbrace{\mathcal{M}^{\mathcal{C}}(\Delta m, F)}_{\lambda = c \mapsto \lambda_c : m \to F_c}$ as $\underbrace{\mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(m,F))}_{\lambda = c \mapsto \{*\} \mapsto \mathcal{M}(m,F)_c}$ $\qquad \mathcal{M}(m, F_c) \overset{\Delta}{=}$

$$\text{Thus} \quad \boxed{\begin{array}{l} \mathcal{M}(m, \lim F) \simeq \mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(m,F)) \\ \mathcal{M}(\mathrm{colim}\, F, m) \simeq \mathrm{Set}^{\mathcal{C}^{op}}(*, \mathcal{M}(F,m)) \end{array}} \qquad m \xrightarrow{\lambda_c} F_c$$

and dually

- As a sign that this reformulation is useful, here is a compact proof that right adjoints preserve limits:

$$\mathcal{M}(m, \underline{R\lim F}) \simeq \mathcal{M}(Lm, \lim F) \simeq \mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(Lm, F))$$

$$\mathcal{M}(m, \underline{\lim RF}) \simeq \mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(m, RF)) \qquad (\mathcal{M}(Lm,F) \cong \mathcal{M}(m,RF))$$

- Replacing $*$ by $W : \mathcal{C} \to \mathrm{Set}$ (resp. by $W : \mathcal{C}^{op} \to \mathrm{Set}$) we arrive at the notion of weighted limits (resp. weighted colimits!)

$$\boxed{\begin{array}{l} \mathcal{M}(m, \lim^W F) \simeq \mathrm{Set}^{\mathcal{C}}(W, \mathcal{M}(m,F)) \\ \mathcal{M}(\mathrm{colim}^W F, m) \simeq \mathrm{Set}^{\mathcal{C}^{op}}(W, \mathcal{M}(F,m)) \end{array}} \qquad m \xrightarrow{\lambda_{c,a}} F_c \quad (a \in W_c)$$

i.e. $\lim^W F$ exists iff $m \mapsto \mathrm{Set}^{\mathcal{C}}(W, \mathcal{M}(m,F)) : \mathcal{C}^{op} \to \mathrm{Set}^{\mathcal{E}}$ is <u>representable</u>

Yet a new notion? No! $\mathrm{colim}^W F$ is an old acquaintance!

(next page)

Weighted colimit as functor tensor product

We have
$$\mathrm{Set}^{C^{op}}[W, \mathcal{M}(F, m)] = \int_c \mathcal{M}(Fc, m)^{Wc}$$
Lecture 0 p. 6
$$= \int_c \mathcal{M}(Wc . Fc, m)$$
coproduct definition
$$= \mathcal{M}\left(\int^c Wc . Fc, m\right)$$
Lecture 0, p. 1

Thus $\boxed{\mathrm{colim}^W F = W \otimes_C F}$

And dually $\lim^W F = \int_c Fc^{Wc}$

Conical limits

$*(c) = \{*\} \in \mathrm{Set}$

The isomorphism $\quad \mathcal{M}(m, \lim F) \simeq \mathrm{Set}^{C}(*, \mathcal{M}(m,F))$ (in Set)

suggests the following definition, for $\mathcal{V}$-categories $\underline{C}, \underline{\mathcal{M}}$, and a $\mathcal{V}$-functor $F : \underline{C} \to \underline{\mathcal{M}}$

{ notation $\underline{\lim}\, F$

Definition For $\underline{\mathcal{M}}, \underline{C}, F$ as above, an enriched limit of F is an object n together with

(note the change: now iso in $\mathcal{V}$) $\qquad *(c) = *$ (unit object of $\mathcal{V}$)

(for all m, natural in m) $\qquad \underline{\mathcal{M}}(m, n) \simeq \underbrace{\underline{\mathcal{V}}}_{\text{cf. lecture 3, p.21}}{}^{\underline{C}}(*, \underline{\mathcal{M}}(m, F))$

recalled next page

If $F : C \to \mathcal{M}$ is an ordinary functor, viewed as a $\mathcal{V}$-functor from the free $\mathcal{V}$-cat. over C cf. lecture 3 p.14, then we write $\lim^* F$ for $\underline{\lim}\, F$, and call it a conical limit of F.

Lemma If $\mathcal{V}$ is symm. mon. closed, and $G : D \to \mathcal{V}$ has a colimit, then $\boxed{\underline{\mathcal{V}}(\mathrm{colim}\, G, w) \simeq \lim \underline{\mathcal{V}}(G, w)}$ naturally in w

Proof We have

$\mathcal{V}(v, \underline{\mathcal{V}}(\mathrm{colim}\, G, w)) \simeq \mathcal{V}(\mathrm{colim}\, G, \underline{\mathcal{V}}(v,w)) \simeq \mathrm{Set}^{C^{op}}(*, \mathcal{V}(G, \underline{\mathcal{V}}(v,w)))$

$\mathcal{V}(v, \lim \underline{\mathcal{V}}(G, w)) \simeq \mathrm{Set}^{C^{op}}(*, \mathcal{V}(v, \underline{\mathcal{V}}(G, w)))$ and we conclude since

$$\mathcal{V}(G, \underline{\mathcal{V}}(v,w))_c \simeq \mathcal{V}(Gc \otimes v, w) \simeq \mathcal{V}(v \otimes Gc, w) \simeq \mathcal{V}(v, \underline{\mathcal{V}}(G,w))_c$$

Exercise Show likewise that if $\underline{\mathcal{M}}$ is a tensored and cotensored $\mathcal{V}$-category, and if $G : D \to \mathcal{M}$ has a colimit, then $\boxed{\underline{\mathcal{M}}(\mathrm{colim}\, G, n) \simeq \lim \underline{\mathcal{M}}(G, n)}$ naturally in n

("likewise" is the hint!)

Proposition If $\underline{\mathcal{M}}$ is tensored, a conical limit is a fortiori an "ordinary" limit, and conversely.

Proof NEXT PAGE

Enriched ends

For $\underline{H} : \underline{C}^{op} \otimes \underline{C} \to \underline{D}$ (enriched functor) we set

enriched end

$$\boxed{\int_{\underline{C}} \underline{H}(c,c) = \lim \Big(\prod_c \underline{H}(c,c) \rightrightarrows \underbrace{\prod_{c,c'} \underbrace{\underline{H}(c,c')}_{D}^{\overbrace{\underline{C}(c,c')}^{V}}}_{D} \Big)}$$

(contrast with $\int_c H(c,c) = \lim \Big(\prod_c H(c,c) \rightrightarrows \prod_{c,c'} \prod_{f : c \to c'} H(c,c') \Big)$ "gone to be enriched")

Thus we can reformulate the object of natural transf.

$$\boxed{\underline{V}^{\underline{D}}(F,G) = \int_{\underline{C}} \underline{D}(Fc, Gc)}$$

One defines enriched coends dually.

Limits versus conical limits

The key is here!

We note that

1 $\underline{V}^{\mathcal{C}}(*, \underline{\mathcal{M}}(m,F)) = \lim\Big(\prod_c \underbrace{\underline{V}(*, \underline{\mathcal{M}}(m,Fc))}_{\underline{\mathcal{M}}(m,Fc)} \rightrightarrows \prod_{c,c'} \underbrace{\underline{V}\big(\coprod_{\mathcal{C}(c,c')} *, \underline{\mathcal{M}}(m,Fc')\big)}_{\prod_{\mathcal{C}(c,c')} \underline{\mathcal{M}}(m,Fc') \text{ lemma p.3}}\Big)$

2 $\mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(m,F)) \simeq \lim\Big(\underbrace{\mathrm{Set}(*, \prod_c \mathcal{M}(m,Fc))}_{\prod_c \mathcal{M}(m,Fc)} \rightrightarrows \underbrace{\mathrm{Set}(*, \prod_{c,c',\mathcal{C}(c,c')} \mathcal{M}(m,Fc'))}_{\prod_{c,c',\mathcal{C}(c,c')} \mathcal{M}(m,Fc')}\Big)$

(noticing $\mathrm{Set}(*,A) \simeq A$)

- If n is a conical limit, then $\mathcal{M}(m,n) = V(*, \underline{V}^{\mathcal{C}}(*, \underline{\mathcal{M}}(m,F)))$.

Since $V(*,-)$ preserves limits, we obtain by 1

$\mathcal{M}(m,n) \simeq \lim\Big(\underbrace{V(*, \prod_c \underline{\mathcal{M}}(m,Fc))}_{\prod_c \mathcal{M}(m,Fc)} \rightrightarrows \underbrace{V(*, \prod_{c,c',\mathcal{C}(c,c')} \underline{\mathcal{M}}(m,Fc'))}_{\prod_{c,c',\mathcal{C}(c,c')} \mathcal{M}(m,Fc')}\Big)$

$\simeq \mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(m,F))$ by 2

- We show conversely that $\lim F$ is conical, by showing

$\underbrace{V(v, \underline{\mathcal{M}}(m, \lim F))}_{\mathcal{M}(v\otimes m, \lim F)} \simeq \underbrace{V(v, \underline{V}^{\mathcal{C}}(*, \underline{\mathcal{M}}(m,F)))}_{\text{since } V(v,-) \text{ preserves limits and by } 1}$

$\simeq \mathrm{Set}^{\mathcal{C}}(*, \mathcal{M}(v\otimes m, F))$

$\lim\Big(\prod_c V(v, \underline{\mathcal{M}}(m,Fc)) \rightrightarrows \prod_{c,c',\mathcal{C}(c,c')} V(v, \underline{\mathcal{M}}(m,Fc'))\Big)$

$\simeq$

by 2

$\lim\Big(\prod_c \mathcal{M}(v\otimes m, Fc) \rightrightarrows \prod_{c,c',\mathcal{C}(c,c')} \mathcal{M}(v\otimes m, Fc')\Big)$

Not all limits are conical

Let $\underline{\mathcal{M}}$ be the 2-category (= Cat-enriched)

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$$

two objects, a b, one 1-morphism f,

$\underline{\mathcal{M}}(a,a) = \underline{\mathcal{M}}(b,b) = \mathbf{1}$ $\quad \underline{\mathcal{M}}(a,b) \simeq \mathbb{N}$ as additive monoid = 1 object cat.

vertical composites of α

The underlying category of $\underline{\mathcal{M}}$ is $\mathbf{2} = a \xrightarrow{f} b$

In $\mathbf{2}$, we have

$$\begin{tikzcd} & b \arrow[dl, "id"'] \arrow[dr, "id"] & \\ b & & b \end{tikzcd}$$

limit cone, i.e. $\underline{b = b \times b}$ [by lack of arrows!]

But b is **not** a conical limit for this diagram.

It would entail by 1 p.4 that

for $C = \overset{\circlearrowleft}{\bullet_0} \quad \overset{\circlearrowleft}{\bullet_1}$ and $F_{\bullet_0} = F_{\bullet_1} = b$

$$\underline{\mathcal{M}}(a,b) \simeq \lim\Big(\prod_c \underline{\mathcal{M}}(a,Fc) \to \prod_{g \in C(c,c')} \underline{\mathcal{M}}(a,Fc') \Big)$$

$$\simeq \prod_c \underline{\mathcal{M}}(a,Fc) \quad \text{[by discreteness of } C\text{]}$$

$$= \underline{\mathcal{M}}(a,b) \times \underline{\mathcal{M}}(a,b)$$

i.e. $\mathbb{N} \simeq \mathbb{N} \times \mathbb{N}$

$\uparrow$

IMPOSSIBLE

since any functor $P : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is determined by $P(1) = (m,n)$ and hence $P(p) = (mp, np)$ for all, which is not full.

Moreover, since $\underline{\mathcal{M}}(a,a) = \mathbf{1} \not\simeq \mathbb{N} \times \mathbb{N}$, $\underset{\cdot}{b} \quad \underset{\cdot}{b}$ has no conical limit in $\underline{\mathcal{M}}$.

A simple example of weighted limit

Take $W : \mathbf{2} \to Set$ "=" $\{0,1\} \to \{*\}$ and $F : \mathbf{2} \to \mathcal{M}$ "=" $f : a \to b$

Then a natural transformation from W to $\mathcal{M}(m, F)$ consists of a function $\{0,1\} \to \mathcal{M}(m,a)$, i.e. two morphisms $g, h : m \to a$ and a morphism $k : m \to b$ s.t. (naturality)

$$\begin{tikzcd} \{0,1\} \arrow[r, "0 \mapsto g \; 1 \mapsto h"] \arrow[d] & \mathcal{M}(m,a) \arrow[d, "f \circ -"] \\ \{*\} \arrow[r, "* \mapsto k"] & \mathcal{M}(m,b) \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} m \arrow[r, "g"] \arrow[d, "h"'] \arrow[dr, "k"] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

We note that k is redundant.

By lecture 0 / p. 10 $Set^C(W, \mathcal{M}(-, F))$ being representable amounts to finding an m_0, and maps π_1, π_2 such that

$$\begin{tikzcd} m_0 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$$

is universal = pullback of f, f! (also called kernel pair)

The use of the weight is an elegant way of encapsulating the repetition of f!

Exercise Synthesise the kernel pair from the formula $\int_c F_c^{W_c}$, with the same F, W as above.

(Hint: one ends up in a rather contrived presentation of $\begin{tikzcd} \cdot \arrow[r] \arrow[d] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$ as a pullback $\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ a \times a \arrow[dr, "f \times f"'] & & b \arrow[dl, "\delta"] \\ & b \times b & \end{tikzcd}$ where $\bullet$ is the same object.)

Morale: The original point of view is worth!

Enriched weighted limits

Enriched weighted limits fill the missing entry in the matrix below

	unenriched	enriched	
limit	$\lim F$	$\underline{\lim}\, F$	
weighted	$\lim^W F$	?	$\underline{\lim}^W F$

DEFINITION 7.4.1. Given a $\mathcal{V}$-functor $F\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ and a $\mathcal{V}$-functor $W\colon \underline{\mathcal{D}} \to \underline{\mathcal{V}}$, the **weighted limit** of F by W, if it exists, is an object $\underline{\lim}^W F$ of $\mathcal{M}$ together with a $\mathcal{V}$-natural isomorphism

$$\underline{\mathcal{M}}(m, \underline{\lim}^W F) \cong \underline{\mathcal{V}}^{\underline{\mathcal{D}}}(W, \underline{\mathcal{M}}(m, F)).$$

Dually, given $F\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ and $W\colon \underline{\mathcal{D}}^{\mathrm{op}} \to \underline{\mathcal{V}}$, the **weighted colimit** of F by W, if it exists, is an object $\underline{\mathrm{colim}}^W F$ of $\mathcal{M}$ together with a $\mathcal{V}$-natural isomorphism

$$\underline{\mathcal{M}}(\underline{\mathrm{colim}}^W F, m) \cong \underline{\mathcal{V}}^{\underline{\mathcal{D}}^{\mathrm{op}}}(W, \underline{\mathcal{M}}(F, m)).$$

Can we find explicit formulas?

A NEW ACQUAINTANCE next page

Enriched functor tensor and cotensor products

Recall the notion of enriched end and coend from lecture 3 p. 22. We can have the computations of page 2.

enriched end

$$\underline{V}^{\underline{C}^{op}}[W, \underline{M}(F, m)] = \int_{\underline{C}} \underline{M}(Fc, m)^{Wc}$$

$$\simeq \int_{\underline{C}} \underline{M}(Wc \otimes Fc, m)$$

cotensor

$$\simeq \underline{M}\left(\int^{\underline{C}} Wc \otimes Fc, m\right)$$

consequence of exercise p. 3

Since this is natural in m, we have

$$\boxed{\underline{\mathrm{colim}}^W F = W \otimes_{\underline{C}} F}$$

enriched functor tensor product

defined as an enriched coend

$$\boxed{W \otimes_{\underline{C}} F = \int^{\underline{C}} Wc \otimes Fc}$$

the underlining here records enrichment of the coend (personal notation!)

enriched functor cotensor product

Dually $\boxed{\underline{\lim}^W F = \{W, F\}}$, where $\boxed{\{W, F\}_{\underline{C}} = \int_{\underline{C}} \{Wc, Fc\}}$

other notation for Fc^{Wc}

~~**Exercise** Show that the proposition p. 3 lifts to weighted limits, i.e., that if $\underline{M}$ is a tensored and cotensored category, if C is an ordinary category and $W: C \to \cancel{M}$ V and $F: C \to M$ are ordinary functors, then~~

~~$$\underline{\lim}^W F = \lim^W F$$~~

STUPID ME: this does not even type-check!
(as W should be both $C \to V$ and $C \to Set$): FORGET IT!

An example of enriched weighted limit

Let D be the unenriched category
$$\begin{tikzcd} & \bullet^{1} \arrow[d] \\ \bullet^{0} \arrow[r] & \bullet^{2} \end{tikzcd}$$
and consider the following functors $W : D \to \mathrm{Cat}$, $F : D \to \mathrm{Cat}$

↑ underlying cat. of Cat

$$\begin{tikzcd} W := & & \mathbb{1} \arrow[d, "1"] \\ \mathbb{1} \arrow[rr, "0"'] & & \mathbb{2} \end{tikzcd} \qquad \begin{tikzcd} F := & & \mathcal{B} \arrow[d, "K"] \\ \mathcal{A} \arrow[rr, "H"'] & & C \end{tikzcd}$$

Cat-enriched coend!

We have
$$\underline{\lim}^{W} F \underset{\Delta}{=} \int_{D} Fc^{Wc} = \int_{D} \underline{\mathrm{Cat}}(Wc, Fc)$$
(cf. p. 9)

It follows that the objects of $\lim^{W} F$ are the natural transformations $\alpha : W \to F$, which amount to triples $(a, b, f : Ha \to Kb)$

α_0 α_1 α_2 naturality

As for the morphisms a natural transformation from a to a' in $\underline{\mathrm{Cat}}(W0, F0)$ is a morphism $h : a \to a'$, idem $k : b \to b'$ and a natural transformation from f to g in $\underline{\mathrm{Cat}}(W2, F2)$ is a commutative square.

Taking into account the identifications induced by the end, we arrive at
$$\begin{tikzcd} Ha \arrow[r, "f"] \arrow[d, "Hh"'] & Kb \arrow[d, "Kk"] \\ Ha' \arrow[r, "f'"] & Kb' \end{tikzcd}$$
i.e. $\lim^{W} F = H/K$

comma category

morphisms = pairs (h, k) s.t. commute

Homotopy colimits are weighted colimits

Recall from lecture 5 p.26 the formula

$$\mathrm{hcolim}\, F \simeq N(-/D) \otimes_D F$$

which we can now rephrase as

$$\boxed{\mathrm{hcolim}\, F \simeq \mathrm{colim}^{N(-/D)} F}$$

This formula unveils the universal property of homotopy colimits (and, dually, of homotopy limits):

hcolim F represents the functor ← this embodies a **UNIVERSAL PROPERTY**

$$\boxed{\mathrm{Set}^{C^{op}}(N(-/D), \mathcal{M}(F, m))}$$

cf. lecture 0 p.10 recalled next page

Exercise Spell out this universal property in the case of homotopy pushouts, thus giving the algebraic facet of the topological exercise of lecture 5, p.25.

Detecting representable functors

A representable functor $C^{op} \to Set$ is a functor F together with an object c_0 of C and a natural isomorphism

$$K : C(-, c_0) \to F$$

Lemma F is representable if and only if there exists an object c_0 of C and an element x_0 of Fc_0 such that the natural transformation $\lambda^{c_0,x_0} : C(-,c_0) \to F$ defined by

$$\left(\lambda^{c_0,x_0}\right)_c \big(\underbrace{f}_{c \to c_0}\big) = \underbrace{Ff x_0}_{\in Fc} \quad \text{is iso}$$

Proof One direction is obvious (qui peut le plus peut le moins).
Conversely, suppose that $K : C(-, c_0) \to F$ is given.
This data is equivalent (Yoneda lemma) to the data of some $x_0 \in Fx_0$, and K is entirely determined by x_0, i.e. $K = \lambda^{c_0,x_0}$.

We say that c_0, x_0 are universal for F.

Weighted / enriched homotopy colimits

We want to apply the machinery of lecture 5 (deformations from bar construction) to weighted colimits.

- One replaces the deformation $\mathcal{M}^D \underset{1}{\overset{B(D,D,-)}{\rightrightarrows}} \mathcal{M}^D$ (forgetting about φ)

  by the deformation
  $$\mathcal{V}^{D^{op}} \times \mathcal{M}^D \underset{1}{\overset{id \times B(D,D,-)}{\rightrightarrows}} \mathcal{V}^{D^{op}} \times \mathcal{M}^D$$
  (here, $\mathcal{V} = \hat{\Delta}$, but see below)

  and one shows that $\mathrm{colim}^- =$ is left deformable, leading to the formula
  $$\boxed{\underline{h\mathrm{colim}}^G F = B(G, D, F)}$$

- But before that, one needs an enriched version of the simplicial bar construction.

DEFINITION Given a small $\mathcal{V}$-category $\underline{\mathcal{D}}$, a tensored $\mathcal{V}$-category $\underline{\mathcal{M}}$, and $\mathcal{V}$-functors $F\colon \underline{\mathcal{D}} \to \underline{\mathcal{M}}$, $G\colon \underline{\mathcal{D}}^{\mathrm{op}} \to \underline{\mathcal{V}}$, the **enriched simplicial bar construction** is a simplicial object $B_\bullet(G, \underline{\mathcal{D}}, F)$ in $\mathcal{M}$ defined by

$$B_n(G,\underline{\mathcal{D}},F) = \coprod_{d_0,\dots,d_n \in \mathcal{D}} \Big(Gd_n \times \underline{\mathcal{D}}(d_{n-1},d_n) \times \cdots \times \underline{\mathcal{D}}(d_0,d_1)\Big) \otimes Fd_0.$$

- It turns out that we can (should) also move from simplicial categories to $\mathcal{V}$-categories. The bar construction is still defined by
  $$B(G,D,F) = \Delta^\bullet \underset{\Delta^{op}}{\otimes} B_\bullet(G,D,F)$$
  enriched

  but now $\Delta^\bullet$ is not the Yoneda embedding $\Delta \to \hat{\Delta}$ anymore, but a given fixed cosimplicial object $\Delta^\bullet : \Delta \to \mathcal{V}$

→ See chapter 9 of Riehl's book or its source Shulman

Homotopy limits and colimits and enriched homotopy theory https://arxiv .org/abs/math/060194

Reflecting on the two facets of homotopy (co)limits

- In lecture 5, we arrived at

$$\mathbb{L}\mathrm{colim}\, F = \mathrm{colim}\, B(D, D, F)$$

which allowed us to use the deformation theory of lecture 4, and hence to guarantee that $\mathbb{L}\mathrm{colim}\_$ is homotopical

and more precisely the left derived functor of colim

- In this lecture, we arrived at

$$\mathbb{L}\mathrm{colim}\, F = \mathrm{colim}^{N(-/D)} F$$

which allowed us to unveil the universality property of homotopy colimits.

We connected the two definitions by proving

$$\mathrm{colim}\, B(D, D, F) \simeq B(*, D, F) \simeq B(*, D, D) \otimes_D F \simeq N(-/D) \otimes_D F$$

lecture 5 p. 14 lecture 5 p. 26

A third facet?

A natural question is whether homotopy colimits are colimits in the homotopy category. Below is a sufficient condition.

The weak equivalences on $\mathcal{M}^D$ are always the pointwise ones. It follows that $\mathcal{M}^D \to (\mathrm{Ho}\,\mathcal{M})^D$ sends weak equivalences to isomorphisms: plainly, if $\alpha : F \to G$ is p.t. α_d is a w.e., for all d, then $\gamma(\alpha_d)$ is an iso!

Therefore we have

$$\begin{tikzcd}
\mathcal{M}^D \arrow[r] \arrow[d, "\gamma_{\mathcal{M}^D}"'] & (\mathrm{Ho}\,\mathcal{M})^D \\
\mathrm{Ho}(\mathcal{M}^D) \arrow[ur]
\end{tikzcd}$$

$\gamma(F) \mapsto \gamma \circ F$

WARNING: limited application!

Works for discrete D (e.g. products)

<u>Proposition</u> If the functor $\mathrm{Ho}(\mathcal{M}^D) \to (\mathrm{Ho}\,\mathcal{M})^D$ above is an iso, then we have $\gamma_{\mathcal{M}}(\mathrm{hlim}\, F) \simeq \lim(\gamma_{\mathcal{M}} \circ F)$ for all $F : D \to \mathcal{M}$.

<u>Proof</u> • We have $\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "\Delta_{\mathcal{M}}"] \arrow[r, phantom, "\perp"] & \mathcal{M}^D \arrow[l, bend left, "\lim"] \end{tikzcd}$. Since Δ is homotopical ($\Delta f_d = f \ \forall f$), Δ is deformable w.r.t. the identity deformation. Hence we can apply the theorem of lecture 4 p.11, so that we have

$$\begin{tikzcd} \mathrm{Ho}\,\mathcal{M} \arrow[r, bend left, "\overline{\Delta_{\mathcal{M}}}"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathcal{M}^D) \arrow[l, bend left, "\overline{\mathrm{hlim}}"] \end{tikzcd}$$

• We have
$$\begin{tikzcd} \mathrm{Ho}\,\mathcal{M} \arrow[r, "\overline{\Delta_{\mathcal{M}}}"] \arrow[rr, bend left, "\Delta_{\mathrm{Ho}\mathcal{M}}"] & \mathrm{Ho}(\mathcal{M}^D) \arrow[r] & (\mathrm{Ho}\,\mathcal{M})^D \end{tikzcd}$$

since $\Delta_{\mathrm{Ho}\mathcal{M}}(\gamma(m))\,d = \gamma(m)$ and $(\to[\overline{\Delta_{\mathcal{M}}}(\gamma(m))])\,d = (\to(\gamma(\Delta_{\mathcal{M}} m)))\,d = (\gamma \circ \Delta_{\mathcal{M}} m)\,d = \gamma(m)$

• Putting this together, we get that $\lim = \overline{\mathrm{hlim}} \circ \to^{-1}$ (both right adjoints of Δ)

$$\begin{tikzcd} \mathrm{Ho}\,\mathcal{M} \arrow[r, bend left] \arrow[r, phantom, "\perp"] \arrow[rr, bend left=40, "\Delta"] & \mathrm{Ho}(\mathcal{M}^D) \arrow[l, bend left, "\overline{\mathrm{hlim}}"] \arrow[r] & (\mathrm{Ho}\,\mathcal{M})^D \arrow[ll, bend left=40, "\lim"] \end{tikzcd}$$

Hence

$$\lim(\gamma \circ F) = \overline{\mathrm{hlim}}\,[\to^{-1}(\gamma \circ F)] = \overline{\mathrm{hlim}}\,[\gamma(F)] = \gamma(\mathrm{hlim}\, F).$$