Lecture 7 Model categories

At last, we introduce the basics of model categories

We also give the definition of simplicial model categories, and we prove most of the assumptions made in lectures 4 and 5.

Lifting problems

Let i and f be arrows in a fixed category $\mathcal{M}$. A **lifting problem** between i and f is simply a commutative square

Note: $A \boxslash B \iff A \subseteq {}^{\boxslash}B \iff B \subseteq A^{\boxslash}$

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "i"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "v"'] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Terminology: RIGHT CLASS of $\mathcal{L}$

LEFT CLASS of $\mathcal{R}$

A **lift** or **solution** is a dotted arrow, as indicated, making both triangles commute. If any lifting problem between i and f has a solution, we say that i has the **left lifting property** with respect to f and, equivalently, that f has the **right lifting property** with respect to i. We use the suggestive symbolic notation $i \boxslash f$ to encode these equivalent assertions.

Suppose $\mathcal{L}$ is a class of maps in $\mathcal{M}$. We write $\mathcal{L}^{\boxslash}$ for the class of arrows that have the right lifting property against each element of $\mathcal{L}$. Dually, we write ${}^{\boxslash}\mathcal{R}$ for the class of arrows that have the left lifting property against a given class $\mathcal{R}$.

Exercise Prove that for all A, B:

- $A \subseteq B \Rightarrow B^{\boxslash} \subseteq A^{\boxslash}$, ${}^{\boxslash}B \subseteq {}^{\boxslash}A$
- $A \subseteq {}^{\boxslash}(A^{\boxslash})$, $A \subseteq ({}^{\boxslash}A)^{\boxslash}$
- $A^{\boxslash} = ({}^{\boxslash}(A^{\boxslash}))^{\boxslash}$, ${}^{\boxslash}A = {}^{\boxslash}(({}^{\boxslash}A)^{\boxslash})$

A diagram whose domain is the ordinal α is called an α-**composite** if, for each limit ordinal $\beta < \alpha$, the subdiagram indexed by β is a colimit cone. An ω-composite might also be called a countable composite. To say that ${}^{\boxslash}\mathcal{R}$ is closed under transfinite composition means that if each arrow in an α-composite between the images of some ordinal and its successor is in ${}^{\boxslash}\mathcal{R}$, then the "composite" arrow from the image of zero to the image of α is also in ${}^{\boxslash}\mathcal{R}$.

colimit cone

$= \omega$-composite of $\to \to \cdots$

Alternative def. inverted

Lemma *Any class of arrows of the form ${}^{\boxslash}\mathcal{R}$ is closed under coproducts, pushouts, transfinite composition, retracts, and contains the isomorphisms.*

Proof

Suppose $i \in {}^{\boxslash}\mathcal{R}$

$$\begin{tikzcd} \cdot \arrow[r, "s_1"] \arrow[rr, bend left, "1"] \arrow[d, "j"'] & \cdot \arrow[r, "r_1"] \arrow[d, "i"] & \cdot \arrow[d, "j"] \\ \cdot \arrow[r, "s_2"'] \arrow[rr, bend right, "1"'] & \cdot \arrow[r, "r_2"'] & \cdot \end{tikzcd} \quad + \quad \begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "j"'] & \cdot \arrow[d, "r \in \mathcal{R}"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r, "s_1"] \arrow[rr, bend left, "1"] \arrow[d, "j"'] & \cdot \arrow[r, "r_1"] \arrow[d, "i"] & \cdot \arrow[r, "u"] \arrow[d, "j"] & \cdot \arrow[d, "r \in \mathcal{R}"] \\ \cdot \arrow[r, "s_2"'] \arrow[rr, bend right, "1"'] & \cdot \arrow[r, "r_2"'] \arrow[urr, dashed] & \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

The rest of the statement is left as Exercise

2b6

An inductive definition of α-composite

and α-composition of morphisms of some class K

α-composites are particular diagrams $D : \alpha \to \mathcal{M}$.

- Any object of $\mathcal{M}$ (i.e. diagram $0 \to \mathcal{M}$) is a 0-composite
- For $\alpha = \operatorname{succ} \beta$, if $D : \beta \to \mathcal{M}$ is a β-composite, then any $D' : \alpha \to \mathcal{M}$ obtained by extending D to α by choosing $D(\beta \leq \alpha)$ in K is an α-composite.
- For $\alpha = \lim \{\beta \mid \beta < \alpha\}$, if D is a diagram indexed by $\{\beta \mid \beta < \alpha\}$ such that, for all $\beta < \alpha$, the subdiagram indexed by $\{\gamma \mid \gamma \leq \beta\}$ is a β-composite, then the colimit cone of D, viewed as a diagram indexed by $\{\beta \mid \beta \leq \alpha\}$, is an α-composite

The composition of an α-composite is the map $D(0 \leq \alpha)$

Remark. For all α, since α is terminal in $\{\alpha \mid \alpha \leq \beta\}$, any diagram $D : \alpha \to M$ is colimit cocone to $D(\alpha)$.

3

Pushout-products, pullback-cotensors, pullback-homs

CONSTRUCTION Consider a two-variable adjunction cf. lecture 3 p.16

$$-\otimes-\colon \mathcal{M}\times\mathcal{N}\to\mathcal{P} \qquad \{-,-\}\colon \mathcal{M}^{\mathrm{op}}\times\mathcal{P}\to\mathcal{N} \qquad \underline{\hom}(-,-)\colon \mathcal{N}^{\mathrm{op}}\times\mathcal{P}\to\mathcal{M}$$

$$\mathcal{P}(m\otimes n,p)\cong\mathcal{N}(n,\{m,p\})\cong\mathcal{M}(m,\underline{\hom}(n,p)).$$

$i: m\to m'$, $j: n\to n'$, $f: p\to p'$

pushout-product

$$\begin{tikzcd}
m\otimes n \arrow[r, "i_*"] \arrow[d, "j_*"'] & m'\otimes n \arrow[d] \arrow[ddr, bend left, "j_*"] & \\
m\otimes n' \arrow[r] \arrow[drr, bend right, "i_*"'] & \cdot \arrow[dr, dashed, "i\hat{\otimes}j"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & m'\otimes n'.
\end{tikzcd}$$

pullback-cotensor

$$\begin{tikzcd}
\{m',p\} \arrow[drr, bend left, "i^*"] \arrow[dr, dashed, "\{\hat{i,f}\}"] \arrow[ddr, bend right, "f_*"'] & & \\
 & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{m,p\} \arrow[d, "f_*"] \\
 & \{m',p'\} \arrow[r, "i^*"'] & \{m,p'\}
\end{tikzcd}$$

pullback-hom

$$\begin{tikzcd}
\underline{\hom}(n',p) \arrow[drr, bend left, "j^*"] \arrow[dr, dashed, "\underline{\hat{\hom}}(j,f)"] \arrow[ddr, bend right, "f_*"'] & & \\
 & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\hom}(n,p) \arrow[d, "f_*"] \\
 & \underline{\hom}(n',p') \arrow[r, "j^*"'] & \underline{\hom}(n,p').
\end{tikzcd}$$

Object of commutative squares

Exercise *Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be classes of maps in $\mathcal{M},\mathcal{N},\mathcal{P}$, respectively. The following lifting properties are equivalent*

$$\mathcal{A}\hat{\otimes}\mathcal{B}\boxslash\mathcal{C} \quad\Leftrightarrow\quad \mathcal{B}\boxslash\{\hat{\mathcal{A},\mathcal{C}}\} \quad\Leftrightarrow\quad \mathcal{A}\boxslash\underline{\hat{\hom}}(\mathcal{B},\mathcal{C}).$$

Remark It follows from the exercise that, say, if $B^{\boxslash}=B'^{\boxslash}$, then $A\hat{\otimes}B\boxslash C \Leftrightarrow A\hat{\otimes}B'\boxslash C$. Indeed

$$A\hat{\otimes}B\boxslash C \Leftrightarrow B\boxslash\{\widehat{A,C}\} \underset{\Delta}{\Leftrightarrow} \{\widehat{A,C}\}\subseteq B^{\boxslash} \Leftrightarrow \{\widehat{A,C}\}\subseteq B'^{\boxslash} \Leftrightarrow A\hat{\otimes}B'\boxslash C$$

$\Uparrow$ assumption

1.

Weak factorization systems

$$\begin{tikzcd} \cdot \arrow[r, "L"] \arrow[rr, bend left, "f"] & \cdot \arrow[r, "R"] & \cdot \end{tikzcd}$$

Definition (wfs) A **weak factorization system** on a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

- (factorization) every arrow can be factored as an arrow of $\mathcal{L}$ followed by an arrow of $\mathcal{R}$,
- (closure) furthermore, $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$. $\Leftrightarrow$ $L \boxslash R$, ${}^{\boxslash}R \subseteq L$, and $L^{\boxslash} \subseteq R$

Retract Lemma *Assume that a morphism $f : X \to Y$ can be factored into $f = pi$.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & T \arrow[ur, "p"'] & \end{tikzcd}$$

If f has the right (left) lifting property with respect to i (to p), then f is a retract of p (of i, respectively).

Proof Information repackaging!

(and dually

$$\begin{tikzcd} \cdot \arrow[r, "i"] \arrow[d, "f"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "id"'] & \cdot \end{tikzcd}$$
)

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, "1_X"] & T \arrow[r, "h"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

Note that we have here a particular kind of retract: $\xrightarrow{id}\xrightarrow{id}$, $\xrightarrow{id}$ = codomain fixed

Proposition In the definition of wfs, we can replace $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$ by:

- $L \boxslash R$ and
- L and R are stable under retracts

Proof $\Rightarrow$ by lemma p.2. $\Leftarrow$ We have to prove ${}^{\boxslash}R \subseteq L$ and $L^{\boxslash} \subseteq R$.

Let $f \in {}^{\boxslash}R$, factorise it as $f = \underbrace{p}_{\in R}\,\underbrace{i}_{\in L}$. Then $f \boxslash p$ (assumption)

$\Rightarrow$ f is a retract of i (lemma)

$\Rightarrow$ $f \in L$ (assumption)

($L^{\boxslash} \subseteq R$ proved dually)

Equivalent definitions of model categories

Let $\mathcal{M}$ be a complete and cocomplete category, with 3 classes of morphisms C (cofibrations), F (fibrations), and W (weak equivalences), where W satisfies the 2-for-3 property.

Proposition Under these assumptions, the following sets of axioms are equivalent:

1. $(C, F\cap W)$ and $(C\cap W, F)$ are w.f.s. — trivial, or acyclic fibrations; trivial, or acyclic cofibrations — most elegant!

2.
- $(C\cap W, F)$ is a w.f.s.
- ${}^{\boxslash}(F\cap W) \subseteq C$, $C^{\boxslash} \subseteq F\cap W$
- $\mathrm{Mor}\,\mathcal{M} = C^{\boxslash}\circ C$

indicates a redundancy

3.
- ${}^{\boxslash}(F\cap W) \subseteq C$, $(C\cap W)^{\boxslash} \subseteq F$
- $\mathrm{Mor}\,\mathcal{M} = F\circ(C\cap W) = (F\cap W)\circ C$, $W = C^{\boxslash}\circ {}^{\boxslash}F$

4.
- C, F, W closed under retracts
- $C \boxslash (F\cap W)$, $(C\cap W)\boxslash F$
- $\mathrm{Mor}\,\mathcal{M} = F\circ(C\cap W) = (F\cap W)\circ C$

Proof
- $1\Rightarrow 2$ Obvious
- $2\Rightarrow 3$ We have to prove $\mathrm{Mor}\,\mathcal{M} = (F\cap W)\circ C$ and $W = C^{\boxslash}\circ {}^{\boxslash}F$
  - $\mathrm{Mor}\,\mathcal{M} = C^{\boxslash}\circ C \subseteq (F\cap W)\circ C \subseteq \mathrm{Mor}\,\mathcal{M}$
  - Let $w\in W$. By assumption $\xrightarrow{w} \; = \; \xrightarrow{\in C} \xrightarrow{\in C^{\boxslash}\subseteq F\cap W} \underset{\text{2-for-3}}{\Rightarrow} \longrightarrow \in C\cap W = {}^{\boxslash}F$
- $3\Rightarrow 1$ We have to prove $C\boxslash F\cap W$ and $C^{\boxslash}\subseteq F\cap W$ (the rest is dual)
  - Let $f\in F\cap W = \xrightarrow{{}^{\boxslash}F}\xrightarrow{C^{\boxslash}} \Rightarrow f\in C^{\boxslash}$ — retract lemma + lemma p.2
  - $\mathrm{id}\in {}^{\boxslash}F \Rightarrow C^{\boxslash}\subseteq C^{\boxslash}\circ {}^{\boxslash}F = W$; $C^{\boxslash}\subseteq (C\cap W)^{\boxslash}\subseteq F$
- $4\Rightarrow 1$ By second prop. p.4 (note that if F, W are closed under retracts, so is $F\cap W$!)
- $1\Rightarrow 4$ We only have to check that W is closed under retracts, which is harder. Next page!

Stability of W under retracts

Let $w \in W$ and f be a retract of w. • We first suppose $f \in F$. Our goal is to prove $f \in F \cap W = C^{\boxslash}$. We write $w = \xrightarrow[w_1]{{}^{\boxslash}F} \xrightarrow[w_2]{C^{\boxslash}}$.

We have

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dd, "f"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "\text{composite}"'] & \bullet \arrow[d, "w_1 \in {}^{\boxslash}F"] \arrow[r] & \bullet \arrow[dd, "f \in F"] \\ & \bullet \arrow[d, "w_2"] \arrow[ur, "\text{lifting!}"'] & \\ \bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

thus $\begin{tikzcd} \bullet \arrow[dr] \arrow[rr, bend left, "\mathrm{id}"] \arrow[rr, phantom, "="] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$, which exhibits f as a retract of w_2

$\Rightarrow f \in C^{\boxslash}$
Lemma p.2

• General case. We write $f = \xrightarrow[i]{{}^{\boxslash}F} \xrightarrow[p]{F}$. • We build the push-out of i, u_0.

$$\begin{tikzcd} \bullet \arrow[r, "u_0"] \arrow[d, "i"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r, "v_0"] \arrow[d, "j"] \arrow[dd, bend left, "w"] & \bullet \arrow[d, "i"] \\ \bullet \arrow[r, "u"] \arrow[d, "p"'] \arrow[ur, phantom, "\mathrm{p.o.}"] & \bullet \arrow[r, "v"] \arrow[d, "q"] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "u_1"'] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "v_1"'] & \bullet \end{tikzcd}$$

• we define v, q by universality $\quad pv = v_1 q$ by universality
• This delimits four commutative squares
• The bottom two squares exhibit p as retract of q
• $i \in {}^{\boxslash}F \Rightarrow j \in {}^{\boxslash}F \subseteq W \Rightarrow q \in W$
Lemma p.2 $\qquad$ 2-for-3
• $p \in W$ (by the first part, since $p \in F$)

$\Downarrow$

$f = p\underset{\in W}{\underset{W}{i}} \in W$ (2-for-3)

In summary:

- A model category is a complete and cocomplete category with C, F, W s.t.
  - W satisfies 2-for-3
  - $(C, F \cap W)$ and $(C \cap W, F)$ are w.f.s.
- In a model category, weak equivalences are stable under retracts

In particular, C and $C \cap W$ are left classes, which by Lemma p.2 implies the assumptions •[2] •[3] •[4] •[5] of lecture 5 p.15 and 18, while the $\xrightarrow{C} \xrightarrow{F \cap W}$ factorisation also (a fortiori) entails •[1].

A recipe for model category structures

We again suppose that $\mathcal{M}$ is complete and cocomplete and that there is a class $\mathcal{W}$ satisfying 2-for-3.

Proposition Suppose that two classes I, J of morphisms of $\mathcal{M}$ are given, s.t.

- $I^{\boxslash} \subseteq J^{\boxslash}$
- ${}^{\boxslash}(I^{\boxslash}) \cap \mathcal{W} \subseteq {}^{\boxslash}(J^{\boxslash})$
- $I^{\boxslash} \subseteq \mathcal{W}$
- ${}^{\boxslash}(J^{\boxslash}) \subseteq \mathcal{W}$
- $\mathrm{Mor}\,\mathcal{M} = J^{\boxslash} \circ {}^{\boxslash}(J^{\boxslash}) = I^{\boxslash} \circ {}^{\boxslash}(I^{\boxslash})$

$C \cap \mathcal{W} = {}^{\boxslash}(J^{\boxslash})$

$\Uparrow$

Then, setting $\boxed{F = J^{\boxslash}}$ and $\boxed{C = {}^{\boxslash}(I^{\boxslash})}$, we have a model structure.

Moreover, we have

- [a] $(I^{\boxslash} \subseteq J^{\boxslash}$ and ${}^{\boxslash}(J^{\boxslash}) \subseteq \mathcal{W}) \Leftrightarrow {}^{\boxslash}(J^{\boxslash}) \subseteq {}^{\boxslash}(I^{\boxslash}) \cap \mathcal{W}$
- [b] In the presence of the other axioms ${}^{\boxslash}(I^{\boxslash}) \cap \mathcal{W} \subseteq {}^{\boxslash}(J^{\boxslash}) \Leftrightarrow J^{\boxslash} \cap \mathcal{W} \subseteq I^{\boxslash}$
- [c] $I \subseteq C$ and $J \subseteq C \cap \mathcal{W}$

Proof We first note that [a] is an obvious consequence of exercise p.2, and that we can put ___ and ___ together, which reads as $\boxed{{}^{\boxslash}F = C \cap \mathcal{W}}$, and hence $\mathrm{Mor}\,\mathcal{M} = J^{\boxslash} \circ {}^{\boxslash}(J^{\boxslash})$ reads as $\mathrm{Mor}\,\mathcal{M} = F \circ (C \cap \mathcal{W})$.

We prove the (rest of the) set 2 of axioms from p.5.

We note that $({}^{\boxslash}F)^{\boxslash} = F$ and $C^{\boxslash} = I^{\boxslash}$ (exercise p.2)

From ${}^{\boxslash}F = C \cap \mathcal{W}$, we get $(C \cap \mathcal{W})^{\boxslash} = ({}^{\boxslash}F)^{\boxslash} = F$. This completes the proof that $(C \cap \mathcal{W}, F)$ is a w.f.s.

Next, we have $I^{\boxslash} \subseteq F \cap \mathcal{W}$ by $I^{\boxslash} \subseteq J^{\boxslash}$ and $I^{\boxslash} \subseteq \mathcal{W}$, hence

${}^{\boxslash}(F \cap \mathcal{W}) \subseteq C$ by def. of C and $C^{\boxslash} \subseteq F \cap \mathcal{W}$ (since $C^{\boxslash} = I^{\boxslash}$ cf. exercise p.2)

Finally, since $C^{\boxslash} = I^{\boxslash}$, $\mathrm{Mor}\,\mathcal{M} = I^{\boxslash} \circ {}^{\boxslash}(I^{\boxslash})$ reads as $\mathrm{Mor}\,\mathcal{M} = C^{\boxslash} \circ C$. Thus we have a model structure. We now prove [b]. One direction follows from the model structure, since $J^{\boxslash} \cap \mathcal{W} \subseteq I^{\boxslash}$ reads as $F \cap \mathcal{W} \subseteq C^{\boxslash}$. For the other direction, we have

$\mathrm{Mor}\,\mathcal{M} = J^{\boxslash} \circ {}^{\boxslash}(J^{\boxslash})$, ${}^{\boxslash}(J^{\boxslash}) \subseteq \mathcal{W} \Rightarrow \mathcal{W} \subseteq (J^{\boxslash} \cap \mathcal{W}) \circ {}^{\boxslash}(J^{\boxslash})$, hence

$J^{\boxslash} \cap \mathcal{W} \subseteq I^{\boxslash} \Rightarrow \mathcal{W} \subseteq I^{\boxslash} \circ {}^{\boxslash}(J^{\boxslash})$ 2-for-3. Then, by the retract lemma p.4, we have:

$f \in {}^{\boxslash}(I^{\boxslash}) \cap \mathcal{W} \Rightarrow f$ is a retract of an element of ${}^{\boxslash}(J^{\boxslash}) \xRightarrow{\text{lemma p.2}} f \in {}^{\boxslash}(J^{\boxslash})$

Finally, we prove [c]. We have $I \subseteq {}^{\boxslash}(I^{\boxslash}) = C$, $J \subseteq {}^{\boxslash}(J^{\boxslash}) = {}^{\boxslash}F = C \cap \mathcal{W}$.

(I, J, W) versus (C, F, W)

complete and cocomplete

Remark Given $\mathcal{M}$, I, J, W, such that

- 2-for-3 holds for W
- $\mathrm{Mor}\,\mathcal{M} = J^{\boxslash} \circ {}^{\boxslash}(J^{\boxslash}) = I^{\boxslash} \circ {}^{\boxslash}(I^{\boxslash})$

• [a] and • [b] of proposition p.7 say that the following three sets of axioms are equivalent:

$${}^{\boxslash}(J^{\boxslash}) \subseteq W, \quad I^{\boxslash} \subseteq J^{\boxslash}, \quad {}^{\boxslash}(I^{\boxslash}) \cap W \subseteq {}^{\boxslash}(J^{\boxslash}) \text{ and } I^{\boxslash} \subseteq W$$

[a] $${}^{\boxslash}(I^{\boxslash}) \cap W = {}^{\boxslash}(J^{\boxslash}) \text{ and } I^{\boxslash} \subseteq W$$

[b] $${}^{\boxslash}(J^{\boxslash}) \subseteq W \text{ and } I^{\boxslash} = J^{\boxslash} \cap W$$

In particular, we have $F = J^{\boxslash}$ and $(F \cap W) = I^{\boxslash}$. Thus, both F and $F \cap W$ are of the form $A^{\boxslash}$ for some A.

- All model categories arise from the above recipe, trivially, by taking $I = C$ and $J = C \cap W$.
- But the intention is to have I, J smaller, typically sets!
Even more importantly, when I, J are sets and when $\mathcal{M}$ is locally presentable, we get $\boxed{\mathrm{Mor}\,\mathcal{M} = J^{\boxslash} \circ {}^{\boxslash}(J^{\boxslash}) = I^{\boxslash} \circ {}^{\boxslash}(I^{\boxslash})}$

FOR FREE!

The small object argument (statement)

All what we have to know about locally presentable categories is that for any set K of objects there exists a limit ordinal κ associated with K such that, for any morphism

$$f : A \to \operatorname*{colim}_{\beta<\kappa} B_\beta$$ (cf. p.2) with $A \in K$,

factors through some B_β, i.e. $f = c_\beta \circ f'$ for some $f' : A \to B_\beta$, where c_β is the β-component of the colimiting cone

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & \operatorname{colim} B_\beta \\ & B_\beta \arrow[u, "c_\beta"'] \end{tikzcd}$$

Proposition Let $\mathcal{M}$ be a locally presentable category, and let I be a set of morphisms of $\mathcal{M}$. Then the following factorisation property holds:

$$\boxed{\operatorname{Mor}\mathcal{M} = I^{\boxslash} \circ {}^{\boxslash}(I^{\boxslash})}$$

Proof next page!

This proposition justifies the following

Definition A cofibrantly generated model category is a complete and cocomplete category, with a class W of morphisms satisfying 2-of-3 and two sets I, J of morphisms satisfying one of the sets of axioms of p.8.

I is called the set of generating cofibrations

J ——————— of generating trivial cofibrations

Small object argument (proof)

- If $f \in I^{\boxslash}$, then $f = f \circ id \in I^{\boxslash} \circ {}^{\boxslash}(I^{\boxslash})$

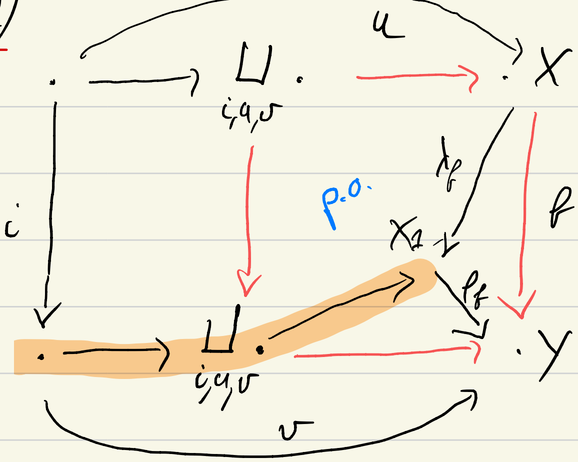
- For fixed $f \in I^{\boxslash}$, we assemble all

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "i"', "\in I"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

(with no lifting ∃ at least one!) as

$$\begin{tikzcd} \coprod_{i,u,v} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d, "f"] \\ \coprod_{i,u,v} \cdot \arrow[r] & \cdot \end{tikzcd}$$

in



We have that λ_f is a push-out of coproducts of i, hence $\lambda_f \in {}^{\boxslash}(I^{\boxslash})$

$X = X_0$

$\lambda_f = \lambda_1$

$f = p_0$

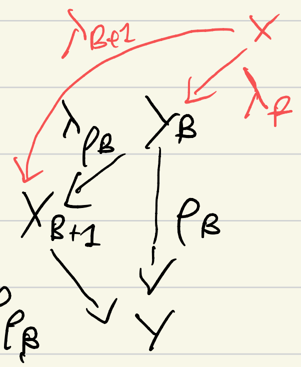
$p_f = p_1$

Y

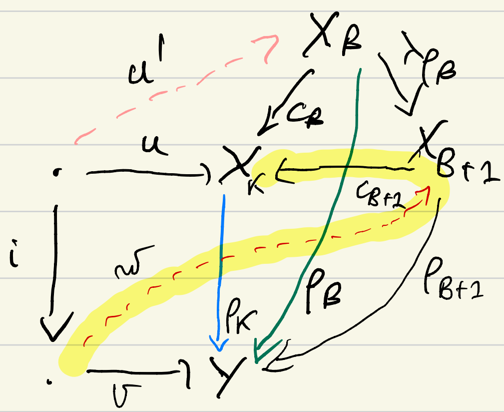
We call $w^f_{i,u,v}$ the arrow $\longrightarrow \nearrow$

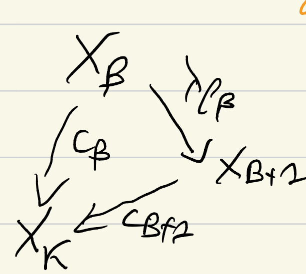
We have $p_f \circ w_{i,u,v} = v$, $\lambda_f \circ u = w^f_{i,u,v} \circ i$

We iterate, with p_f in place of f, transfinitely, and stop at κ associated with the set of domains of arrows in I: for each γ, we have $f = p_\gamma \circ \underbrace{\lambda_\gamma}_{\in {}^{\boxslash}(I^{\boxslash})}$ (transfinite composition of λ_f's)



$p_{\beta+1} = p_{p_\beta}$





- u factors as $c_\beta \circ u'$ (locally presentable) $\Rightarrow$ $p_\beta \circ u' = v \circ i$
- $p_\beta = p_\kappa \circ c_\beta$ (def. of p_κ by universality)
- By instantiating above $f \rightsquigarrow p_\beta$, $u \rightsquigarrow u'$, we get $w^{p_\beta}_{i,u',v}$ (abbreviated as w) $= \dashrightarrow$ s.t. $w \circ i = \lambda_{p_\beta} \circ u'$, $p_{p_\beta} \circ w = v$
- We have

$$(c_{\beta+1} \circ w) \circ i = c_{\beta+1} \circ (w \circ i) = c_{\beta+1} \circ (\lambda_{p_\beta} \circ u') = (c_{\beta+1} \circ \lambda_{p_\beta}) \circ u' = c_\beta \circ u' = u$$

$$p_\kappa \circ (c_{\beta+1} \circ w) = \underbrace{(p_\kappa \circ c_{\beta+1})}_{\text{universality of colimit}} \circ w = p_{\beta+1} \circ w = p_{p_\beta} \circ w = v$$

Therefore, $p_\kappa \in I^{\boxslash}$, and $f = \underbrace{\xrightarrow{\lambda_\kappa}}_{\in {}^{\boxslash}(I^{\boxslash})} \underbrace{\xrightarrow{p_\kappa}}_{\in I^{\boxslash}}$ provides a decomposition

A bonus of the small object argument

By analysing the proof on p.10, we have in fact proved something tighter:

$$\boxed{\mathrm{Mor}\,\mathcal{M} = I^{\boxslash} \circ \mathrm{cell}(I)}$$

where

- $\mathrm{cell}(I)$ is the class of transfinite compositions of pushouts of coproducts of elements of I.

This gives us a Bonus! We define further notations (for a class $A \subseteq \mathrm{Mor}\,\mathcal{M}$):

- $\mathrm{ret}(A)$ is the class of retracts of elements of A.
- $\mathrm{ret}'(A)$ is the class of retracts with fixed codomain (cf. p.4) of elements of A.
- $\overline{I}$ is the smallest class containing I and closed under the constructions listed in lemma p.2

  SATURATION of I

Proposition Under the assumptions of the proposition p.9, we have, additionally:

$$\boxed{{}^{\boxslash}(I^{\boxslash}) = \overline{I} = \mathrm{ret}(\mathrm{cell}(I)) = \mathrm{ret}'(\mathrm{cell}(I))}$$

Proof $\mathrm{ret}'(\mathrm{cell}(I)) \subseteq \mathrm{ret}(\mathrm{cell}(I)) \subseteq \overline{I} \underset{\text{lemma p.2}}{\subseteq} {}^{\boxslash}(I^{\boxslash}) \subseteq \mathrm{ret}'(\mathrm{cell}(I))$.

retract lemma + ${}^{\boxslash}(I^{\boxslash}) \subseteq I^{\boxslash} \circ \mathrm{cell}(I)$

OTHER BONUS NEXT PAGE

Functorial cofibrant replacement

In a model category, for every object m, we have $0 \to m = 0 \xrightarrow{C} \bullet \xrightarrow{F \cap W} m$ for some $\rightarrow$ and $\rightarrow$. The middle object $\bullet$ is a cofibrant replacement of m.

Proposition In a cofibrantly generated model category, cofibrant replacement can be made functorial, and $\rightarrow$, $\rightarrow$ natural. see p. 14

(In an arbitrary model category, cofibrant replacement is functorial "up to homotopy")

Proof sketch We show that at each step of the small object argument, we have a canonical way to functorialise. We specialise the picture of p. 10 with $f = 0 \to m$, and we write m' for the middle object in

$$0 \longrightarrow m' \xrightarrow{f} m$$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[dd, "i"', "\in I"] \arrow[rrr, bend left, "u"] & \coprod_{i,u,v} \cdot \arrow[rr] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\
& & m' \arrow[dr, "f"] & \\
\cdot \arrow[r] \arrow[rrr, bend right, "v"'] & \coprod_{i,u,v} \cdot \arrow[ur] \arrow[rr] \arrow[uur, phantom, "p.o."] & & m
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r] \arrow[dd, "i"', "\in I"] \arrow[rrr, bend left, "u"] & \coprod_{i,u,v'} \cdot \arrow[rr] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\
& & n' \arrow[dr, "f"] & \\
\cdot \arrow[r] \arrow[rrr, bend right, "f \circ v"'] & \coprod_{i,u,v'} \cdot \arrow[ur] \arrow[rr] \arrow[uur, phantom, "p.o."] & & n
\end{tikzcd}$$

Let $f : m \to n$. We seek to define $f' : m' \to n'$ canonically. For this, for every i, u, v, we need

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "i"'] & 0 \arrow[d, "(\text{fixed})"] \\
\cdot \arrow[r, "v''"'] & n'
\end{tikzcd}$$

The uniform $\downarrow$ is provided by initiality

$\xrightarrow{v''}$ is provided by the $(i, u, f \circ v)$ component of $\nearrow$

One can prove more, namely that $f f' = f f$ (hint: maps out of a pushout)

Then one can iterate (modulo a variation of the argument)

and derive

$$\begin{tikzcd}
m' \arrow[r, "f'"] \arrow[d] & n' \arrow[d] \\
m'' \arrow[r, "f''"] \arrow[d] & n'' \arrow[d] \\
m \arrow[r, "f"] & n
\end{tikzcd}$$

etc!

Functoriality follows from the canonicity of the choice made.

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also called classical ⇝ <u>Quillen model structure on $\hat{\Delta}$</u>

Proofs can be found in many sources (e.g. Najib Idrissi's notes https://Idrissi.eu/en/20-21-homotopie/homotopie.pdf)

- cofibrations are the monomorphisms (i.e. morphisms that are pointwise injective $\Rightarrow$ all objects are cofibrant
- weak equivalences are the morphisms $f: X \to Y$ whose topological realisation is a <u>weak homotopy equivalence</u>, i.e., induces isomorphisms $\pi_n(X, x) \to \pi_n(Y, f(x))$ for all n, $x \in X$.
- $F = \{ \underbrace{\Lambda^n_k \hookrightarrow \Delta^n \mid 0 \leq k \leq n}_{\text{all horns}} \}^{\boxslash}$ (cf. lecture 2 p.11)

Fibrant objects are called Kan complexes

The model structure is cofibrantly generated, with
- $J =$ all horns and
- $I = \{ \partial \Delta^n \hookrightarrow \Delta^n \}$

For a contrast, <u>Joyal model structure</u> on $\hat{\Delta}$ has
- $C =$ monomorphisms (unchanged), but
- quasi-categories as fibrant objects

cf. lecture 2 p.5

in arbitrary model categories ⤳ **Cylinder objects**

(good) cylinder

We write $[id, id] : A+A \to A$ as $A+A \xrightarrow[C]{[i_1,i_2]} cyl(A) \xrightarrow[W]{p} A$

A *left homotopy* H from $f : A \to X$ to $g : A \to X$ is a map $H : cyl(A) \to X$ s.t.

we do not require $F \cap W$!

$$\begin{tikzcd} A+A \arrow[r, "{[i_1,i_2]}"] \arrow[dr, "{[f,g]}"'] & cyl(A) \arrow[d, "H"] \\ & X \end{tikzcd}$$

Exercise Show that this defines a reflexive and symmetric relation.

As an application of this notion, we show that "cofibrant replacement is functorial up to left homotopy" (in contrast with p. 12).

Lemma If $g \in F \cap W$ and if $g f_1$ and $g f_2$ are left homotopic, then f_1 and f_2 are left homotopic.

Proof We can display the assumption as

$$\begin{tikzcd} A+A \arrow[r, "{[f_1,f_2]}"] \arrow[d, "{[i_1,i_2]}"', "C"] & B \arrow[d, "g"', "F\cap W"] \\ cyl(A) \arrow[r] \arrow[ur] & C \end{tikzcd}$$

The lifting provides the desired homotopy

Now we fix a choice of $0 \xrightarrow{C} QX \xrightarrow[q_X]{F\cap W} X$ for all X. Let $f : X \to Y$.

Then we have a lifting

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "c"'] & QY \arrow[d, "q_Y", "F\cap W"'] \\ QX \arrow[r, "q_X"'] \arrow[ur, dashed] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

which is a good candidate for "Qf" making q natural

black stuff fixed

Proposition All arrows $\longrightarrow$ such that

$$\begin{tikzcd} QX \arrow[r] \arrow[d, "q_X"'] & QY \arrow[d, "q_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

are left homotopic.

Proof Let $\tilde{f}_1, \tilde{f}_2$ be two such arrows. Then

$q_Y \tilde{f}_1 = q_Y \tilde{f}_2 \Rightarrow q_Y\tilde{f}_1, q_Y\tilde{f}_2$ left hom. $\Rightarrow \tilde{f}_1, \tilde{f}_2$ left hom.

exercise above lemma above

Remark In the situation of the proposition, we have

$\gamma(f)$ is iso $\iff \gamma(\longrightarrow)$ is iso (2-for-3 for isos)

f w.e. $\iff \longrightarrow$ w.e. (2-for-3 in $\mathcal{M}$)

Fibrant - cofibrant replacement

One can define dually a notion of right homotopy, starting from

$$\begin{tikzcd} A \arrow[rr, "\langle id, id \rangle"] \arrow[dr, "C\cap W"'] & & A\times A \\ & path(A) \arrow[ur, "F"'] & \end{tikzcd}$$

One defines fibrant replacement $X \xrightarrow{C\cap W} RX \xrightarrow{F} 1$, functorial up to right homotopy.

Let M_{cf} be the full subcategory of cofibrant and fibrant objects of M.

One can prove the following.

- 1 In M_{cf}, left homotopy coincides with right homotopy and is an equivalence relation and a congruence $\sim$, and therefore $RQ : M \to M_{cf}/\sim$ is a functor.

- 2 For every $f \in Mor\, M_{cf}$, f is a homotopy equivalence (i.e., iso in $M_{cf}/\sim$) $\iff f \in W$

Hence, by the remark p.14, RQ sends weak equivalences to isomorphisms in $M_{cf}/\sim$.

- 3 The induced functor $\xrightarrow{\widetilde{RQ}}$ is in fact an equivalence of categories.

$$\begin{tikzcd} M \arrow[r, "RQ"] \arrow[d, "\gamma"'] & M_{cf}/\sim \\ M[W^{-1}] \arrow[ur, "\widetilde{RQ}"'] & \end{tikzcd}$$

For details, see e.g. Dwyer-Spalinski's notes
https://math.jhu.edu/~eriehl/616-s16/Dwyer-Spalinski.pdf
or my notes on model categories (on curien.galene.org)

Assuming this, we can prove

Proposition If M is a model category, then M is saturated, i.e. for all f, $\gamma(f)$ iso $\Rightarrow f$ w.e.

Proof $\gamma(f)$ iso $\underset{\text{iso preservation}}{\Rightarrow}$ $(\overbrace{\widetilde{RQ}\gamma}^{RQ}) f$ is iso $\underset{2}{\Rightarrow}$ $RQf \in W$ $\underset{\text{remark p.14}}{\Rightarrow}$ $f \in W$

cf. lecture 3 p.18

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Simplicial model categories

two-variable adjunction

cf. lecture 3 p.17

Definition A simplicial model category is a model category equipped with an action $-\otimes- : \hat{\Delta} \times \mathcal{M} \to \mathcal{M}$ that has right adjoints in the two variables (hence we have a tensored and cotensored simplicial category) such that the associated pushout-product bifunctor $-\hat{\otimes}-$ (cf. p.3) maps pairs of cofibrations to a cofibration that is trivial if either of the domain cofibrations are.

In formulas: $\boxed{C \hat{\otimes} C \boxslash (F\cap W) \quad C \hat{\otimes} (C\cap W) \boxslash F \quad (C\cap W) \hat{\otimes} C \boxslash F}$

Equivalent to (cf. p.3) $C \boxslash \widehat{\hom}(C, F\cap W)$ $\quad C \boxslash \widehat{\hom}(C\cap W, F)$ $\quad (C\cap W) \boxslash \widehat{\hom}(C, F)$

i.e. $\widehat{\hom}(C, F\cap W) \subseteq F\cap W$ $\quad \widehat{\hom}(C\cap W, F) \subseteq F\cap W$ $\quad \widehat{\hom}(C,F) \subseteq F$

Consequences

A special case of

$$\begin{tikzcd}
m\otimes n \arrow[r, "i_*"] \arrow[d, "j_*"'] & m'\otimes n \arrow[d] \arrow[ddr, bend left, "j_*"] & \\
m\otimes n' \arrow[r] \arrow[drr, bend right, "i_*"'] & \cdot \arrow[dr, dashed, "\underline{i\hat{\otimes} j}"] & \\
& & m'\otimes n'.
\end{tikzcd}$$

is $(i: 0 \to m)$

$$\begin{tikzcd}
0 \cong 0\otimes n \arrow[r] \arrow[d, "id"'] & m\otimes n \arrow[d, "id"] \arrow[ddr, bend left, "m\otimes j"] & \\
0 \cong 0\otimes n' \arrow[r] \arrow[drr, bend right] & m\otimes n \arrow[dr, dashed, "\underline{m\otimes j}"] & \\
& & m\otimes n'.
\end{tikzcd}$$

Since every object in $\hat{\Delta}$ is cofibrant cf. p.12, we obtain

- 2 Cofibrations and trivial cofibrations of $\mathcal{M}$ are preserved by tensoring with any simplicial set.

cf. (A2) and (A3) of lecture 5 p.12

By further instantiating j as $0 \to n$, and since $m\otimes 0 \cong 0$, we also get.

- 3 Cofibrant objects of $\mathcal{M}$ are preserved by tensoring with any simplicial set.

We also have, for symmetric reasons $(i\otimes n = i \hat{\otimes} (0 \to n))$

- 1 If n is cofibrant, then cofibrations and trivial cofibrations of $\hat{\Delta}$ are preserved by $-\otimes n$
- 1' (Dually) (since $\hom(0,p) \cong 1$), for any simplicial set K, $-^K$ preserves fibrations and trivial fibrations

Left Quillen bifunctors

More generally

Definition A bifunctor $-\otimes=$ between model categories is a **left Quillen bifunctor** if it preserves colimits in both variables and if the associated pushout-product bifunctor $-\hat{\otimes}=$ maps pairs of cofibrations to a cofibration that is acyclic if either of the domain cofibrations are.

This terminology is justified by the fact that (cf • [1] and • [2] p.16)
If $-\otimes=$ is a left Quillen bifunctor, and if m (resp. n) is cofibrant, then $m\otimes-$ (resp. $-\otimes n$) is a left Quillen functor
lecture 5 p.16

$\hat{\Delta}$ is a simplicial model category

assumed in lecture 5

The following result is classical (see Goerss-Jardine I-4.2 or Land's notes 3.29). mutatis mutandis
Remember the sets C of cofibrations (all monos) and J of generating trivial cofibrations (all horn inclusions) of $\hat{\Delta}$ (cf. p.13)

saturation (cf. p.11)

Lemma We have $\overline{J} = \overline{C\hat{\otimes}J}$

Exercise Let A,B,C,D be sets, and $A\subseteq B\subseteq D$, $A\subseteq C\subseteq D$. Show that the p.o. of $A\subseteq B$ and $A\subseteq C$ is $B\cup C$ and that $[B\subseteq D, C\subseteq D] = (B\cup C\subseteq D)$.

Proposition $\hat{\Delta}$ is a simplicial model category

Proof That $C\hat{\otimes}C\subseteq C$ follows from the exercise.
By the lemma, we have $C\hat{\otimes}J \subseteq \overline{J} = {}^{\boxslash}(J^{\boxslash})$ i.e. $C\hat{\otimes}J \boxslash F$
prop. p.11 $\qquad J^{\boxslash} = F$

Since $J^{\boxslash} = F = (C\cap W)^{\boxslash}$, we get $C\hat{\otimes}(C\cap W)\boxslash F$ by the remark p.3.

Simplicial homotopy as left homotopy

Proposition[1] If X is a cofibrant object of a simplicial model category, Then $\Delta^1\otimes X$ is a cylinder object:

$$X+X \xrightarrow{c} \Delta^1\otimes X \xrightarrow[(\rho^0\otimes id)]{W} \Delta^0\otimes X \simeq X$$

using X cofibrant

Proof We can read $X+X$ as $(\Delta^0+\Delta^0)\otimes X$. Then, since $\Delta^0+\Delta^0 \xrightarrow[{[d^0,d^1]}]{} \Delta^1$ is mono = cofibration, by •[1] p.16 we have that $\longrightarrow$ is a cofibration.
We can also read $X+X$ as $(\Delta^0\otimes X)+(\Delta^0\otimes X)$. Then read as $(\partial\Delta^1\subseteq\Delta^1)\otimes id$
we read $\longrightarrow$ as $[d^0\otimes id, d^1\otimes id]$. and obtain $\longrightarrow\longrightarrow = [id, id]$ since $\rho^0 d^0 = \rho^0 d^1 = id$. Moreover, since, say, d^0 is a horn inclusion, by •[1] p.16 we get that $d^0\otimes id \in C\cap W$, and from $\xrightarrow{d^0\otimes id}\longrightarrow = id$, we get $\longrightarrow \in W$ by 2-for-3

We go back to model categories for what below:

Proposition[2] If $f,g: X\to Y$ are homotopic in a model category, then f,g are mapped to the same morphism in $Ho\mathcal{M}$.

Proof (This is of course part of •[3] on p.15). With the notation of p.14, we have $p\in W$, $pi_0 = pi_1 = id$
It follows that $\gamma(i_0)=\gamma(i_1)=\gamma(p)^{-1}$, and that
$$\gamma(f)=\gamma(H)\gamma(i_0)=\gamma(H)\gamma(i_1)=\gamma(g).$$

Proposition[3] In a model category, homotopy equivalences are weak equivalences.

Proof By the proposition 2, if f,g form a homotopy equivalence, then $\gamma(f)\gamma(g)=id$ and $\gamma(g)\gamma(f)=id$, hence $\gamma(f)$ is an isomorphism. It follows that $f\in W$ by saturation (cf. p.15).

Note that here we exploit saturation to prove the proposition while we proved saturation "the other way around" from a similar proposition restricted to fibrant-cofibrant objects

Simplicial homotopies are weak homotopies

We come back to simplicial model categories

Proposition 2' If $f, g : X \to Y$ are simplicially homotopic, then f, g are mapped to the same morphism in $\mathrm{Ho}\,\mathcal{M}$.

Proof. If X is cofibrant, we conclude by Proposition 1 and 2 p. 18.

If X is arbitrary, we can take $q : \tilde{X} \to X \in W$ (cofibrant)

Then fq and gq are simplicially homotopic, hence by the first part of the proof $\gamma(f)\gamma(q) = \gamma(g)\gamma(q)$ and hence $\gamma(f) = \gamma(g)$.

cf. (A1) of lecture 5 p. 12

Proposition 3' In a simplicial model category, simplicial homotopy equivalences are weak equivalences.

Proof: Like Proposition 3 from Proposition 2 on last page.

The Reedy model structure (sketch)

We are interested in simplicial objects of $\mathcal{M}$, i.e. $X \in \mathcal{M}^{\Delta^{op}}$.
Recall from lecture 5 p.17 the n-th latching object $L_n X = (sk_{n-1} X)_n$
and the n-th latching map $L_n X \to X_n$. Dually, there are matching
objects and matching maps $X_n \to M_n X$. The following theorem is classical
(see e.g. Hovey's book). SKETCHED IN THE FOLLOWING SLIDES
These notions can be generalised by replacing Δ by any Reedy category.

DEFINITION A **Reedy category** is a small category $\mathcal{D}$ equipped with

(i) a degree function assigning a non-negative integer to each object,
(ii) a wide subcategory $\overrightarrow{\mathcal{D}}$ whose non-identity morphisms strictly raise degree, and
(iii) a wide subcategory $\overleftarrow{\mathcal{D}}$ whose non-identity morphisms strictly lower degree

so that every arrow factors uniquely as an arrow in $\overleftarrow{\mathcal{D}}$ followed by one in $\overrightarrow{\mathcal{D}}$.

This notion is self-dual: $\mathcal{D}$ Reedy $\Rightarrow$ $\mathcal{D}^{op}$ Reedy. Δ and Δ^{op} are Reedy.
$(\deg([n]) = n)$

THEOREM REEDY mod. struct. *Let $\mathcal{M}$ be a model category, and let $\mathcal{D}$ be a Reedy category. There is a model structure on $\mathcal{M}^{\mathcal{D}}$ with pointwise weak equivalences, whose cofibrations are the maps $X \to Y$ such that each relative latching map $L^d Y \coprod_{L^d X} X^d \to Y^d$ is a cofibration in $\mathcal{M}$, and whose fibrations are the maps $X \to Y$ such that each relative matching map $X^d \to Y^d \times_{M^d Y} M^d X$ is a fibration in $\mathcal{M}$.*

In lecture 5, we assumed that geometric realisation $|-| : \mathcal{M}^{\Delta^{op}} \to \mathcal{M}$ is left Quillen.
This result is consequence of the following prop. and lemma (cf p.17):

Proposition (Hirschhorn's book 18.4.11) Let $\mathcal{D}$ be a Reedy category, and let $\mathcal{M}$ be a simplicial model category. Then the functor tensor product

$$- \otimes_{\mathcal{D}} - : \underbrace{\hat{\Delta}^{\mathcal{D}^{op}}}_{\text{Reedy}} \times \underbrace{\mathcal{M}^{\mathcal{D}}}_{\text{Reedy}} \to \mathcal{M} \text{ is a left Quillen bifunctor.}$$

DETAILS in the next slides

Lemma $\Delta^{\bullet} : \Delta \to \hat{\Delta}$ is Reedy cofibrant.

Hence the proposition instantiated with $\mathcal{D} = \Delta^{op}$ yields that $\Delta^{\bullet} \otimes_{\Delta^{op}} - = |-|$ is left Quillen, which proves the theorem p.17 of lecture 5.

Heuristics: constructing simplicial objects by induction

- To factorise, we will need to construct the object in the middle.
  construction by induction, taking care of enforcing naturality
- We shall also build morphisms by induction (again in the factorisation, in the liftings)

We want to specify $X : \Delta^{op} \to \mathcal{M}$

base
- Pick an object X_0 of $\mathcal{M}$ $\rightsquigarrow$ functor $\Delta^{op}_{\leq 0} \to \mathcal{M}$

ind.
- Suppose we have defined X up to $n-1$ $\rightsquigarrow$ functor $\Delta^{op}_{\leq n-1} \to \mathcal{M}$
  - Pick an object X_n of $\mathcal{M}$. We need, for each $i, j < n$

  maps $X_f : X_i \to X_n$ for all $f : [n] \to [i]$ and

  maps $X_g : X_n \to X_j$ for all $g : [j] \to [n]$

  Let $i_{n-1,n} : \Delta_{\leq n-1} \to \Delta_{\leq n}$ (inclusion)

  Let, for $Y : \Delta^{op}_{\leq n-1}$
  $$L_n Y = \left(\mathrm{Lan}_{i_{n-1,n}} Y\right)_n = \int^{i<n} \underbrace{\Delta^{op}_{\leq n}(i_{n-1,n}(i), n)}_{\Delta([n],[i])} \cdot Y_i$$

  Dually $M_n Y = (\mathrm{Ran}_{i_{n-1,n}} Y)_n$

  So we can repackage: we need two maps

  $$L_n X \to X_n \to M_n X$$

  (for X as constructed at this stage)

  $$\begin{tikzcd} L_n X \arrow[r] \arrow[rr, bend right, "\tau_n"'] & X_n \arrow[r] & M_n X \end{tikzcd}$$

  (τ_n defined via the maps $g \circ f : [j] \to [i]$ $(i, j < n)$)

providing a factorisation of the canonical map $L_n X \to M_n X$.
(to ensure functoriality)

Latching and matching objects

Exercise Show that for F, K, L as in the picture, with L full and faithful, then

$$(\mathrm{Lan}_{LK}F)L = \mathrm{Lan}_K F$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"] & \mathcal{E} \\ \mathcal{D} \arrow[d, "L"] & \\ \mathcal{D}' & \end{tikzcd}$$

Applying the exercise to

$$\begin{tikzcd} \Delta^{op}_{\leq n-1} \arrow[r, "i_{n-1}"] \arrow[d, "i_{n-1,n}"] \arrow[dd, bend right=60, "i_{n-1}"'] & \Delta^{op} \arrow[r, "X"] & \mathcal{M} \\ \Delta^{op}_{\leq n} \arrow[d, "i_n"] & & \\ \Delta^{op} & & \end{tikzcd}$$

and recalling the definition of $sk_n X = \mathrm{Lan}_{i_n}(X i_n)$ (lecture 2 p.22)

we get $(sk_{n-1}X)\, i_n = \mathrm{Lan}_{i_{n-1,n}}(X i_{n-1})$

$\Leftarrow$ previous slide

This justifies to define the <u>n-th latching object</u> of X

by $\boxed{L_n X = ((sk_{n-1}X)\, i_n)_n = (sk_{n-1}X)_n}$

(and dually, $M_n X = (cosk_{n-1}X)_n$)

Heuristics: constructing maps between simplicial objects

Suppose $B, X \in \mathcal{M}^{\Delta^{op}}$. We want to build a map $h : B \to X$

- Depth Specify a map $h_0 : B_0 \to X_0$
- ind. Specify a map $h_n : B_n \to X_n$ such that

$$\begin{tikzcd} L_n B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & B_n \arrow[r] \arrow[d, "h_n"] \arrow[dr, phantom, "\square"] & M_n B \arrow[d] \\ L_n X \arrow[r] & X_n \arrow[r] & M_n X \end{tikzcd}$$

commutes

Suppose moreover that we want h_n to be a lifting in

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & X_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed, "h_n"] & Y_n \end{tikzcd}$$

We can package the two requirements $\circ$ by asking $\circ$ in

$$\boxed{\begin{tikzcd} A_n \sqcup_{L_n A} L_n B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & X_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed, "h_n"] & Y_n \times_{M_n Y} M_n X \end{tikzcd}}$$

(and similarly for $\square$)

(In details:

$$\begin{tikzcd} L_n A \arrow[r] \arrow[d] & A_n \arrow[d] \arrow[ddr, bend left] & \\ L_n B \arrow[r] \arrow[drr, bend right] & A_n \sqcup_{L_n A} L_n B \arrow[dr] & \\ & & B_n \end{tikzcd}$$

(we have by naturality)

$$\begin{tikzcd} L_n A \arrow[r] \arrow[d] \arrow[dd, bend right] & A_n \arrow[d] \arrow[ddr, bend left] & \\ L_n B \arrow[r] \arrow[d] & A_n \sqcup_{L_n A} L_n B \arrow[dr] & \\ L_n X \arrow[rr] & & X_n \end{tikzcd}$$

(we have by nat. and by functoriality)

The Reedy model structure: proof

Proposition: For C, F, W as in p.20, we have that $F \cap W$ is defined as F, replacing "fibration" by "trivial fibration" (idem $C \cap W$).

PROOF next page!

Remark The requirement for $f: X \to Y$ to be a (trivial) (co)fibration, for the component at 0, boils down to f_0 (trivial) (co)fibration, since $L_0 X = 0$ implies $X_0 \sqcup_{L_0 X} L_0 Y = X_0$

- We want to factorise $g: X \to Y$ as $\underset{i \in C}{X \rightarrowtail} Z \underset{h \in F \cap W}{\overset{\sim}{\twoheadrightarrow}} Y$

basic (cf. remark) Choose a factorisation $X_0 \overset{i_0}{\rightarrowtail} Z_0 \overset{h_0}{\underset{\sim}{\twoheadrightarrow}} Y_0$ (in $\mathcal{M}$)

ind. Choose a factorisation of the induced map $X_n \sqcup_{L_n X} L_n Z \to Y_n \times_{M_n Y} M_n Z$ in $\mathcal{M}$

and define i_n and h_n

$$
\begin{tikzcd}
 & X_n \sqcup_{L_n X} L_n Z \arrow[rr] \arrow[dr, tail] & & Y_n \times_{M_n Y} M_n Z \arrow[dr] & \\
X_n \arrow[ur] \arrow[rr, "i_n"] & & Z_n \arrow[ur, two heads, "\sim"] \arrow[rr, "h_n"] & & Y_n
\end{tikzcd}
$$

Then i, h are cofibrations and trivial cofibrations in $\mathcal{M}^{\Delta^{op}}$ by construction.
(using the proposition)

- That $C \boxslash (F \cap W)$ follows readily from the discussion p.23: the definition of Reedy fibrations and cofibrations serves exactly that purpose

- Closure of F and C under retracts:

This follows from the observation that if the map $g: \boldsymbol{X} \to \boldsymbol{Y}$ is a retract of the map $h: \boldsymbol{W} \to \boldsymbol{Z}$, then for every n the relative latching map $\boldsymbol{X}_n \amalg_{\mathrm{L}_n \boldsymbol{X}} \mathrm{L}_n \boldsymbol{Y} \to \boldsymbol{Y}_n$ is a retract of the relative latching map $\boldsymbol{W}_n \amalg_{\mathrm{L}_n \boldsymbol{W}} \mathrm{L}_n \boldsymbol{Z} \to \boldsymbol{Z}_n$ and the relative matching map $\boldsymbol{X}_n \to \boldsymbol{Y}_n \times_{\mathrm{M}_n \boldsymbol{Y}} \mathrm{M}_n \boldsymbol{X}$ is a retract of the relative matching map $\boldsymbol{W}_n \to \boldsymbol{Z}_n \times_{\mathrm{M}_n \boldsymbol{Z}} \mathrm{M}_n \boldsymbol{W}$.

Proof of proposition p.24

• If $\forall_\alpha\ X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \xrightarrow{\text{latch}_\alpha} Y_\alpha$ is a trivial cofibration, we need to prove that $\forall_\alpha X_\alpha \to Y_\alpha$ is a w.e. We actually prove that it is a triv cof. More generally, we show that

•$_1$ $\forall_\alpha\ \text{latch}_\alpha \square S \Rightarrow \forall_\alpha (X_\alpha \to Y_\alpha) \square S$ for any S (singleton)

By considering

$$\begin{tikzcd} L_\alpha X \arrow[r] \arrow[d] & X_\alpha \arrow[d] \arrow[ddr, bend left] & \\ L_\alpha Y \arrow[r] & X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \arrow[dr, "\text{latch}_\alpha"'] & \\ & & Y_\alpha \end{tikzcd}$$

it is enough to show

•$_2$ $(\forall \beta<\alpha\ \text{latch}_\beta \square S) \Rightarrow L_\alpha X \to L_\alpha Y \square S$

(We write $\beta<\alpha$ if $\deg\beta < \deg\alpha$)

<u>Exercise</u> Show that if $F, G : I \to C$ and $\alpha : F \to G$ then

$\alpha_i \square S\ \forall i \Rightarrow (\operatorname{colim} F \to \operatorname{colim} G) \square S$, for any S.

Proof of •$_2$: By the exercise, it is enough to prove $(X_\beta \to Y_\beta) \square S$ for all $\beta<\alpha$.

The idea is to proceed like p.23. For this to work, we need a reinforced statement

•$_3$ $(\forall \beta<\alpha\ \text{latch}_\beta \square S) \Rightarrow \underbrace{X \to Y \square \text{cot}\overset{= E\to B}{S}}$ considered on $C_{<\deg(\alpha)}$

We have then that assuming

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \text{cot} E \arrow[d] \\ Y \arrow[r] & \text{cot} B \end{tikzcd}$$

on the $\deg\beta - 1$ filtration

we construct

$$\begin{tikzcd} X_\beta \sqcup_{L_\beta X} L_\beta Y \arrow[r] \arrow[d] & E \arrow[d] \\ Y_\beta \arrow[r] \arrow[ur, dashed, "\text{by assumption}"] & B \end{tikzcd}$$

allowing to extend

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \text{cot} E \arrow[d] \\ Y \arrow[r] & \text{cot} B \end{tikzcd}$$

on the $\deg\beta$ filtration

(For the base case, we have $\text{latch}_\beta = X_\beta \to Y_\beta$, and hence we are done by assumption.)

• In the converse direction, considering again

$$\begin{tikzcd} L_\alpha X \arrow[r] \arrow[d] & X_\alpha \arrow[d] \arrow[ddr, bend left] & \\ L_\alpha Y \arrow[r] & X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \arrow[dr, "\text{latch}_\alpha"'] & \\ & & Y_\alpha \end{tikzcd}$$

it is enough to prove that $L_\alpha X \to L_\alpha Y$ is a trivial cofibration, but this then works fine since we can assume by induction that the latching maps for $\beta<\alpha$ are trivial cofibrations and use •$_2$. The base case is trivial since $L_\alpha X$ and $L_\alpha Y$ are then both $\perp$.

Reedy model structure when $\mathcal{M}$ is a simplicial model category

To get the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$ (and actually on $\mathcal{M}^{\mathcal{C}}$ for any Reedy category), we just needed a model category structure on $\mathcal{M}$. We now assume that $\mathcal{M}$ is a simplicial model category. Then we have

Hirschhorn 18.4

THEOREM (The Reedy diagram homotopy lifting extension theorem). Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a simplicial model category.

(1) *If $i\colon \boldsymbol{A} \to \boldsymbol{B}$ is a Reedy cofibration of $\mathcal{C}$-diagrams in $\mathcal{M}$ and $p\colon X \to Y$ is a fibration in $\mathcal{M}$, then the map of $\mathcal{C}^{\mathrm{op}}$-diagrams of simplicial sets*

$$\operatorname{Map}(\boldsymbol{B}, X) \longrightarrow \operatorname{Map}(\boldsymbol{A}, X) \times_{\operatorname{Map}(\boldsymbol{A},Y)} \operatorname{Map}(\boldsymbol{B}, Y)$$

is a Reedy fibration that is a Reedy weak equivalence if either of i or p is a weak equivalence.

(2) *If $i\colon A \to B$ is a cofibration in $\mathcal{M}$ and $p\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a Reedy fibration of $\mathcal{C}$-diagrams in $\mathcal{M}$, then the map of $\mathcal{C}$-diagrams of simplicial sets*

$$\operatorname{Map}(B, \boldsymbol{X}) \longrightarrow \operatorname{Map}(A, \boldsymbol{X}) \times_{\operatorname{Map}(A,\boldsymbol{Y})} \operatorname{Map}(B, \boldsymbol{Y})$$

is a Reedy fibration that is a weak equivalence if either i or p is a weak equivalence.

where $\operatorname{Map}(\boldsymbol{B},X)$ is defined by $\operatorname{Map}(\boldsymbol{B},X)_a = \underbrace{\underline{\mathcal{M}}(\boldsymbol{B}_a, X)}_{\in \hat{\Delta}}$

For the proof, we shall need the following

Exercise

If $\boldsymbol{B}$ is a $\mathcal{C}$-diagram in $\mathcal{M}$ and X is an object of $\mathcal{M}$, then $\operatorname{Map}(\boldsymbol{B}, X)$ is a $\mathcal{C}^{\mathrm{op}}$-diagram of simplicial sets and for every object α of $\mathcal{C}$ there is a natural isomorphism $\mathrm{M}_\alpha \operatorname{Map}(\boldsymbol{B}, X) \approx \operatorname{Map}(\mathrm{L}_\alpha \boldsymbol{B}, X)$.

(Hint: $\underline{\mathcal{M}}(-,X)$ turns limits into colimits.)

Notation: Hirschhorn also uses $\operatorname{Map}(Y,X)$ for $\underline{\mathcal{M}}(Y,X)$

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SEE p.30 FOR A HIGH LEVEL VIEW of what is going on here

PROOF. We will prove part 1; the proof of part 2 is similar.
We must show that for every object α of $\mathcal{C}$ the map of simplicial sets

$U_\alpha \to V_\alpha \times_{M_\alpha V} M_\alpha U$

1

$$\underbrace{\mathrm{Map}(\boldsymbol{B}_\alpha, X)}_{U_\alpha} \longrightarrow \quad \text{Pullback of} \left(\begin{array}{c} \mathrm{Map}(\boldsymbol{A}_\alpha, X) \times_{\mathrm{Map}(\boldsymbol{A}_\alpha, Y)} \mathrm{Map}(\boldsymbol{B}_\alpha, Y) \\ \big\downarrow \\ \mathrm{M}_\alpha\big(\mathrm{Map}(\boldsymbol{A}, X) \times_{\mathrm{Map}(\boldsymbol{A}, Y)} \mathrm{Map}(\boldsymbol{B}, Y)\big) \\ \big\uparrow \\ \mathrm{M}_\alpha \mathrm{Map}(\boldsymbol{B}, X) \end{array} \right) \qquad \begin{array}{c} V_\alpha \\ \downarrow \\ M_\alpha V \\ \uparrow \\ M_\alpha U \end{array}$$

" relative matching map

is a fibration that is a weak equivalence if either i or p is a weak equivalence. Exercise p.26 implies that this map is isomorphic to the map

$$\mathrm{Map}(\boldsymbol{B}_\alpha, X) \longrightarrow \quad \text{Pullback of} \left(\begin{array}{c} \mathrm{Map}(\boldsymbol{A}_\alpha, X) \times_{\mathrm{Map}(\boldsymbol{A}_\alpha, Y)} \mathrm{Map}(\boldsymbol{B}_\alpha, Y) \\ \big\downarrow \\ \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{A}, X) \times_{\mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{A}, Y)} \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{B}, Y) \\ \big\uparrow \\ \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{B}, X) \end{array} \right),$$

and the codomain of this map is the limit of the diagram

$$\begin{tikzcd}
\mathrm{Map}(\boldsymbol{A}_\alpha, X) \arrow[r] \arrow[d] & \mathrm{Map}(\boldsymbol{A}_\alpha, Y) \arrow[d] & \mathrm{Map}(\boldsymbol{B}_\alpha, Y) \arrow[l] \arrow[d] \\
\mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{A}, X) \arrow[r] & \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{A}, Y) & \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{B}, Y) \arrow[l] \\
& \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{B}, X) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

useless

So we can repackage:

$$\begin{array}{c} \mathrm{Map}(Q,X) \\ \downarrow \\ \text{Pullback of} \\ \left(\begin{array}{c} \mathrm{Map}(P,X) \\ \downarrow \\ \mathrm{Map}(P,Y) \\ \uparrow \\ \mathrm{Map}(Q,Y) \end{array} \right) \end{array}$$

$$\mathrm{Map}(\boldsymbol{B}_\alpha, X) \longrightarrow \quad \text{Pullback of} \left(\begin{array}{c} \overbrace{\mathrm{Map}(\boldsymbol{A}_\alpha, X) \times_{\mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{A}, X)} \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{B}, X)}^{\mathrm{Map}(A_\alpha \sqcup_{L_\alpha A} L_\alpha B, X)} \\ \big\downarrow \\ \underbrace{\mathrm{Map}(\boldsymbol{A}_\alpha, Y) \times_{\mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{A}, Y)} \mathrm{Map}(\mathrm{L}_\alpha \boldsymbol{B}, Y)}_{\mathrm{Map}(A_\alpha \sqcup_{L_\alpha A} L_\alpha B, Y)} \\ \big\uparrow \\ \mathrm{Map}(\boldsymbol{B}_\alpha, Y) \end{array} \right).$$

$= \widehat{\hom}(P \to Q, p : X \to Y)$

2 Since $\underbrace{\boldsymbol{A}_\alpha \amalg_{\mathrm{L}_\alpha \boldsymbol{A}} \mathrm{L}_\alpha \boldsymbol{B}}_{P} \to \underbrace{\boldsymbol{B}_\alpha}_{Q}$ is a cofibration that is a weak equivalence if i is a weak equivalence (by assumption and Prop. p.24 (two)) the result now follows from Exercise p.3 $\square$

$p : X \to Y \in$ class C $\quad$ $P \to Q \in$ class B

{ if p fibration, take $A = C \cap W$: $\widehat{\hom}(P \to Q, X \to Y) \in F$ iff
$A \boxslash \widehat{\hom}(P \to Q, X \to Y)$ iff $\underbrace{A \hat{\otimes} (P \to Q)}_{\text{trivial cof. by the simplicial model structure on } M} \boxslash (X \to Y)$

$A = C$

If i is a w.e., then take $A = C$ and use

Completing the proof of the theorem in lecture 5 p.17

Exercise Show that for $\mathcal{M}$ tensored, $G: D^{op} \to V$, $F: D \to \mathcal{M}$, $m \in \mathcal{M}$, we have $\mathcal{M}(G \otimes_D F, m) = V^{D^{op}}(G, \underbrace{\mathrm{Map}(F, m)}_{d \mapsto \mathcal{M}(Fd, m)})$

applied to $\otimes = \otimes_D : V^{D^{op}} \times \mathcal{M}^D \to \mathcal{M}$ $\{\mathcal{M} \to V^{D^{op}}, \mathcal{N} \to \mathcal{M}^D, P \to \mathcal{M}\}$

- By the exercise p.3, part (1) of the theorem on p.27 and instantiated reformulates as:

If $i: A \to B$ is a Reedy cofibration on $\mathcal{M}^{\Delta^{op}}$ and $j: \mathbf{K} \to \mathbf{L}$ is a Reedy cofibration in $\hat{\Delta}^{\Delta}$, then the pushout-product

$$\mathbf{L} \underset{\Delta^{op}}{\otimes} A \sqcup_{\mathbf{K} \underset{\Delta^{op}}{\otimes} A} \mathbf{K} \underset{\Delta^{op}}{\otimes} B \to \mathbf{L} \underset{\Delta^{op}}{\otimes} B$$

is a cofibration in $\mathcal{M}$ that is a weak equivalence if either i or j is a Reedy weak equivalence.

- Specialising j to $id: \Delta^{\bullet} \to \Delta^{\bullet}$, we get that if $i: A \to B$ is a Reedy cofibration on $\mathcal{M}^{\Delta^{op}}$, then

$$|A| = \Delta^{\bullet} \otimes A \longrightarrow \Delta^{\bullet} \otimes B = |B| \text{ is a trivial}$$

cofibration. Therefore geometric realisation is a fortiori left Quillen.

What have we done on p. 27 and 28 ?

- Our goal was to establish on p. 29. Thanks to Exercise p. 3, we translated this into the statement on p. 27.
- To prove amounts to prove the relative matching maps $_1$ are cofibrations. We were able to show that $_1$ reformulates as $\widehat{\mathrm{Hom}}(-,-)$, where $-$, $-$ are a cofibration and a fibration, by assumption on i, p, respectively.

So our goal reformulates as $(C \cap W) \boxslash \widehat{\mathrm{Hom}}(-,-)$, which by Exercise 3 (different instantiation and the other way around) then follows from the axiom of simplicial model categories.