lecture B Categorical interpretation of extensional type theory

In defining the category of contexts in lecture A, we highlighted that a categorical semantics

- should be invariant under α-conversion
- gives a prominent role to substitution

This motivate a syntax that is

- variable-free (de Bruijn notation)
- features substitution as a constructor rather than an operation defined by induction on the size of terms

While studying the interpretation of λ-calculus in $\overbrace{\text{Cartesian Closed Categories}}^{CCC}$,

CALCULI of EXPLICIT SUBSTITUTIONS

Abadi - Cardelli - Curien - Lévy

arose

A core syntax of dependent explicit substitutions

$$
\begin{aligned}
&\sigma ::= \text{base types, possibly dependent} \mid \sigma[s] \\
&\Gamma ::= \emptyset \mid (\Gamma, \sigma) \\
&M ::= 1 \mid M[s] \mid \text{cases given by the signature} \\
&s ::= id \mid \uparrow \mid M \cdot s \mid s \circ s
\end{aligned}
$$

$\Gamma \vdash x : A$

- The typing judgements are: $\sigma : U$ in lecture A

$$\Gamma \;\texttt{context} \qquad \Gamma \vdash \sigma \;\texttt{type} \qquad \Gamma \vdash M : \sigma \qquad \Gamma' \vdash s : \Gamma$$

- The typing rules are:

$$\frac{}{\emptyset \;\texttt{context}} \qquad \frac{\Gamma \vdash \sigma \;\texttt{type}}{\Gamma, \sigma \;\texttt{context}} \qquad \frac{\Gamma' \vdash s : \Gamma \quad \Gamma \vdash \sigma \;\texttt{type}}{\Gamma' \vdash \sigma[s] \;\texttt{type}}$$

$$\frac{}{\Gamma, \sigma \vdash 1 : \sigma[\uparrow]} \qquad \frac{\Gamma' \vdash s : \Gamma \qquad \Gamma \vdash M : \sigma}{\Gamma' \vdash M[s] : \sigma[s]}$$

$$\frac{}{\Gamma \vdash id : \Gamma} \qquad \frac{}{\Gamma, \sigma \vdash \uparrow : \Gamma} \qquad \frac{\Gamma_1 \vdash s_1 : \Gamma_2 \qquad \Gamma_2 \vdash s_2 : \Gamma_3}{\Gamma_1 \vdash s_2 \circ s_1 : \Gamma_3}$$

$$\frac{\Gamma' \vdash s : \Gamma \qquad \Gamma \vdash \sigma \;\texttt{type} \quad \Gamma' \vdash M' : \sigma[s]}{\Gamma' \vdash M' \cdot s : \Gamma, \sigma}$$

- Definitional equalities (you should read = as =)

$$
\begin{aligned}
&\sigma[s][t] = \sigma[s \circ t] \\
&\sigma[id] = \sigma \\
\\
&1[M \cdot s] = M \\
&M[s][t] = M[s \circ t] \\
&M[id] = M
\end{aligned}
$$

$$
\begin{aligned}
&\uparrow \circ (M \cdot s) = s \\
&(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3) \\
&id \circ s = s \circ id = s \\
&(M \cdot s) \circ t = M[t] \cdot (s \circ t) \\
&1[s] \cdot (\uparrow \circ s) = s
\end{aligned}
$$

Warning: this is a first try!

Definitional equalities should be typed too

- $\vdash \Gamma = \Delta \qquad \Gamma \vdash \sigma = \tau \qquad \Gamma \vdash M = N : \sigma \qquad \Gamma' \vdash s = t : \Gamma$

- $$\frac{\Gamma'' \vdash t : \Gamma' \qquad \Gamma' \vdash s : \Gamma \qquad \Gamma \vdash \sigma \;\mathtt{type}}{\Gamma'' \vdash \sigma[s][t] = \sigma[s \circ t]}$$

etc.

- (not exhaustive)

$$\frac{\Gamma \vdash \sigma = \sigma'}{\Gamma \vdash \sigma' = \sigma} \qquad \frac{\Gamma \vdash \sigma = \sigma'}{\vdash (\Gamma, \sigma) = (\Gamma, \sigma')} \qquad \frac{\Gamma' \vdash s_1 = s_2 : \Gamma \quad \Gamma \vdash \sigma}{\Gamma' \vdash \sigma[s_1] = \sigma[s_2]}$$

cf. lecture A p9

$$\frac{\vdash \Gamma = \Gamma' \quad \Gamma \vdash M : \sigma}{\Gamma' \vdash M : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash \sigma = \sigma'}{\Gamma \vdash M : \sigma'}$$

$$\frac{\vdash \Gamma = \Gamma' \quad \Gamma \vdash s : \Delta}{\Gamma' \vdash s : \Delta} \qquad \frac{\Gamma \vdash s : \Delta \quad \Gamma \vdash \Delta = \Delta'}{\Gamma \vdash s : \Delta'}$$

Locally cartesian closed categories

A locally cartesian closed category (LCCC) is a category $\mathbb{C}$

- with a terminal object, and s.t.
- all slice categories $\mathbb{C}/c$ are cartesian closed.

$\Leftarrow$ in particular, $\mathbb{C}$ is a CCC

This is equivalent to the fact of requiring the existence of two successive adjoints to all functors

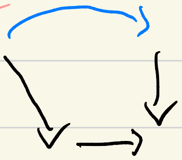
$$k \circ \_ : \mathbb{C}/c_1 \to \mathbb{C}/c_2 \quad \text{for } k : c_1 \to c_2$$

$$\boxed{\underbrace{k \circ \_}_{\text{Opportunely renamed } \Sigma k \text{ (we shall see why!)}} \dashv k^* \dashv \Pi k}$$

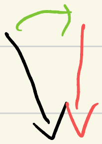
For the time being, we focus on the first adjunction: $\forall f, g$

$$\frac{\mathbb{C}/c_2(k \circ f,\ g)}{\mathbb{C}/c_1(f,\ k^* g)}$$

$\Leftarrow$ commutative square



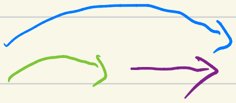
$\Leftarrow$ commutative triangle



$\longrightarrow$ counit

↳ commutative triangle

$$\begin{tikzcd}
{} \arrow[r] \arrow[rr, bend left] \arrow[dr, "f"'] & {} \arrow[r] \arrow[d, "k^*g"] & {} \arrow[d, "g"] \\
 & {} \arrow[r, "k"'] & {}
\end{tikzcd}$$



i.e., we have: $k \circ \_$ has a left adjoint $\Longleftrightarrow$ $\mathbb{C}$ has pullbacks

Exercise Prove the equivalence between the two definitions of LCCC.

The general scheme of the interpretation

- contexts are mapped to objects of $\mathbb{C}$
- substitutions are mapped to morphisms of $\mathbb{C}$

up to def. equality

cf. category of contexts

lecture A p. 24

- types are interpreted as objects in the slice category. More precisely if $\Gamma \vdash \sigma$ type, then $\sigma \in Ob(\mathbb{C}/_\Gamma)$, i.e. $\sigma : \bullet \to \Gamma$

(interpretation of)

- context extension Γ, σ is interpreted by the domain of σ, and σ, viewed as a morphism in $\mathbb{C}$, interprets $\Gamma, \sigma \vdash \downarrow : \Gamma$
- terms are interpreted as sections

$$\Gamma' \vdash s : \Gamma \qquad \Gamma \vdash \sigma \text{ type} \qquad \Gamma \vdash M : \sigma$$

$$\begin{tikzcd} \Gamma & \Gamma' \arrow[l, "s"'] \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, bend left, "M"] & (\Gamma, \sigma) \arrow[l, bend left, "\uparrow = \sigma"] \end{tikzcd}$$

$\sigma(x)$

- as disjoint union of all fibers $\sigma(x)$ i.e. an "meta" $\Sigma x : \Gamma . \sigma(x)$

x-fiber of Γ, σ

$\|$

$M(x) \in \sigma(x)$

$\uparrow$

x

$x \in \Gamma$

(a bit like in a sequent $\underbrace{A_1 \ldots A_n}_{\text{conjunction}} \overset{\text{implication}}{\vdash} \overbrace{B_1 \ldots B_n}^{\text{disjunction}}$ we have an implicit top level

Interpretation of the core syntax

$$\begin{tikzcd}
\Gamma,\sigma \arrow[d, bend right, "\uparrow"'] & \Gamma',\sigma[s] \arrow[l] \arrow[d, bend right, "\uparrow"'] \arrow[dl, phantom, "p.b. \lrcorner"] \\
\Gamma \arrow[u, bend right, "M"'] & \Gamma' \arrow[l, "s"'] \arrow[u, bend right, "{M[s]=\langle id, M\circ s\rangle}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma,\sigma \arrow[d, "\uparrow"'] & \Gamma,\sigma,\sigma[\uparrow] \arrow[l] \arrow[d, bend right, "\uparrow"'] \arrow[dl, phantom, "p.b. \lrcorner"] \\
\Gamma & \Gamma,\sigma \arrow[l, "\uparrow"] \arrow[u, bend right, "{1=\langle id, id\rangle}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma,\sigma \arrow[d, "\uparrow"'] & \Gamma',\sigma[s] \arrow[l, "s'"] \arrow[d, bend right, "\uparrow"'] \arrow[dl, phantom, "\lrcorner"] \\
\Gamma & \Gamma' \arrow[l, "s"'] \arrow[ul, "{M'\cdot s = s'\circ M'}"] \arrow[u, bend right, "M'"']
\end{tikzcd}$$

Remark

in particular for $s = id$ we can pick $s' = id$ and get that $M' \cdot id$ and M' receive the same interpretation

What about definitional equalities?

The following equalities are OK:

$$\uparrow \circ (M \cdot s) = s$$
$$(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3)$$
$$id \circ s = s \circ id = s$$

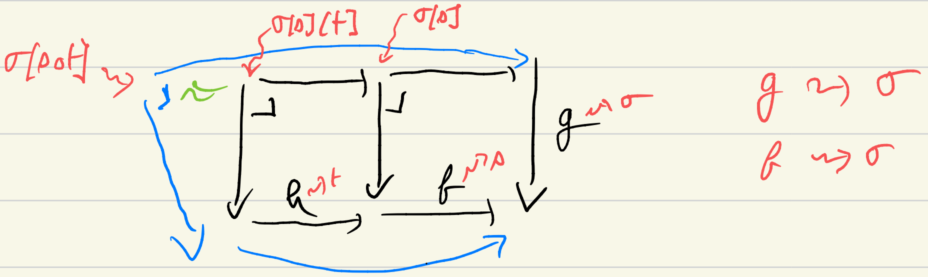
obvious

$$\uparrow \circ (M' \cdot s) = \underbrace{\uparrow \circ s'}_{s \circ \underbrace{\uparrow \quad}_{id}} \circ M'$$

BUT...

Substitution up to isomorphism

However, the type conversion rule $\boxed{\sigma[s][t] \doteq \sigma[s\circ t]}$ cannot be modelled by an equality, since in an LCCC the composition of chosen pullbacks is only isomorphic to the chosen pullback of the composite, in general. For example, in **Set**, the chosen pullback of $(f : b \to a, g : c \to a)$ is given by $\{(y,z) \in b \times c \mid fy = gz\}$, so that, for $h : b' \to b$, the pullback of g along $f\circ h$ is a subset of $b' \times c$, while the pullback along h of the pullback along f of g is a (isomorphic) subset of $b' \times (b \times c)$.



This also impacts some term or substitution equalities:

$$\dfrac{\Gamma,\sigma \vdash 1 : \sigma[\uparrow] \qquad \dfrac{\Gamma' \vdash s : \Gamma \quad \Gamma' \vdash M' : \sigma[s]}{\Gamma' \vdash M'\cdot s : \Gamma,\sigma}}{\Gamma' \vdash \underbrace{1[M'\cdot s]}_{\|?} : \sigma[\uparrow][M'\cdot s]}$$

MISMATCH!

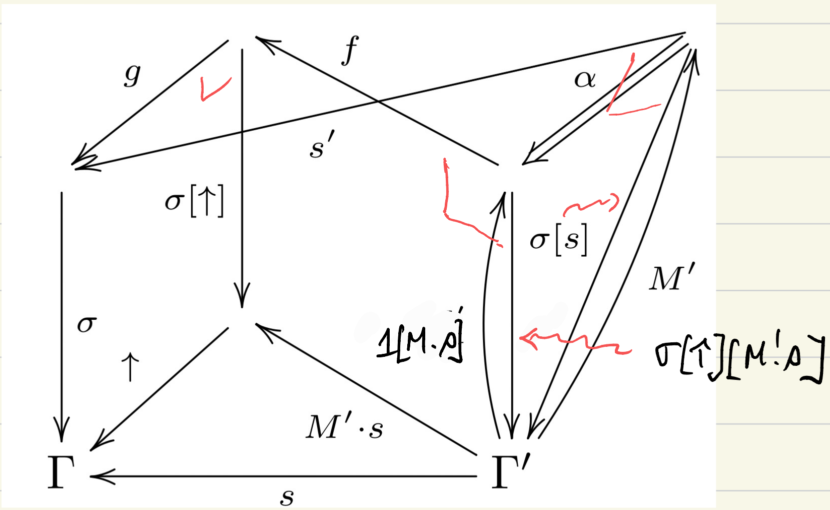
$\wr\wr$

$\sigma[\uparrow\circ(M'\cdot s)]$

$M' : \sigma[s]$ // cf. p. 6

But we can prove $\alpha \circ M' = 1[M'\cdot s]$
where α is the iso $\sigma[s] \to \sigma[\uparrow][M'\cdot s]$

Exercise Prove this.
(Hint: chase the picture)



Type theory with explicit coercions

- New constructor $M ::= \ldots \mid \mathsf{c}(M, \sigma, \sigma')$
- New judgement $\Gamma \vdash \sigma \cong \sigma'$

$$\frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash \sigma \cong \sigma'}{\Gamma \vdash \mathsf{c}(M, \sigma, \sigma') : \sigma'} \qquad \frac{\Gamma \vdash \sigma = \sigma'}{\Gamma \vdash \sigma \cong \sigma'} \qquad \frac{\Gamma' \vdash s : \Gamma \quad \Gamma \vdash \sigma \cong \sigma'}{\Gamma' \vdash \sigma[s] \cong \sigma'[s]}$$

+ symmetry and transitivité

- Equations revisited:

$$\sigma[s][t] \cong \sigma[s \circ t]$$
$$\sigma[id] = \sigma$$

← can be forced by choice

$$\mathsf{c}(1[M \cdot s], \sigma[\uparrow][M \cdot s], \sigma[s]) = M$$
$$\mathsf{c}(M[s][t], \sigma[s][t], \sigma[s \circ t]) = M[s \circ t]$$

$$\uparrow \circ (M \cdot s) = s$$
$$(s_1 \circ s_2) \circ s_3 = s_1 \circ (s_2 \circ s_3)$$
$$id \circ s = s \circ id = s$$
$$(M \cdot s) \circ t = \mathsf{c}(M[t], \sigma[s][t], \sigma[s \circ t]) \cdot (s \circ t)$$
$$\mathsf{c}(1[s], \sigma[\uparrow][s], \sigma[\uparrow \circ s]) \cdot (\uparrow \circ s) = s$$

$$\mathsf{c}(\mathsf{c}(M, \sigma_1, \sigma_2), \sigma_2, \sigma_3) = \mathsf{c}(M, \sigma_1, \sigma_3)$$
$$\mathsf{c}(M, \sigma, \sigma) = M$$
$$\mathsf{c}(M, \sigma_1, \sigma_2)[t] = \mathsf{c}(M[t], \sigma_1[t], \sigma_2[t])$$

Grothendieck Fibrations

We now formulate a more general framework than categories with pull-backs, with as benefit a conceptual space to distinguish

$$\Gamma \vdash \sigma \text{ type} \quad \text{and} \quad \Gamma, \sigma \vdash \uparrow : \Gamma$$

Definition A Grothendieck fibration is the data of two categories $\mathbb{E}$ and $\mathbb{B}$ and a functor $p : \mathbb{E} \to \mathbb{B}$ such that for every triple $(X, B, u : X \to pB)$ there exists a morphism $u_B : \bullet \longrightarrow B$ s.t.

- $p(u_B) = u$
- for all pairs of $\searrow$ and $\rightarrow$ as below s.t. $p(\searrow) = \rightarrow \xrightarrow{u}$

$$\begin{tikzcd} {} \arrow[dr] \arrow[rr, bend left=10] & & {} \\ & \bullet \arrow[r, "u_B"'] \arrow[d, dotted, no head] & B \arrow[d, dotted, no head] \\ {} \arrow[r] & X \arrow[r, "u"'] & pB \end{tikzcd}$$

total category

$(\mathbb{E})$

base category

$(\mathbb{B})$

$\exists! \rightarrow$ s.t. $p(\rightarrow) = \rightarrow$ and

$$\begin{tikzcd} {} \arrow[dr] \arrow[rr] & & {} \\ & {} \arrow[r] & {} \end{tikzcd}$$

commutes

Terminology
- We write $B[u]$ for the domain of u_B.

  ← substitution!

- u_B is called a (hyper) cartesian lifting of u.
- If $pA = X$, we say that A lies over X.
- A fibration is cloven when a choice for all cartesian liftings is given.

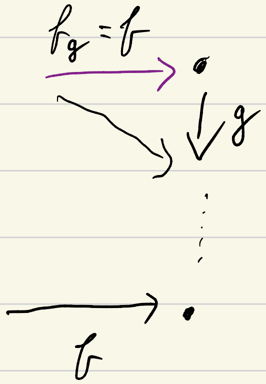
  ← scindée

- It is split if

$$u : X \to Y, v : Y \to Z, p(C) = Z \Rightarrow C[v][u] = C[v \circ u] \text{ and } v_C \circ u_{C[v]} = (v \circ u)_C$$

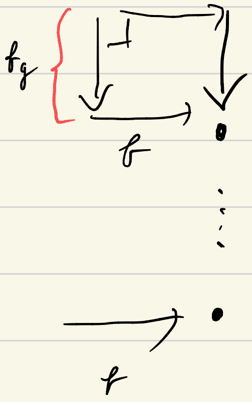
Two basic examples of fibrations

- **dom** : $\mathbb{C}^{\to} \to \mathbb{C}$ is a split fibration for any category $\mathbb{C}$, as well as its restrictions $\mathtt{dom}_X : \mathbb{C}/X \to \mathbb{C}$ (for every object X of $\mathbb{C}$).



Every morphism is cartesian!

- **cod** : $\mathbb{C}^{\to} \to \mathbb{C}$ is a fibration if and only if $\mathbb{C}$ has pullbacks (and hence this fibration is usually non-split).



The notion of fibration can be "synthesized":

Exercise Show that a fibration $p : \mathbb{E} \to \mathbb{B}$ allows to define a pseudo-functor $\tilde{p} : \mathbb{B} \to Cat$, where $\tilde{p}(X) = p^{-1}(X)$ ← fiber over X:

objects: B s.t. $pB = X$

morphisms f s.t. $pf = id$

($Ff \circ Fg$ naturally equivalent to $F(f \circ g)$, in a coherent way)

Conversely, show that a pseudofunctor $F : \mathbb{B} \to Cat$ gives rise to a fibration $\tilde{F} : \mathbb{E} \to \mathbb{B}$, where

$Ob\, \mathbb{E} = \{(X, B) \mid B \in Ob\, FX\}$

↑ Grothendieck construction

reminiscent of el F for a presheaf!

Comprehension categories

$$\begin{tikzcd}
\mathbb{E}_1 \arrow[r, "F_t"] \arrow[d] & \mathbb{E}_2 \arrow[d] \\
\mathbb{B}_1 \arrow[r, "F_b"] & \mathbb{B}_2
\end{tikzcd}$$

A morphism between two fibrations $p_1 : \mathbb{E}_1 \to \mathbb{B}_1$ and $p_2 : \mathbb{E}_2 \to \mathbb{B}_2$ is a pair $F = (F_t, F_b)$ of functors such that $p_2 \circ F_t = F_b \circ p_1$ and F_t maps cartesian morphisms to cartesian morphisms. It is called *strict* if it maps the chosen cartesian morphisms of p_1 to chosen cartesian morphisms of p_2.

notion due to BART JACOBS

A comprehension structure on a fibration $p : \mathbb{E} \to \mathbb{B}$, where $\mathbb{B}$ has pullbacks and has a terminal object (i.e., has all finite limits), is given by a morphism from p to **cod** over $\mathbb{B}$

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "P"] \arrow[dr, "p"'] & & \mathbb{B}^{\to} \arrow[dl] \\
& \mathbb{B} &
\end{tikzcd}$$

- We write, for $f \in \mathbb{E}(A,B)$

$$\begin{tikzcd}
{} \arrow[r, "P_f"] \arrow[d, "P(A)"'] & {} \arrow[d, "P(B)"] \\
{} \arrow[r, "pf"'] & {}
\end{tikzcd}$$

- For A in $\mathbb{E}$ over X, we write $P(A) : (X, A) \to X$

The diagrams interpreting the core syntax are adapted as follows (the only change is that types are interpreted as local objects, i.e. objects of $\mathbb{E}$ over Γ, i.e.,

$$\begin{tikzcd}
\Gamma \arrow[r, bend left, "M"] & (\Gamma, \sigma) \arrow[l, bend left, "\uparrow = \sigma"]
\end{tikzcd}$$

is replaced by

$$\begin{tikzcd}
\sigma \arrow[d, dash, dotted] & \\
\Gamma \arrow[r, bend left, "M"] & (\Gamma, \sigma) \arrow[l, bend left, "\uparrow = \mathcal{P}(\sigma)"]
\end{tikzcd}$$

Context extension is interpreted via the comprehension structure:

$$\text{dom}\,\underline{P(\sigma) = \Gamma, \sigma}$$

$$\begin{tikzcd}
\Gamma, \sigma \arrow[dd, bend right, "\uparrow"'] & & \Gamma', \sigma[s] \arrow[ll, "\mathcal{P}(s_\sigma)"'] \arrow[dd, bend right, "\uparrow"'] \\
\sigma \arrow[d, dash, dotted] & & \sigma[s] \arrow[d, dash, dotted] \\
\Gamma \arrow[uu, bend right, "M"'] & & \Gamma' \arrow[ll, "s"'] \arrow[uu, bend right, "M[s] = \langle id, M \circ s \rangle"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma, \sigma \arrow[dd, bend right, "\uparrow"'] & & \Gamma, \sigma, \sigma[\uparrow] \arrow[ll, "\mathcal{P}(\uparrow_\sigma)"'] \arrow[dd, bend right, "\uparrow"'] \\
\sigma \arrow[d, dash, dotted] & & \sigma[\uparrow] \arrow[d, dash, dotted] \\
\Gamma & & \Gamma, \sigma \arrow[ll, "\uparrow"'] \arrow[uu, bend right, "1 = \langle id, id \rangle"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma, \sigma \arrow[dd, bend right, "\uparrow"'] & & \Gamma', \sigma[s] \arrow[ll, "\mathcal{P}(s_\sigma)"'] \arrow[dd, bend right, "\uparrow"'] \\
\sigma \arrow[d, dash, dotted] & & \sigma[s] \arrow[d, dash, dotted] \\
\Gamma & & \Gamma' \arrow[ll, "s"'] \arrow[uull, "M' \cdot s"] \arrow[uu, bend right, "M'"']
\end{tikzcd}$$

Why moving from LCCCs to the framework of filtrations?

- More generality, but not so much...
- Gives us access to powerful stratification tools

Interpreting Π-types and Σ-types

We revert to LCCC's for simplicity. Remember that $\Sigma_k \dashv k^* \dashv \Pi_k$ (cf. p.4)

Then
- $\Sigma_\sigma \tau$ is interpreted as $\sigma \circ \tau$ ← whence the notation anticipated
  $\Pi_\sigma \tau$ ——— $\Pi_\sigma \tau$

$$\begin{tikzcd} \Gamma, \Pi_\sigma\tau \arrow[d] & \Gamma,\sigma,\tau \arrow[d] \\ \Gamma & \Gamma,\sigma \arrow[l] \end{tikzcd}$$

variable-free notation

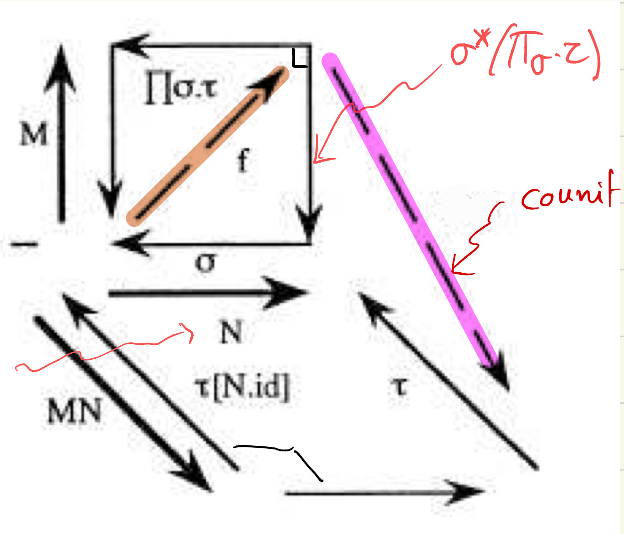
- $\lambda\sigma.M$ is interpreted "as usual":

$$\begin{array}{c} M \in \overbrace{\mathcal{C}/_{\Gamma,\sigma}(k^*\overset{\underset{\sim}{Id}}{Id}}^{\text{section}}, \tau) \\ \hline\hline \lambda\sigma.M \in \mathcal{C}/_\Gamma(Id, \Pi_\sigma\tau) \end{array}$$

one can chose Id as pull-back of Id

$$\begin{tikzcd} \cdot \arrow[rr, "\lambda\sigma M"] \arrow[dr, "id"'] & & \cdot \arrow[dl, "\Pi_\sigma\tau"] & \cdot \arrow[rr, "M"] \arrow[dr, "id"'] & & \cdot \arrow[dl, "\tau"] \\ & \cdot & & & \cdot \arrow[lll, "\sigma"] & \end{tikzcd}$$

- For application, the rule is $$\frac{\Gamma \vdash M : \Pi_\sigma\tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau[N.Id]}$$



$\sigma^*(\Pi_\sigma.\tau)$

counit

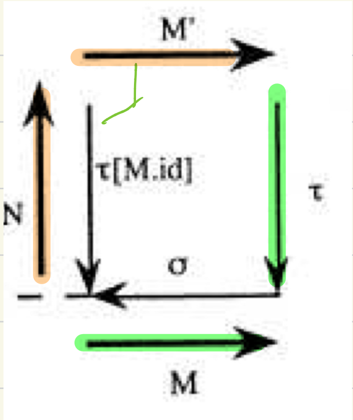
identified with N.id

cf. remark p.6

$\nearrow = \langle M, N \rangle$

$MN = \langle id, \nearrow, \searrow \rangle$

- (M,N) (for $M:\sigma$, $N:\tau[M.id]$) is interpreted as $M' \circ N$ below



We have

$$\sigma \circ \underbrace{\tau \circ M'}_{} \circ N$$
$$\underbrace{M}_{id} \circ \underbrace{\tau[M.id]}_{id}$$

- $\mathrm{Ind}_{\Sigma_\sigma.\tau}\, u\, c = \alpha \circ c$

$$\begin{tikzcd} \Pi_{\sigma\circ\tau} u & & & \\ & {} \arrow[ul, "\bar{\alpha}"'] & {} & {} \\ & \cdot \arrow[u, "\Pi_\sigma\Pi_\tau u"] \arrow[u, bend left, "c"] & \cdot \arrow[u, leftarrow, "\Pi_\tau u"] \arrow[l, "\sigma"] & \cdot \arrow[u, leftarrow, "u"] \arrow[l, "\tau"] \end{tikzcd}$$

where α (iso) is inferred by adjunction + Yoneda

$$\mathcal{C}/_{\Gamma,\sigma,\tau}(b[\sigma\circ\tau], u) \simeq \mathcal{C}/_\Gamma(b, \Pi_{\sigma\circ\tau}u)$$
$$\wr$$
$$\mathcal{C}/(\Gamma,\sigma,\tau)(b[\sigma][\tau], u) \simeq \mathcal{C}/_\Gamma(b, \Pi_\sigma\Pi_\tau u)$$

Contexts as arrows to the terminal object

If $\vdash A$ type, then A is interpreted as an an object of $\mathcal{C}$, or more precisely as an object of $\mathcal{C}/1$

The terminal object interprets the empty context

$$1 \xleftarrow{A}$$

If $A \vdash B$ type, then $\vdash A, B$ ctx

$$1 \xleftarrow{A} \xleftarrow{B} \qquad\qquad 1 \xleftarrow{A} \xleftarrow{B}$$

Etc $\Gamma \vdash \sigma$ type $\qquad \vdash \Gamma, \sigma$ ctx

$$\underbrace{1 \leftarrow \cdots \leftarrow}_{\Gamma} \xleftarrow{\sigma} \qquad\qquad \underbrace{1 \leftarrow \cdots \leftarrow \leftarrow}_{\Gamma,\sigma}$$

With this colour code, the formation of $\Sigma \sigma . \tau$ is

$$\underbrace{1 \leftarrow \cdots \leftarrow}_{\Gamma} \xleftarrow{\sigma} \xleftarrow{\tau} \rightsquigarrow \underbrace{1 \leftarrow \cdots \leftarrow}_{\Gamma} \xleftarrow{\sigma \circ \tau}$$

Isomorphisms and definitional equalities for Π and Σ types

"vertical" version of the composition of pullback squares

We add new type isomorphisms

$$\text{BECK-CHEVALLEY conditions}\left\{\begin{aligned}(\Sigma\sigma.\tau)[s] &\simeq \Sigma\sigma[s].\tau[1\cdot s\circ\uparrow]\\(\Pi\sigma.\tau)[s] &\simeq \Pi\sigma[s].\tau[1\cdot s\circ\uparrow]\end{aligned}\right.$$

Exercise Derive the latter isomorphism in the framework of LCCC's.

In the framework of comprehension categories, these isomorphisms have to be postulated.

Exercise Consider the equations (without coercions) associated with Π-types and Σ-types in the syntax with explicit substitutions:

$$(\lambda\sigma.M)N = M[N.id]$$

$$\left.\begin{aligned}(MN)[s] &= M[s]N[s]\\(\lambda\sigma.M)[s] &= \lambda\sigma.M[1\cdot s\circ\uparrow]\\(M,N)[s] &= (M[s],N[s])\end{aligned}\right\}\text{These equalities take care of step-by-step substitution}$$

$$ind_{\Sigma\sigma.\tau}\,u\,(a,b) = u\,a\,b \qquad (\Gamma,\sigma,\tau,u \text{ ctx } \Gamma\vdash a:\sigma,\ \Gamma\vdash b:\tau[a\cdot id])$$

Insert the right coercions to make them type-check.

Theorem The interpretation of the syntax with explicit coercions (with Π and Σ types) in LCCC's (and more generally in comprehension categories with extra structure) is sound, i.e., all equations (with coercions inserted) are valid.

The proof goes back to SEELY's seminal paper Locally cartesian closed categories and type theory

Extensional type theory

The interpretation of identity types in LCCC's (or comprehension categories with "identity type structure") validates the following reflection rule

$$\boxed{\frac{\Gamma \vdash p : a =_A b}{\Gamma \vdash a \equiv b : A}}$$

($\dfrac{\Gamma \vdash p : \mathrm{Id}_A(a,b)}{\Gamma \vdash a = b : A}$ in the notation used here)

"propositionally equal ⟹ definitionally equal"

In fact, it also validates

$$\boxed{\frac{\Gamma \vdash p : a =_A a}{\Gamma \vdash p \equiv (\mathrm{refl}\, a) : a =_A a}}$$

We call this strong reflection ⤳

Note that, then, the induction principle becomes tautological, since applying C or $(\mathrm{ind}\, C\, c)$ to x, y, p forces $x \equiv y$ by the presence of p, and then $p \equiv (\mathrm{refl}\, x)$ by the second rule.

Exercise Show, using the induction principle $\mathrm{Ind}_{=_A}$, that strong reflection in fact follows from reflection. (Hint: consider $C\, x\, y\, p = \mathrm{Id}(p, (\mathrm{refl}\, x))$.)

Terminology We call comprehension categories ++ the comprehension categories equipped with the additional structure allowing to interpret Σ-type, Π-types and extensional identity types.

LCCC's are models of extensional type theory

In our variable-free notation, we have $\Gamma, A, A[\uparrow] \vdash \mathrm{Id}_A$ type

We interpret $\mathrm{Id}A$ as $\langle id, id\rangle$ i.e.

$$\begin{tikzcd}
 & & \Gamma,A \arrow[lld, bend right, "id"'] \arrow[ld] \arrow[ldd, bend left, "id"] \\
\Gamma,A \arrow[d] & \Gamma,A,A[\uparrow] \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & \\
\Gamma & \Gamma,A \arrow[l] &
\end{tikzcd}$$

for the sake of highlighting the arguments better

We shall write A for $A[\uparrow]$ sloppily below

For $\Gamma \vdash a : A$, we get

- $\mathrm{Id}_A(a,b)$ as a pull-back
- $(\mathrm{refl}\ a)$ as $\langle id, a\rangle$

$$\begin{tikzcd}
\Gamma,\mathrm{Id}_A(a,a) \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma,A \arrow[d] \\
\Gamma \arrow[u, bend left] \arrow[r, "(a,a)"'] & \Gamma,A,A
\end{tikzcd}$$

justified as follows: $\xrightarrow[id]{}\xrightarrow[(a,a)]{} = \xrightarrow[a]{}\longrightarrow$ holds, because, further composing with either projection $\longrightarrow$ we get $\underbrace{\xrightarrow[id]{}\underbrace{\xrightarrow{a,a}\longrightarrow}_{a}}_{a} = \underbrace{\xrightarrow[a]{}\underbrace{\longrightarrow\longrightarrow}_{id}}_{a}$

$(a,a) \circ id = a \circ$

- Strong definitional extensionality holds:

$$\begin{tikzcd}
 & & \Gamma,A \arrow[dl, dash] \arrow[dd] \\
\Gamma,\mathrm{Id}_A(a,b) \arrow[urr] \arrow[dd] \arrow[r, phantom, "\lrcorner", very near start] & \Gamma,A & \\
 & & \Gamma,A,A \arrow[ul] \\
\Gamma \arrow[uu, bend left, "p"] \arrow[uur, "a"] \arrow[urr, "(a,b)"'] & &
\end{tikzcd}$$

We have $\nearrow\longrightarrow /\!/ = \nearrow\longrightarrow\nwarrow\downarrow = \underbrace{\nearrow\downarrow}_{id}\underbrace{\nearrow}_{\xrightarrow[a]{}}\nwarrow = a$

choosing the other projection $\nwarrow$

$= \cdots\cdots \qquad \cdots = b$

- Hence $a = b$
- Moreover p has the same specification as refl_A since

$\xrightarrow[p]{}\longrightarrow = id$ assumption

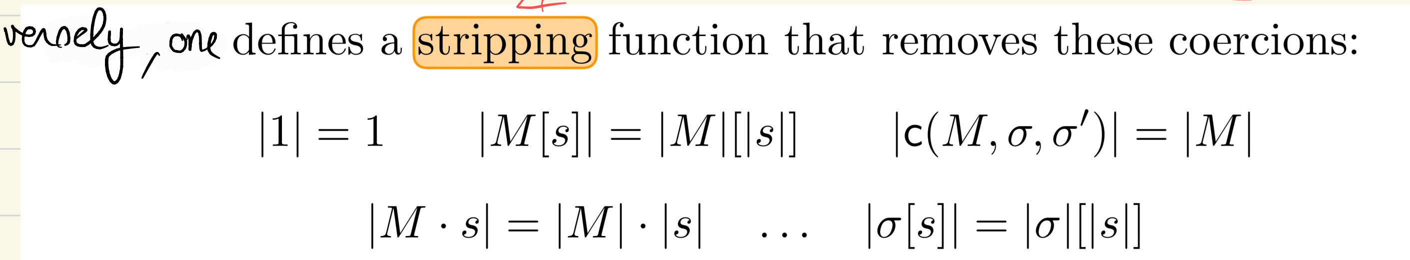
$\xrightarrow[p]{}\longrightarrow = a$ argued above

Hence $p = \mathrm{refl}\ a$

Coherence theorem

We annotate with $\boxed{\vdash_e}$ the judgements of the type theory with explicit coercions.

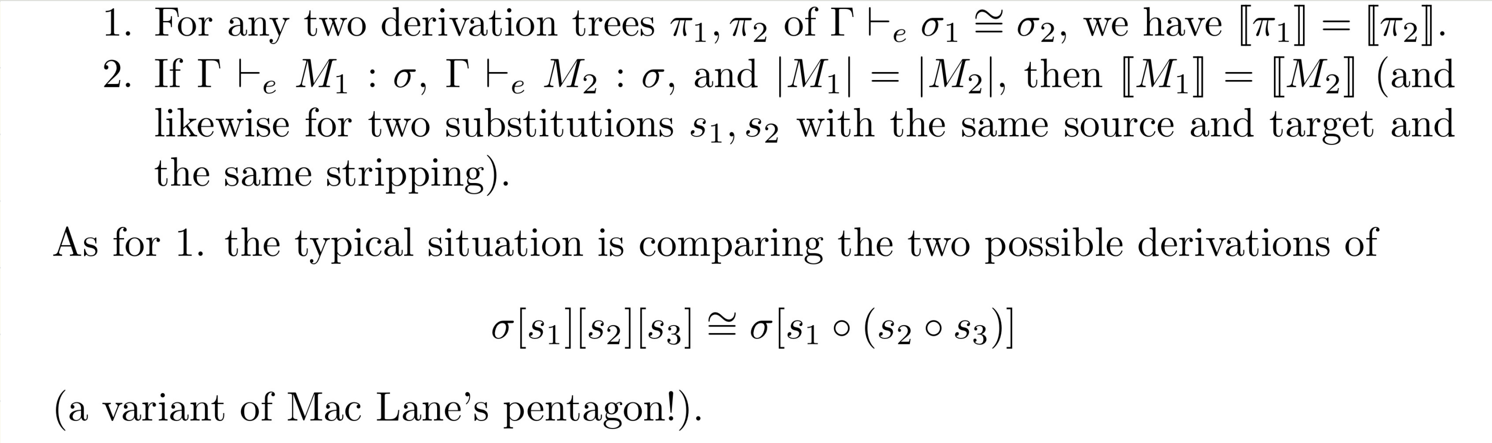
Every proof of a judgement in ordinary type theory can be "edited" to become a proof in the type theory with explicit coercions. ↳ and one recovers the original judgement by

Conversely, one 
defines a stripping function that removes these coercions:

$$|1| = 1 \qquad |M[s]| = |M|[|s|] \qquad |\mathsf{c}(M,\sigma,\sigma')| = |M|$$

$$|M \cdot s| = |M| \cdot |s| \qquad \ldots \qquad |\sigma[s]| = |\sigma|[|s|]$$

Coherence theorem (Curien 1990)



1. For any two derivation trees π_1, π_2 of $\Gamma \vdash_e \sigma_1 \cong \sigma_2$, we have $[\![\pi_1]\!] = [\![\pi_2]\!]$.
2. If $\Gamma \vdash_e M_1 : \sigma$, $\Gamma \vdash_e M_2 : \sigma$, and $|M_1| = |M_2|$, then $[\![M_1]\!] = [\![M_2]\!]$ (and likewise for two substitutions s_1, s_2 with the same source and target and the same stripping).

As for 1. the typical situation is comparing the two possible derivations of

$$\sigma[s_1][s_2][s_3] \cong \sigma[s_1 \circ (s_2 \circ s_3)]$$

(a variant of Mac Lane's pentagon!).

The original proof was given using rewriting techniques, like the proof of coherence of monoidal categories given in lecture 3.

But one can prove the theorem via strictification and we begin by doing this for monoidal categories as a warming up.

↳ JOYAL-STREET

Two strategies for addressing the mismatch cf. p.7

- Change the syntax $\rightsquigarrow$ explicit coercions

  P.-L. Curien, Substitution up to isomorphism, Fundamenta Informaticae 19 (1993)

  https://www.irif.fr/~curien/substitution.pdf

- Change the model: from LCCC's (non split) $\rightsquigarrow$ split fibrations

  M. Hofmann, On the interpretation of type theory in locally cartesian closed categories LNCS 933 (1994)

  citeseerx.ist.psi.edu/viewdoc/download?doi=10.1.1.54.4410&rep=rep1&type=pdf

- The two approaches fit in a nice conceptual picture

  see p.23

  P.-L. Curien, R. Garner and M. Hofmann, Revisiting the categorical interpretation of dependent type theory, TCS 546 (2014)

  https://www.irif.fr/~curien/CGH-Glynn-anniv-2013.pdf

Free strict monoidal categories

Remember the syntax of Lecture 3, p. 3, starting from a set $\mathbf{X}$ of object variables, with object terms and canonical terms & etc... These data define a monoidal category $\mathrm{Free}(\mathbf{X})$ that deserves its name: for every monoidal category $\mathbb{C}$, and every function $e : \mathbf{X} \to \mathrm{Ob}\,\mathbb{C}$, there exists a unique strict monoidal functor $[\![-]\!]^e_{\mathbb{C}} : \mathrm{Free}(\mathbf{X})$ extending e.

$$\left.\begin{array}{l}|(X \otimes Y) \otimes Z| \\ |X \otimes (Y \otimes Z)|\end{array}\right\} = XYZ$$

Let us see what $[\![\_]\!]^\rho_{\mathbb{C}}$ looks like when $\mathbb{C}$ is strict. We define a "stripping" function $X \mapsto |X|$ from object terms to words over X defined by forgetting the parentheses and the I's:

$$|T_1 \otimes T_2| = |T_1||T_2| \qquad |I| = \epsilon \text{ (the empty word)}$$

We have the following properties (for the second and the third one, $\mathbb{C}$ is supposed strict):

- if $M : T \to T'$ is a well-typed term, then $|T| = |T'|$;
- if $|T| = |T'|$ then $[\![T]\!]^\rho_{\mathbb{C}} = [\![T']\!]^\rho_{\mathbb{C}}$;
- for any well-typed term M, one has $\boxed{[\![M]\!]^\rho_{\mathbb{C}} = id}$.

Remark In fact, we have that X^*, viewed as a discrete category, is strict monoidal, and hence the map $\subseteq : X \to X^*$ sending x to the word x of length 1 extends to a strict monoidal functor

$$\boxed{\begin{array}{l}|-| : \mathrm{Free}(X) \to X^* \\ \quad \shortparallel \\ [\![-]\!]^{\subseteq}_{X^*}\end{array}}$$

Strictly, we complete stripping by setting $|M| = id$, and we have

$$[\![M]\!] = \varepsilon\{|M|\} = \varepsilon\{id\} = id$$

see p. 23 for this notation

Proof architecture

Here, monoidal means strong monoidal, cf lecture 3 p.7

- Let us now revert to monoidal categories. Let $(F, \beta) : \mathbb{C} \to \mathbb{C}'$ be a monoidal functor between monoidal categories (i.e., $\beta : FC_1 \otimes FC_2 \to F(C_1 \otimes C_2)$ are natural isos satisfying the obvious compatibility conditions). The freeness of $\mathit{Free}(X)$ "lives" in the category $\mathbb{M}\mathit{onCat}_s$ of monoidal categories and *strict* monoidal functors (i.e. those for which $\beta = id$, and hence $F\alpha = \alpha$, $F\lambda = \lambda$, and $F\rho = \rho$). Hence we cannot directly relate $[\![\_]\!]^{\rho}_{\mathbb{C}}$ and $[\![\_]\!]^{F\rho}_{\mathbb{C}'}$. But the following "glueing property" can be proved: for all T there exists an isomorphism $\gamma_{\rho,T} : F([\![T]\!]^{\rho}_{\mathbb{C}}) \to [\![T]\!]^{F\rho}_{\mathbb{C}'}$, such that for every $M : T \to T'$ the following coherence equation holds:

$$\begin{tikzcd} F[\![T]\!]_{\mathbb{C}} \arrow[r, "F[\![M]\!]_{\mathbb{C}}"] \arrow[d, "\gamma"'] & F[\![T']\!]_{\mathbb{C}} \arrow[d, "\gamma"] \\ {[\![T]\!]_{\mathbb{C}'}} \arrow[r, "{[\![M]\!]_{\mathbb{C}'}}"] & {[\![T']\!]_{\mathbb{C}'}} \end{tikzcd}$$

$$\gamma_{\rho,T'} \circ F[\![M]\!]^{\rho}_{\mathbb{C}} = [\![M]\!]^{F\rho}_{\mathbb{C}'} \circ \gamma_{\rho,T} \,.$$

$$\begin{tikzcd} \mathit{Free}(X) \arrow[d, "{[\![\,]\!]}"] \arrow[dd, bend right, "{[\![\,]\!]}"'] \\ \mathbb{C} \arrow[d, "F"] \\ \mathbb{C}' \end{tikzcd}$$

The isomorphisms $\gamma_{\rho,T}$ are defined by induction on the structure of T (the "types"), and then the coherence equation is proved by induction on the structure of the morphism terms M.

- Finally, the most crucial part of Joyal-Street's proof relies on the existence of a faithful monoidal functor $F : \mathbb{C} \to \mathbb{C}^s$, where $\mathbb{C}^s$ is a strict monoidal category associated with $\mathbb{C}$ (next page). In fact, F is a monoidal equivalence, but we do not need it.

- If in (B) we take F to be the faithful functor from $\mathbb{C}$ to $\mathbb{C}^s$ (given in (C)), we see that if $M_1, M_2 : T \to T'$, then (using (A)):

$$F[\![M_1]\!]^{\rho}_{\mathbb{C}} = \gamma^{-1}_{\rho,T'} \circ [\![M_1]\!]^{F\rho}_{\mathbb{C}'} \circ \gamma_{\rho,T} = \gamma^{-1}_{\rho,T'} \circ \gamma_{\rho,T} = \gamma^{-1}_{\rho,T'} \circ [\![M_2]\!]^{F\rho}_{\mathbb{C}'} \circ \gamma_{\rho,T} = F[\![M_2]\!]^{\rho}_{\mathbb{C}}$$

from which $[\![M_1]\!]^{\rho}_{\mathbb{C}} = [\![M_2]\!]^{\rho}_{\mathbb{C}}$ follows by faithfulness.

Joyal - Street strictification

It remains to define $\mathbb{C}^s$ and the faithful embedding. The objects of $\mathbb{C}^s$ are pairs (E,δ), where E is (just) an endofunctor on $\mathbb{C}$ and $\delta : E(A)\otimes B \to E(A\otimes B)$ is an iso, natural in A, B, and is required to commute appropriately with α. The tensor product is given by

$$(E,\delta)\otimes(E',\delta') = (E\circ E', E\delta'\circ\delta)\ .$$

This tensor product is strict: composition of functors is associative!

The functor $F : \mathbb{C} \to \mathbb{C}^s$ takes A to $(A\otimes \_, \alpha)$. It is faithful since one may recover $f : A \to B$ from $F(f)_I$.

$$\alpha : \underbrace{(A\otimes B)}_{EB}\otimes C \to \underbrace{A\otimes(B\otimes C)}_{E(B\otimes C)}$$

Remark A more conceptual view of this construction is obtained by switching dimensions by one and considering our monoidal category as a bicategory (let us still call it $\mathbb{C}$) with just one object $\star$. We may now consider the Yoneda embedding

$$\mathcal{Y} : \mathbb{C} \to \mathbf{Cat}^{\mathbb{C}^{op}}\ .$$

One observes that $\mathbb{C}^s$, again considered as a bicategory, is the full sub-bicategory of $\mathbf{Cat}^{\mathbb{C}^{op}}$ having $\mathcal{Y}(\star)$ as (unique) object, and that F is the corestriction of $\mathcal{Y}$ to $\mathbb{C}^s$. Under these glasses, the strictness of $\mathbb{C}^s$ follows from the following sequence of observations: **Cat** is a 2-category, that is, a strict bicategory, and therefore so is $\mathbf{Cat}^{\mathbb{C}^{op}}$, and therefore so is $\mathbb{C}^s$. In two words, Yoneda strictifies!

Three large categories

$\mathbf{Mon} \rightsquigarrow$ { monoidal categories
monoidal functors

$\mathbf{Mon_s} \longrightarrow$ { monoidal categories
strict monoidal functors

- Free(X) lives here as free over X

$\mathbf{SMon_s} \dashrightarrow$ { strict monoidal categories
strict monoidal functors

- X^* (discrete strict monoidal) lives here as free over X

$$\begin{tikzcd}
X^* \arrow[rrdd, dashed, "\underline{\{\!\{-\}\!\}^{F\circ\rho}_{\mathbb{C}'}}"'] & Free(X) \arrow[l, "|-|"'] \arrow[rd, "{[\![-]\!]^{\rho}_{\mathbb{C}}}"] \arrow[rdd, "\underline{[\![-]\!]^{F\circ\rho}_{\mathbb{C}'}}" description] & \\
 & & \mathbb{C} \arrow[d, squiggly, "F"] \\
 & & \mathbb{C}' = \mathbb{C}^s
\end{tikzcd}$$

commutes

cf. p.21

commutes up to coherent isos ✓

different notation to stress different flavours of freeness!

END of warming up!

Back to comprehension categories

The goal is to imitate Joyal-Street's proof, replacing monoidal categories by fibrations, comprehension, comprehension ff.
So we need a way to strictify, i.e. to associate a split fibration with a fibration.

The Giraud-Bénabou construction

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration. We define $p' : \mathbb{E}' \to \mathbb{B}$ as follows:

- Ob $\mathbb{E}'$ consists of pairs $(X, \varphi : \mathrm{dom}_X \to p)$. We set $p'(X,\varphi) = X$

object of $\mathbb{B}$

morphism of fibrations over $\mathbb{B}$ i.e., $\forall f : g_1 \to g_2$ in $\mathbb{B}/X$, $\varphi(f)$ is cartesian

$$\begin{tikzcd} \varphi \arrow[d, dash, dotted] \\ X \end{tikzcd} \qquad \begin{tikzcd} & \varphi(f) \arrow[d, dash, dotted] \\ X & Y \arrow[l, "f"'] \end{tikzcd}$$

- $\mathbb{E}'[(X,\phi),(Y,\psi)]$ consists of the pairs (t,μ) where $t \in \mathbb{B}[X,Y]$ and μ is a natural transformation from ϕ to $\psi \circ \mathtt{dom}_t$ over $\mathbb{B}$, i.e. $\mu_s : \phi(s) \to \psi(t \circ s)$ and $p(\mu_s) = id_Z$ for all $s : Z \to X$.

$$\begin{tikzcd} \varphi(s) \to \psi(t\circ s) \arrow[d, dash, dotted] \\ Z \arrow[r, "s"] & X \arrow[r, "t"] & Y \end{tikzcd}$$

- We check that p' is a fibration: for $t : X \to Y$ we set $\phi[t] = \phi \circ \mathtt{dom}_t$, i.e. $\phi[t](s) = \phi(t \circ s)$ for all $s : Z \to X$, and $t_\phi = (t, id)$. Moreover, p' is split, by the associativity of composition in $\mathbb{B}$.

Synthesising $\psi = \varphi[t]$:

$$\begin{tikzcd} & \psi \arrow[d, dash, dotted] & \varphi \arrow[d, dash, dotted] \\ Z \arrow[r, "s"'] & X \arrow[r, "t"'] & Y \end{tikzcd}$$

We look for $\psi(s)$, and $\varphi(t\circ s)$ is a candidate

$$\begin{tikzcd} \psi(s) \arrow[d, dash, dotted] \\ Z \end{tikzcd}$$

- This carries over to comprehension structures:

$$\mathcal{P}'(\phi) = \mathcal{P}(\phi(id)) \qquad \mathcal{P}'(t,\mu) = \mathcal{P}(\psi(t : t \to id) \circ \mu_{id})$$

- Finally, we define $F : \mathbb{E} \to \mathbb{E}'$ over $\mathbb{B}$ by

$$\begin{tikzcd} & A[f] \arrow[d, dash, dotted] \\ X & Y \arrow[l, "f"'] \end{tikzcd}$$

$$F(A) = (X, \phi), \ \text{ where } \phi(f : Y \to X) = A[f] .$$

For $f : A_1 \to A_2$, we set $\overset{pA}{\overset{\shortparallel}{F}}(f) = (u, \mu)$ where $u = p(f)$ and where, for every $v : X \to X_1 = p(A_1)$, μ_v is the unique morphism from $A_1[v]$ to $A_2[u \circ v]$ such that $(u \circ v)_{A_2} \circ \mu_v = (f \circ v_{A_1})$.

This functor is faithful since one can recover f from μ_{id}.

Like for Joyal-Street, the objective of defining F guides the definition of p'.

Mutatis mutandis

One defines two classifying comprehension categories ++ cf. p.16

$\boxed{\mathrm{Synt}^e}$ and $\boxed{\mathrm{Synt}}$, based respectively on $\vdash_e$ and $\vdash$

- Base objects are equivalence classes of contexts.
- A base morphism from a base object with representative Γ to a base object with representative Δ is an equivalence class of a substitution $\Gamma \vdash s : \Delta$.
- Local objects are equivalence classes of types. The fibration functor p sends a class with representative σ well-typed over Γ to the class of Γ.
- A morphism from the local object with representative σ well-formed over Γ to the local object with representative τ well-formed over Δ is a pair $([s],[t])$ of equivalence classes, where $\Gamma \vdash s : \Delta$, $(\Gamma,\sigma) \vdash t : (\Delta,\tau)$, and $\uparrow \circ\, t = s \circ \uparrow$ is provable.

equivalence classes quotient w.r.t equalities, not isos

$\mathbf{Fib}, \mathbf{ML} \rightsquigarrow$ { fibrations, comprehension categories ++ / morphisms of fibrations ++ }

$\mathbf{Fib}_s, \mathbf{ML}_s \longrightarrow$ { fibrations, comprehension categories ++ / strict morphisms of fibrations ++ } • Synt^e lives here

$\mathbf{SFib}_s, \mathbf{SML}_s \dashrightarrow$ { split fibrations on comprehension categories ++ / strict morphisms of fibrations ++ } • Synt lives here

$$\begin{tikzcd}
\mathrm{Synt}^e \arrow[dr, "{[\![-]\!]_1}"] \arrow[ddr, "{[\![-]\!]_2}"'] & \\
& p_1 \arrow[d, squiggly, "F"] \\
& p_2
\end{tikzcd}$$

CURIEN

natural glueing isos γ

$$\begin{tikzcd}
\mathrm{Synt} \arrow[ddr, dashed, "{\{\!\!\{-\}\!\!\}}"'] & \mathrm{Synt}^e \arrow[l, "|-|"'] \arrow[dd, "{[\![-]\!]}"] \\
& \\
& p
\end{tikzcd}$$

HOFMANN

split

Let $\Gamma \vdash_e M_1 : \sigma$, $\Gamma \vdash_e M_2 : \sigma$ s.t. $|M_1| = |M_2|$. Then

$$[\![M_1]\!] = \gamma_{\Gamma,\sigma} \circ \{\!\!\{|M_1|\}\!\!\} \circ \gamma_\Gamma^{-1} = \gamma_{\Gamma,\sigma} \circ \{\!\!\{|M_2|\}\!\!\} \circ \gamma_\Gamma^{-1} = [\![M_2]\!]$$