# Lecture C Homotopy type theory

(Lecture A, p.23)

Recall the induction principle for identity types, informally:

- if we have a type family $C(x,y,p)$ depending on $x : A$, $y : A$, $p : x =_A y$,
- if we have a term $c(x)$ depending on $x : A$ of type $C(x,x,(\text{refl}\, x))$

← (historical name for $\text{ind}_{=_A} C\, c$)

- then we have a term $(J\, c)(x,y,p)$ depending on $x : A$, $y : A$, $p : x =_A y$ of type $C(x,y,p)$, s.t. $(J\, c)\, x\, x\, (\text{refl}\, x) \equiv c\, x$

Idea: to define $q\, o\, p$ it is enough to define $q\, o\, \text{refl}$ (as $q$) and then use induction

## Elementary properties involving identity types

The following properties are easy applications of path induction:

- Paths can be inverted and composed.
- These operations have the structure of a weak groupoid, with `refl` as identity.

↪ more on this to come!

- Functions are functors: if $A : U$, $B : U$, if $f : A \to B$, and if $x, y : A$ and $p : (x =_A y)$, then we can define

$$\overbrace{\text{ap}_f\, x\, y\, p}^{\text{“}fp\text{”}} : ((f\, x) =_B (f\, y))$$

by path induction, setting $\text{ap}_f\, x\, x\, (\texttt{refl}\, x) :\equiv \texttt{refl}_{(f\, x)}$.

But what if $f : \Pi_{x:A}(B\, x)$ is a dependent function? We have that $(f\, x) : (B\, x)$ and $(f\, y) : (B\, y)$ do not live in the same type... So we must do something first.

# Transport

This is probably the most ubiquitous application of path induction. Consider a type family $C : A \to U$. We can define the type family

$$D\, x\, y\, p :\equiv (C\, x) \to (C\, y)$$

Then

$$\mathtt{transport}^C :\equiv (\mathtt{ind}_{=_A}\, D\, (\lambda x.\mathtt{id}_{(C\,x)})) : \Pi_{x,y:A}\Pi_{p:(x=_A y)}(C\, x) \to (C\, y)$$

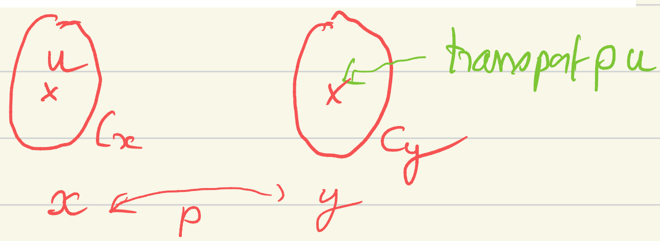
We often write $(\mathtt{transport}^C\, p)$ for $(\mathtt{transport}^C\, x\, y\, p)$

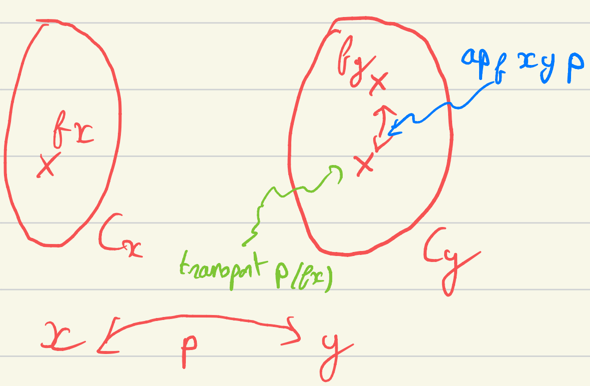
In English, given $p : (x =_A y)$ and $u : (C\, x)$, $(\mathtt{transport}\, p\, u)$ **transports** $u$ from $(C\, x)$ to $(C\, y)$.

We can then define, for $f : \Pi_{x:A}(C\, x)$, $x, y : A$ and $p : (x =_A y)$:

$$\mathtt{ap}_f\, x\, y\, p : ((\mathtt{transport}\, p\, (f\, x)) =_{(B\, y)} (f\, y))$$

by $\mathtt{ap}_f\, x\, x\, (\mathtt{refl}_x) :\equiv \mathtt{refl}_{(f\, x)}$.




## Some properties of transport

- We have, by definition, for $C : A \to U$, $x : A$ and $u : (C\, x)$:

$$(\mathtt{transport}^C\, (\mathtt{refl}_x)\, u) \equiv u$$

One can show by path induction that transport respects composition of paths:

$$(\mathtt{transport}^C\, q) \circ (\mathtt{transport}^C\, p) \sim (\mathtt{transport}^C\, (p \cdot q))$$

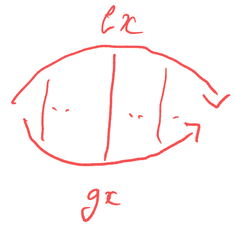
- If $C : B \to U$, $f : A \to B$, $p : (a_1 =_A a_2)$ and $u : (C\, (f\, a_1))$, then

$$\mathtt{transport}^C\, (\mathtt{ap}_f\, p)\, u = \mathtt{tansport}^{C \circ f}\, f\, u$$

- If $A : U$ and $C : A \to U$ is defined by $C\, x :\equiv (a_1 =_A x)$ ($a_1$ fixed), then, for $p : (a_2 =_A a_3)$ and $q : (C\, a_2)$, we have:

$$\mathtt{transport}^C\, p\, q = q \cdot p$$

# Equivalence of types




We set (pointwise equality)

$$f \sim_{A \to B} g :\equiv \Pi_{x:A}(f\,x =_B g\,x)$$

An inhabitant of $f \sim_{A\to B} g$ is called a homotopy from $f$ to $g$.
The following is non-official home-cooked notation!
We say that $A, B$ are equivalent (notation $\stackrel{\sim}{\longleftrightarrow}$) via $f : A \to B$ if there is an inhabitant in $(\mathtt{qinv}\, f)$, where

homotopy equivalence!

$$\mathtt{qinv}\, f :\equiv \Sigma_{g:B\to A}(f \circ g \sim \mathtt{id}) \times (g \circ f \sim \mathtt{id})$$

We shall also consider a weaker notion: two types $A$ and $B$ are logically equivalent (notation $A \longleftrightarrow B$) if we can exhibit $f : A \to B$ and $g : B \to A$.

logical implication

Exercise: Show that $\Pi_{f:0\to 0}(f \sim \mathtt{id})$ is inhabited.

## An example of logical equivalence

We set

$$\mathtt{biinv}\, f :\equiv (\Sigma_{g:B\to A}(f \circ g \sim \mathtt{id})) \times (\Sigma_{h:B\to A}(h \circ f \sim \mathtt{id}))$$

We have

$$(\mathtt{qinv}\, f) \longleftrightarrow (\mathtt{biinv}\, f)$$

Proof: In one direction, if $g$ is a quasi-inverse, then take $g, h$ to be both $g$
Conversely, we note that if $f \circ g \sim \mathtt{id}$ and $h \circ f \sim \mathtt{id}$, then (using the properties listed in the previous slide)

$$g \sim (h \circ f) \circ g \sim h \circ (f \circ g) \sim h$$

and hence $g \circ f \sim h \circ f \sim \mathtt{id}$, establishing $(\mathtt{qinv}\, f)$ via $g$.

# Characterising some identity types

- Product types: One can show that

$$(x =_{A \times B} y) \overset{\sim}{\longleftrightarrow} ((\mathtt{pr}_1\, x =_A \mathtt{pr}_1\, y) \times (\mathtt{pr}_2\, x =_B \mathtt{pr}_2\, y))$$

- 0-ary case: One can show that $(x =_1 y) \overset{\sim}{\longleftrightarrow} 1$ and for U

For function types (and Π-types), we need additional **axioms**, which are:

- Function extensionality: We assume

$$(f =_{A \to B} g) \overset{\sim}{\longleftrightarrow} (f \sim_{A \to B} g)$$

- Univalence: We set

$$A \cong B :\equiv \Sigma_{f : A \to B} (\mathtt{biinv}\, f)$$

and write witnesses of this type as $(f, g, h, \eta, \epsilon)$. Then we assume

$$(A =_U B) \overset{\sim}{\longleftrightarrow} (A \cong B)$$

where in both cases the equivalence is via the canonical morphism from the equality type to its characterisation.

It can be proved that univalence implies function extensionality.

Dependent version

$$x =_{\Sigma x : A . B} y \overset{\sim}{\longleftrightarrow} \Sigma_{p : \mathrm{pr}_1 x =_A \mathrm{pr}_1 y}\ \mathrm{transport}\ p\ \mathrm{pr}_2 x =_{B(\mathrm{pr}_1 y)} \mathrm{pr}_2 y$$

## Canonical morphisms (function extensionality, univalence)

• The family of canonical morphisms from $(f =_{A\to B} g)$ to $(f \sim_{A\to B} g)$ is defined by path induction: when $f$ and $g$ coincide, then $\lambda x.\mathtt{refl}_x : (f \sim_{A\to B} f)$.

• The canonical morphism $\mathtt{idtoeq} : (A =_U B) \to (A \cong B)$ is defined using transport for the family $\mathtt{id} : U \to U$. Let $p : (A =_U B)$. We have

$$
\begin{array}{l}
\mathtt{transport}^{\mathtt{id}_U}\, p : A \to B \\
\mathtt{transport}^{\mathtt{id}_U}\, p^{-1} : B \to A \\
(\mathtt{transport}^{\mathtt{id}_U}\, p) \circ (\mathtt{transport}^{\mathtt{id}_U}\, p^{-1}) \sim (\mathtt{transport}^{\mathtt{id}_U}\, \mathtt{refl}_B) :\equiv \mathtt{id}_B \\
(\mathtt{transport}^{\mathtt{id}_U}\, p^{-1}) \circ (\mathtt{transport}^{\mathtt{id}_U}\, p) \sim (\mathtt{transport}^{\mathtt{id}_U}\, \mathtt{refl}_A) :\equiv \mathtt{id}_A
\end{array}
$$

Therefore we can set $\mathtt{idtoeq}\, p$ to be

$$((\mathtt{transport}^{\mathtt{id}_U}\, p), (\mathtt{transport}^{\mathtt{id}_U}\, p^{-1}), (\mathtt{transport}^{\mathtt{id}_U}\, p^{-1}), \eta, \epsilon)$$

where $\eta, \epsilon$ witness the above two pointwise equalities.

## The equations of univalence

We recall that the univalence axiom says that $(A =_U B) \overset{\sim}{\longleftrightarrow} (A \cong B)$ via $\mathtt{idtoeq}$. Unrolling the definition, it says that we assume the existence of

$\mathtt{ua} : (A \cong B) \to (A =_U B)$, such that

$\mathtt{idtoeq} \circ \mathtt{ua} \sim \mathtt{id}_{(A\cong B)}$. Hence, for $e = (f, g, h, \eta, \epsilon)$, we have, in particular:

$$
\begin{array}{l}
\mathtt{transport}^{\mathtt{id}_U}\, (\mathtt{ua}\, e) = f \\
\mathtt{transport}^{\mathtt{id}_U} (\mathtt{ua}\, e)^{-1} = g = h
\end{array}
$$

Slowly, we have

$$\mathtt{transport}\, (\mathtt{ua}\, e) = \mathtt{pr}_1(\mathtt{idtoeq}\, (\mathtt{ua}\, e)) = (\mathtt{pr}_1\, e) = f$$

# An application of univalence

In usual mathematics:

- If $f$ is a bijection from $A$ to $B$, and if we have a semigroup structur on $A$, with multiplication $m : A \times A \to A$, we can define a binary operation on $B$, by setting

$$m'(b_1, b_2) = f(m(f^{-1}(b_1), f^{-1}(b_2)))$$

- We can then prove that $m'$ is associative, and hence that we have defined a semigroup structure on $B$.

In **univalent** mathematics, the perspective is different. We define

$$\texttt{Semigroup}\, A :\equiv \Sigma_{m:(A\times A\to A)} \Pi_{x,y:A} (m(m(x,y),z) =_A m(x, m(y,z)))$$

- If $e : (A \cong B)$ and $(m, a) : (\texttt{Semigroup}\, A)$, then

$$\texttt{transport}^{\texttt{Semigroup}} (\mathrm{ua}\, e)\, (m, a)$$

is automatically a semigroup structure on $B$.

- We then need to do a bit of computation to unroll this structure, and to discover that it is indeed equal to some $(m', a')$ where $m'$ is the one constructed above

We define, for $e = (f, g, h, \eta, \epsilon)$:

$$m' :\equiv \mathrm{pr}_1\, (\texttt{transport}^{\texttt{Semigroup}} (\mathrm{ua}\, e)\, (m, a))$$

We have (with $R :\equiv \lambda X.((X \times X) \to X$ and $Q :\equiv \lambda X.(X \times X))$:

$$\begin{aligned}
& m'(b_1, b_2) \\
&\quad = (\texttt{transport}^R\, (\mathrm{ua}\, e)\, m)\, (b_1, b_2) \\
&\quad = \texttt{transport}^{\mathrm{id}_U} (\mathrm{ua}\, e)\, m(\texttt{transport}^Q\, (\mathrm{ua}\, e)^{-1}\, (b_1, b_2)) \\
&\quad = (f\; m(\texttt{transport}^Q\, (\mathrm{ua}\, e)^{-1}\, (b_1, b_2))) \\
&\quad = (f\; m((\texttt{transport}^{\mathrm{id}_U}\, (\mathrm{ua}\, e)^{-1}\, b_1), \texttt{transport}^{\mathrm{id}_U}\, (\mathrm{ua}\, e)^{-1}\, b_2))) \\
&\quad = (f\, (m\, ((g\, b_1), (g\, b_2))))
\end{aligned}$$

# h-levels

- We define $\boxed{\mathrm{Prop}(A) :\equiv \prod_{x,y:A} x =_A y}$ "a proposition has at most one element, in which case it is true"
- We define $\boxed{\mathrm{Set}(A) :\equiv \prod_{x,y:A} \mathrm{Prop}(x =_A y)} \equiv \prod_{x,y:A} \prod_{p,q : x =_A y} p =_{x=_A y} q$
- We define $\boxed{\text{is-}n\text{-type}(A) :\equiv \begin{cases} \mathrm{Prop}(A) & (n=-1) \\ \prod_{x,y:A} \text{is-}(n-1)\text{-type}(x =_A y) & (n \geq 0) \end{cases}}$

Thus $\mathrm{Prop} = \text{is-}(-1)\text{-type}$, $\mathrm{Set} = \text{is-}0\text{-type}$.

Exercise Justify the names Groupoid and (weak) 2-groupoids for is-1-type, is-2-type, respectively

Hence the choice of 0 for Set (and hence $-1$ for Prop) is made to match geometrical/homotopical dimensions

If is-n-type(A) is inhabited, we say that A is of h-level n

- We can in fact start from level $-2$!
- We define $\boxed{\mathrm{isContr}(A) :\equiv \sum_{x:A} \prod_{y:A} x =_A y}$ "A has exactly one element"

contractible — center of contraction

Proposition We have a logical equivalence $\mathrm{Prop}(A) \longleftrightarrow \prod_{x,y:A} \mathrm{isContr}(x =_A y)$

The following exercises indicate the steps for this proof.

By the proposition, we can extend h-levels downwards by 1 by setting $\text{is-}(-2)\text{-type} = \mathrm{isContr}$

Exercise[1] Exhibit a logical implication $\mathrm{isContr}(A) \to \mathrm{Prop}(A)$ (Hint: use transitivity of propositional equality)

Exercise[2] Exhibit a logical implication $\mathrm{Prop}(A) \to \mathrm{Set}(A)$ (Hint: fix some $z_0 : A$ and establish that (using last property p.2) one has, for $c : \mathrm{Prop}(A)$ and for all $x, y, p : x =_A y$: $p = c_{z_0,y} \circ (c_{z_0,x})^{-1}$).

Exercise[3] Exhibit a logical implication $\mathrm{Prop}(A) \times A \to \mathrm{isContr}(A)$

Proof $\leftarrow$ If $d : \prod_{x,y:A} \mathrm{isContr}(x =_A y)$, then take the first projection of $d_{xy}$.

$\to$ From $c : \mathrm{Prop}(A)$ we get $c' : \mathrm{Set}(A)$ by exercise 2, hence $c'_{xy} : \mathrm{Prop}(x =_A y)$, and we apply exercise 3 to $(c'_{xy}, c_{xy})$ THUS

Exercise

Let $P : A \to U$ be a type family. Show that:

(i) If each $P(x)$ is contractible, then $\Sigma_{x:A} P(x)$ is equivalent to $A$.

(ii) If $A$ is contractible with center $a$, then $\Sigma_{x:A} P(x)$ is equivalent to P(a).

| -2 | -1 | 0 | 1 | 2 | ... |
|---|---|---|---|---|---|
| isContr | Prop | Set | Groupoid | 2-groupoid | |

## Truncation principles

We set

alternative notation for $a =_A b$

$$\underline{A}_0() = A$$
$$\underline{A}_1(a,b) = \underline{A}(a,b) = \overbrace{Id_A(a,b)}$$
$$\underline{A}_2(a,b;p,q) = Id_{\underline{A}(a,b)}(p,q) \quad \ldots$$

$$\frac{\Gamma \vdash p : \underline{A}_{n+1}(a_1,b_1;\ldots;a_{n+1},b_{n+1})}{\Gamma \vdash a_{n+1} = b_{n+1} : \underline{A}_n(a_1,b_1;\ldots;a_n,b_n)}\; n-\text{truncation}$$

Note that 0-truncation amounts to collapse the intensional and extensional equalities (cf. lecture B p. 16)

These principles form a strict hierarchy (one can find a weak $n$-groupoid model that validates $n$-truncation and invalidates $n-1$-truncation).

# A glimpse of synthetic homotopy theory

The circle $S^1$ is the type defined as follows.

- $S^1 : U$
- Constructors : $\mathrm{base} : S^1$

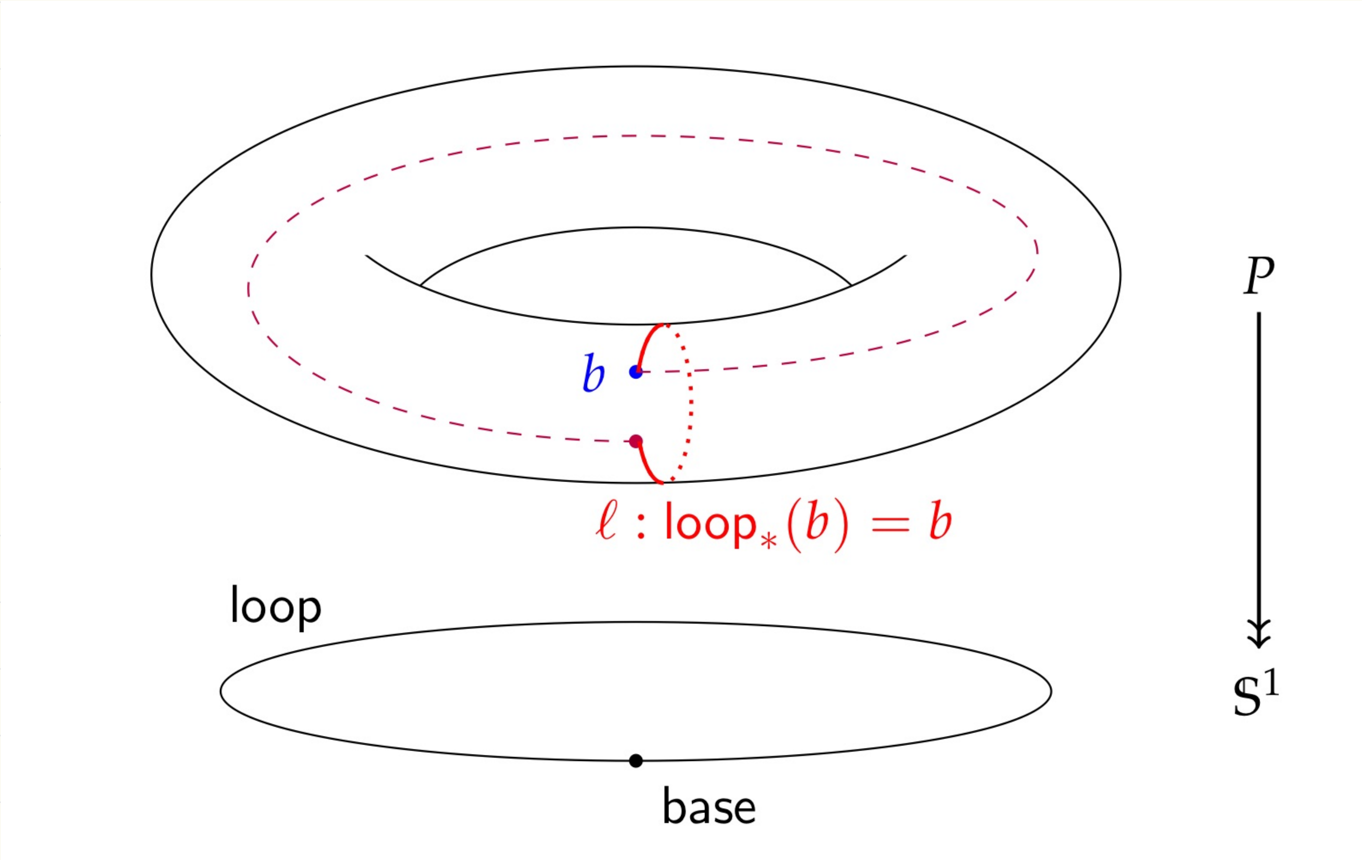
  $\mathrm{loop} : \mathrm{base} =_{S^1} \mathrm{base}$
- Induction principle

note that the presence of a **SPECIFIC** element of $\mathrm{base} =_{S^1} \mathrm{base}$ does not contradict the idea that refl is the only generic elt of $=_{S^1}$.

$$\boxed{\mathrm{ind}_{S^1} : \prod_{P : S^1 \to U} \left(\left(\sum_{b : P(\mathrm{base})} \big(\underbrace{\mathrm{transport}^P\, \mathrm{loop}}_{\text{also written } \mathrm{loop}_*}\, b =_{P(\mathrm{base})} b\big)\right) \to \left(\prod_{x : S^1} P x\right)\right)}$$

with associated reductions

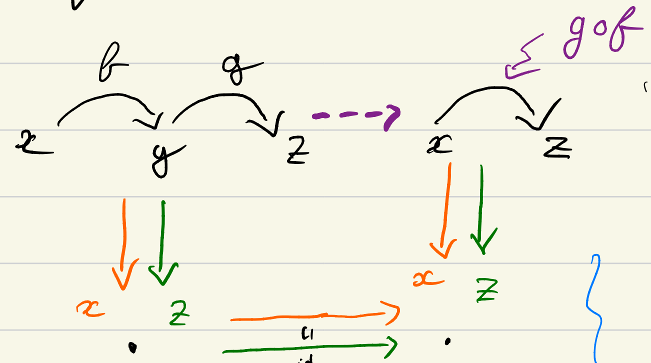
$$\boxed{\begin{aligned} &\mathrm{ind}_{S^1}\, P\, (b, \ell)\ \mathrm{base} \equiv b \\ &\mathrm{ap}_{\mathrm{ind}_{S^1} P (b,\ell)}\ \mathrm{base}\ \mathrm{base}\ \mathrm{loop} \equiv \ell \end{aligned}}$$



# Towards a higher-categorical structure

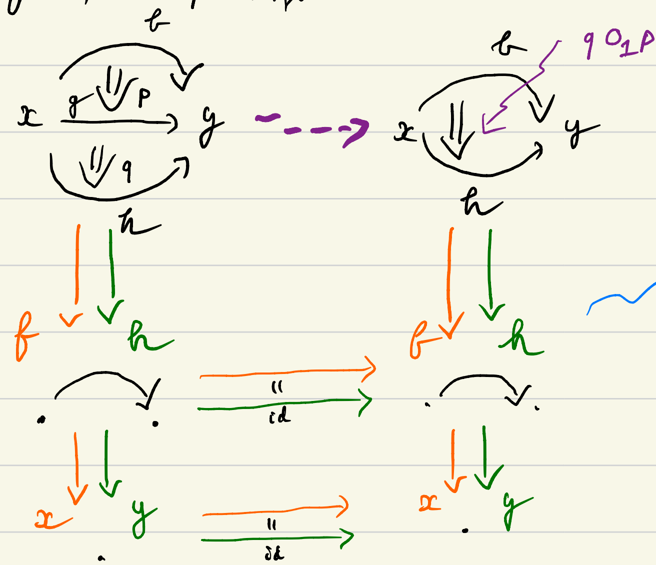
- Composition of 1-morphisms

$x:A, y:A, z:A, f: x =_A y, g: y =_A z$




- Vertical composition of 1-morphisms

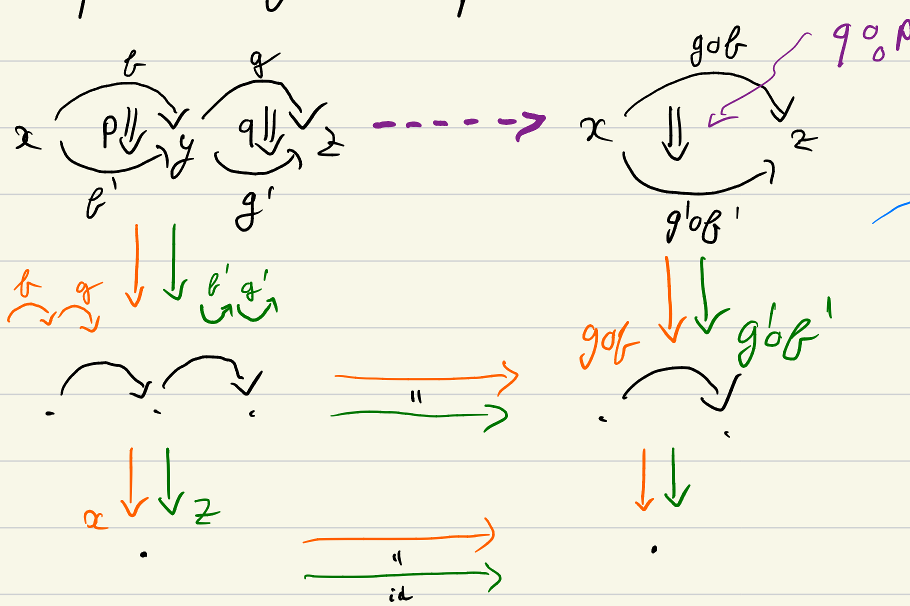
$x:A, y:A, f: x =_A y, \ldots,$
$p: f = g, q: g = h$




obtained by path induction

- Horizontal composition of 1-morphisms

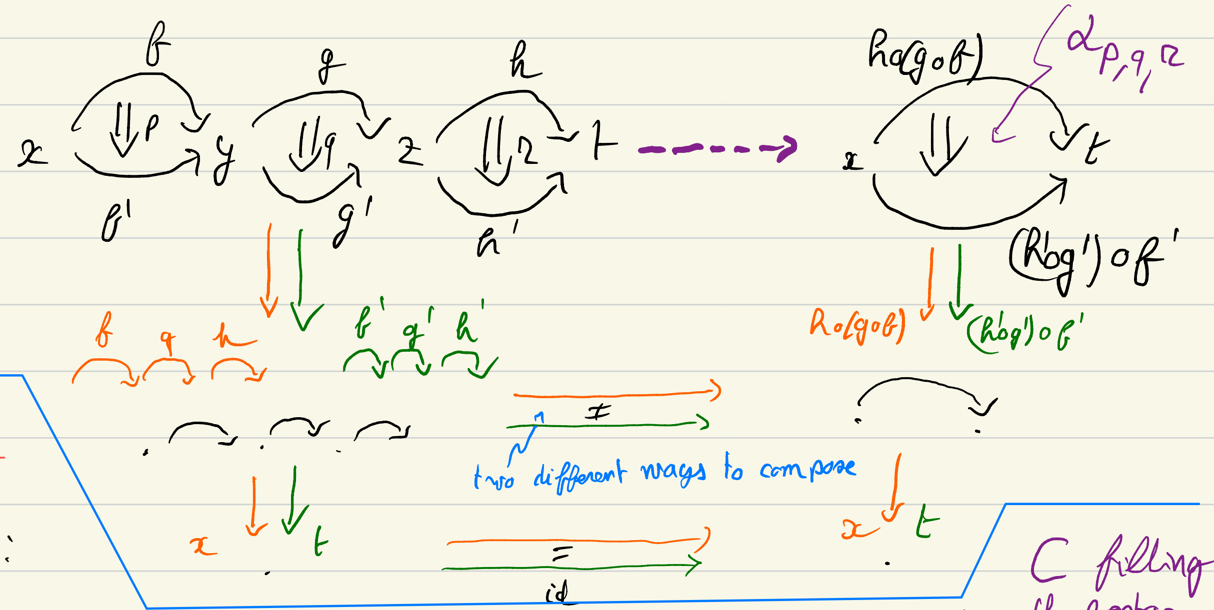
$x:A, \ldots,$
$p: f = f', q: g = g'$




- Identities are provided by refl

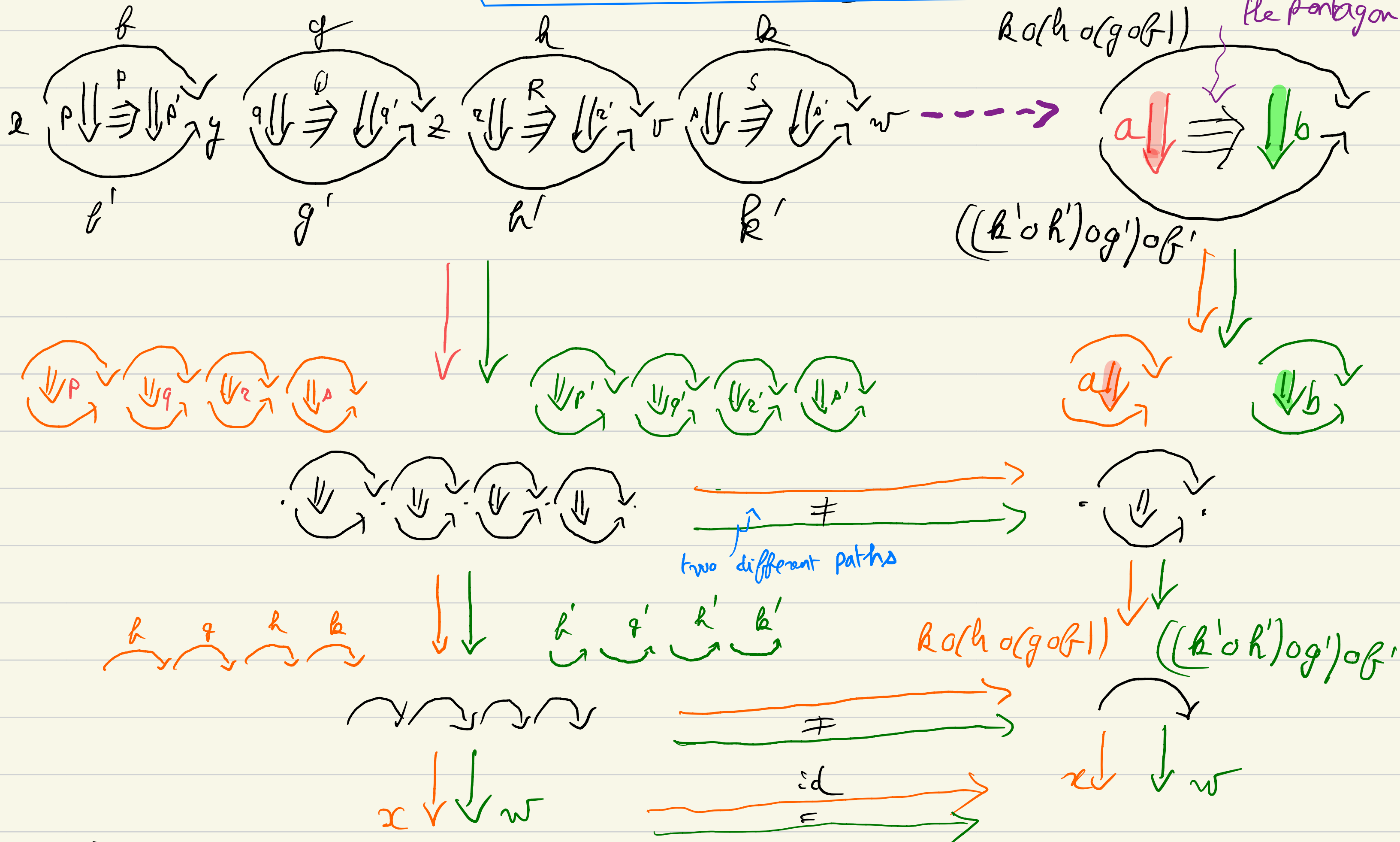
# Towards Coherence

- **Associativity up to iso:**

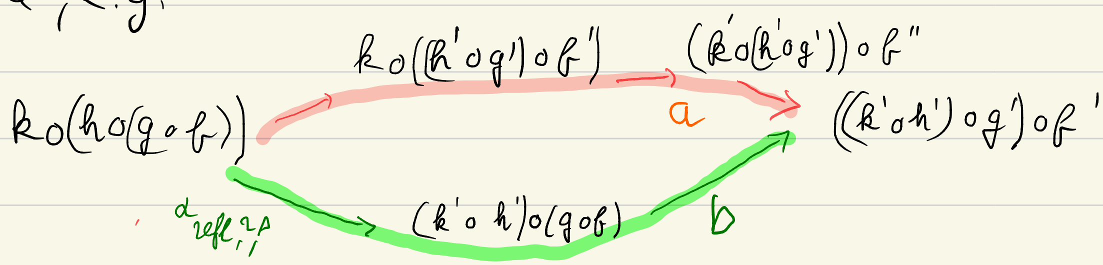



Maclane pentagon ⇝

- **One level up:**




where, e.g.



# Globular sets

The category $\mathbb{G}$ of globes has

- natural numbers as objects
- morphisms $(n-1) \overset{\sigma^n}{\underset{\tau^n}{\rightrightarrows}} n$ for all $n \geq 1$, subject to

$\sigma_n \circ \sigma_{n-1} = \tau_n \circ \sigma_{n-1}$ and $\sigma_n \circ \tau_{n-1} = \tau_n \circ \tau_{n-1}$.

inducing the globular equality in $C^{\mathbb{G}^{op}}$ ← globular objects

$$\begin{tikzcd} \cdot \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] & \cdot \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

- source of source = source of target
- target of source = target of target

We write $\mathbb{G}set = Set^{\mathbb{G}^{op}}$ (category of globular sets)

For a globular set $C_\bullet = C_0 \overset{s}{\underset{t}{\leftleftarrows}} C_1 \leftleftarrows C_2 \cdots \in Ob\, \mathbb{G}set$,

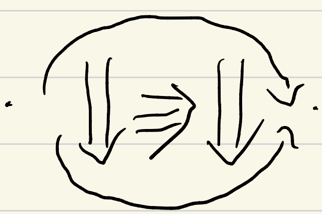
we use the same notation as p. 8, e.g., for $\alpha \in C_2$, we write

$$\alpha \in C_2(\underset{sf=sg}{x}, \underset{tf=tg}{y}; \underset{s\alpha}{f}, \underset{t\alpha}{g})$$

Exercise Show that there are only two morphisms in $\mathbb{G}(m,n)$
(hint: ... does not matter in $\ldots \circ \sigma^n$).

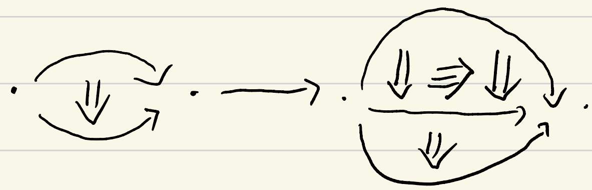
Exercise Give an explicit description of $y: \mathbb{G} \to \mathbb{G}set$

Hint: here is a picture for $y_3$ (the n-globe):



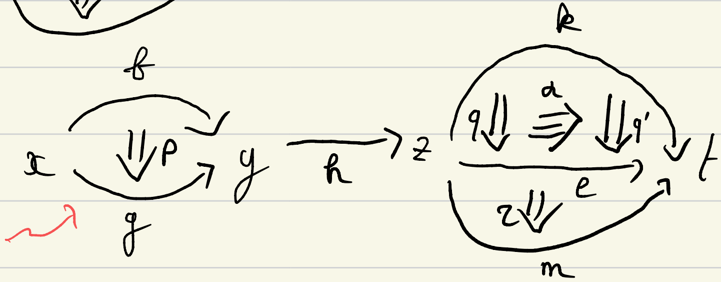
# Globular pasting diagrams

- As diagrams



- As contexts (up to $\alpha$-conversion):

names are distinct

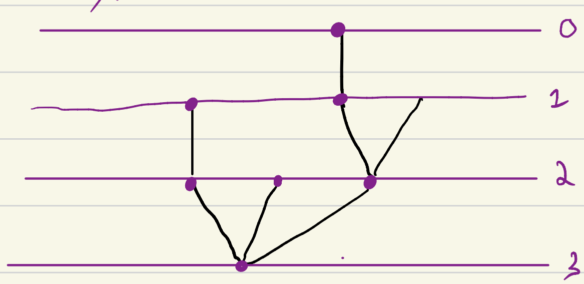



$$\Gamma = x : A\,, y : A\,, z : A\,, t : A,$$
$$f : \underline{A}(x,y)\,, g : \underline{A}(x,y)\,, h : \underline{A}(y,z)\,, k : \underline{A}(z,t)\,, l : \underline{A}(z,t)\,, m : \underline{A}(z,t),$$
$$p : \underline{A}_2(x,y;f,g)\,, q : \underline{A}_2(z,t;k,l)\,, r : \underline{A}_2(z,t;l,m),$$

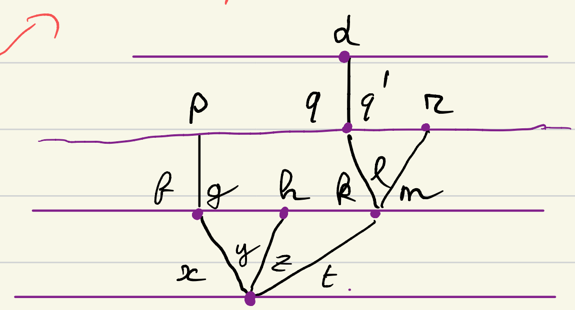
$$q' : \underline{A}_2(z,t;k,e)\,,\ \alpha : \underline{A}_3(z,t;k,e;q,q')$$

- As trees (Batanin trees)

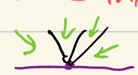
leveled, planar




the same, with names




rationale for name insertion



By convention $pd(-1) = \emptyset$

- As code for trees $pd(n) = pd(n-1)^*$ free monoid

for $u = a_1 \ldots a_p \in pd(n)$, we write $[a_1 \ldots a_p]_n$ to stress levels.

$$[[[\,]_1]_2[\,]_2[[[\,]_0]_1[\,]_1]_2]_3$$

Further, to accommodate names, we write $[|a_1| \ldots |a_p|]_n$

the same, with names

$$[\overset{x}{|}[\overset{f}{|}[\overset{p}{|}]_1\overset{g}{|}]_2\overset{y}{|}[\overset{h}{|}]_2\overset{z}{|}[\overset{k}{|}[\overset{q}{|}[\overset{\alpha}{|}]_0\overset{q'}{|}]_1\overset{\ell}{|}[\overset{r}{|}]_1\overset{m}{|}]_2\overset{t}{|}]_3$$

- Notation:

$\boxed{\text{for } \pi \text{ a notation for a Batanin tree, we write } \Gamma^{\pi}_{A} \text{ for the associated context, and } \Gamma^{n} \text{ for } \Gamma^{y_n}}$

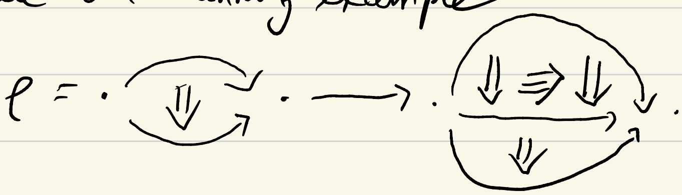
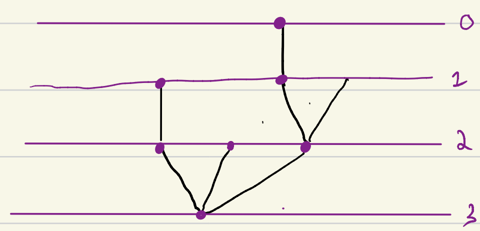
omitted when clear

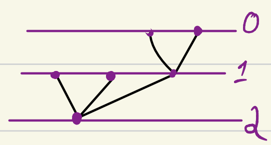
cf. second exercise p.12

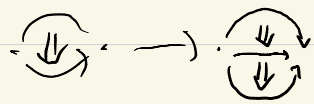
# The globular set $\begin{cases}\text{of pasting diagrams} \\ \text{of } \underline{\text{a}} \text{ pasting diagram}\end{cases}$

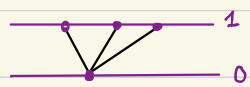
- We get a globular set pd by setting $s_n = t_n = \partial_n$ defined by
  - $\partial_1 = pd(1) \to pd(0) = \{[\,]_0\}$ ⟵ at level 0!
  - $\partial_n([a_1 \dots a_p]_n) = [\partial_{n-1}a_1 \dots \partial_{n-1}a_p]_{n-1}$

Recall our running example

$\rho =$  $=$ 


$\partial\rho =$ 

$=$ 

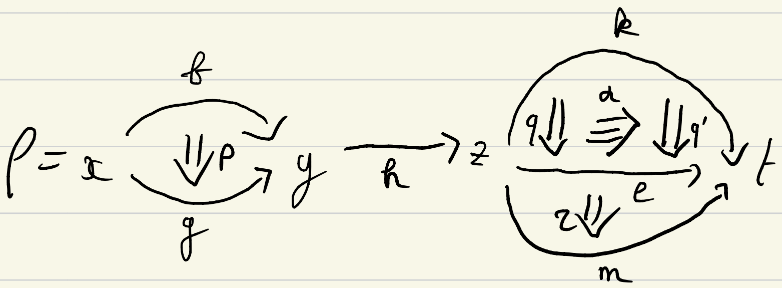
$\partial\partial\rho =$ 

$= \cdot \to \cdot \to \cdot \to \cdot$

$\partial\partial\partial\rho =$ 

$= \cdot$

- Each $\pi \in pd(n)$ gives rise to a globular set $\hat{\pi}$. For our running example:




$\hat{\rho}(0) = \{x, y, z, t\}$

$\hat{\rho}(1) = \{f, g, h, k, \ell, m\}$

$\hat{\rho}(2) = \{p, q, q', r\}$

$\hat{\rho}(3) = \{\alpha\}$

$sq = k \quad tq = \ell$ etc...

<u>Exercise</u> Define $\hat{\pi}$ by induction, for $\pi = [\overset{x_0}{|}\,\pi_{x_0x_1}\,\overset{x_1}{|} \cdots \overset{x_{n-1}}{|}\,\pi_{x_{n-1}x_n}\,\overset{x_n}{|}]$.

# A general pattern

For a pasting diagram $p \in pd(n)$ we consider *(in first approximation)* $p$-ladder diagrams of the form

$$\begin{tikzcd}
\Gamma^{p} \arrow[rr] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & & \Gamma^{n} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
\Gamma^{\partial p} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & & \Gamma^{n-1} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
\vdots & \vdots & \vdots \\
\Gamma^{0} \arrow[rr, shift left] \arrow[rr, shift right, "= \\ id"'] & & \Gamma^{0}
\end{tikzcd}$$

in the category of contexts

and where all rectangles

□ □ □ □ □ □

commute

where all $\downarrow$ and $\downarrow$ are determined by $p$ and $n$

We actually consider ladders **UNDER** $\Gamma^0$ (see picture p. 17)

Here $\downarrow s$ is the projection that removes all the highest dimensional declarations as well as their "target", e.g. $x:A, y:A, p:x=_A y \xrightarrow{x} A$, while $\downarrow t$ is $x:A, y:A, p:x=_A y \xrightarrow{y} A$ (check this on p. 11)

**Theorem** For all $\pi$ ← COHERENCE THEOREM

any two parallel $\partial\pi$-ladders (i.e., sharing the same $\downarrow\downarrow$) there exists a filling in the form of a $\pi$-ladder

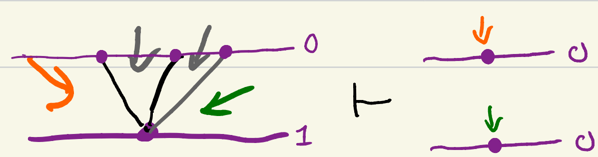
$$\begin{tikzcd}
\Gamma^{\pi} \arrow[rr, dashed] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & & \Gamma^{n+1} \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
\Gamma^{\partial\pi} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & & \Gamma^{n} \arrow[d, shift right, "t"'] \arrow[d, shift left, "t"] \\
\Gamma^{\partial\partial\pi} \arrow[rr, shift left] \arrow[rr, shift right] \arrow[d, shift right] \arrow[d, shift left] & & \Gamma^{n-1} \arrow[d, shift right] \arrow[d, shift left] \\
\vdots & \vdots & \vdots \\
\Gamma^{0} \arrow[rr, shift left] \arrow[rr, shift right, "= \\ id"'] & & \Gamma^{0}
\end{tikzcd}$$

WARNING: The bijection $\pi \mapsto \Gamma^{\pi}$ is not functorial:

- unique $s = t : \pi \to \partial\pi$
- distinct $s, t : \Gamma^{\pi} \to \Gamma^{\partial\pi}$

e.g. $x:A, y:A, p:x =_A y \vdash \dfrac{x:A}{y:A}$

The general pattern is (with the iconography of p. 13)



# Proof strategy for the theorem (cf. p.15)

- We set $\partial\Gamma^n$ = the context $\Gamma^n$ minus its unique top dimensional declaration

e.g. $\partial(x:A, y:A, p : x =_A y) = x:A, y:A$, with associated projection $\Gamma^n \vdash q_n : \partial\Gamma^n$ (e.g. $q_1 = (x,y)$)
↖ cf. lecture A

- $\partial\Gamma^0 = 1$ (empty context = terminal)

Exercise [0] Prove that ($n \geq 1$)

$$\begin{tikzcd} & \partial\Gamma^n \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\ \Gamma^{n-1} \arrow[dr, "q_{n-1}"'] & & \Gamma^{n-1} \arrow[dl, "q_{n-1}"] \\ & \partial\Gamma^{n-1} & \end{tikzcd}$$

and $q_n = \langle s, t \rangle$

Exercise [1] Prove that, for all $n$,

$$\begin{tikzcd} \Gamma^n \arrow[dr, "q_n"] \arrow[dd, bend right, "s"'] & \Gamma^{n-1} \arrow[l, "r_n"'] \arrow[r, "r_n"] \arrow[d, "\langle id, id \rangle"] \arrow[ddl, dashed, "id"'] \arrow[ddr, dashed, "id"] & \Gamma^n \arrow[dl, "q_n"'] \arrow[dd, bend left, "t"] \\ & \partial\Gamma^n \arrow[dl] \arrow[dr] & \\ \Gamma^{n-1} \arrow[dr, "q_{n-1}"'] & & \Gamma^{n-1} \arrow[dl, "q_{n-1}"] \\ & \partial\Gamma^{n-1} & \end{tikzcd}$$

where, say, $r_1 = (x, x, \mathrm{refl}\ x)$

$x,y,p,q$ $\partial\Gamma^2$

$\Gamma^1$ $x,y,p$ $\qquad$ $x,y,q$ $\Gamma^1$

$x,y$ $\partial\Gamma^1$

Exercise [2] Prove that for $\pi = \left[ \overset{x_0}{|} \pi_1 \overset{x_1}{|} \cdots \overset{x_{p-1}}{|} \pi_p \overset{x_p}{|} \right]_n$

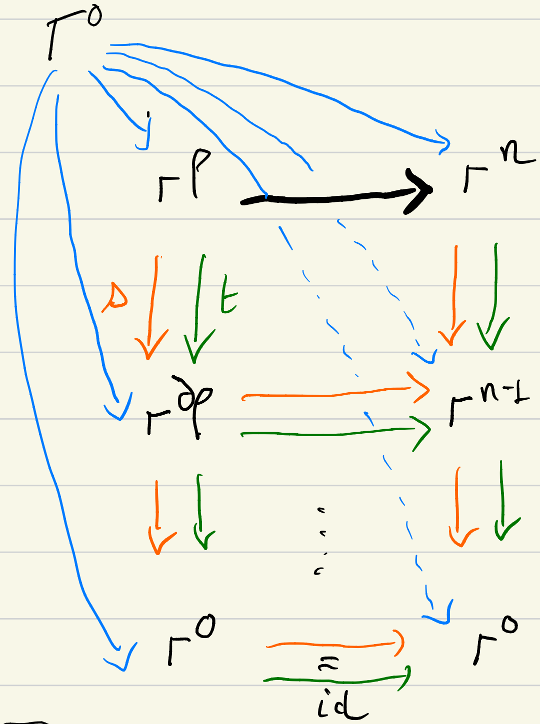
one can obtain $\Gamma_A^\pi$ as the limit of the diagram

$$\begin{tikzcd} \Gamma^{\pi_1}_{x_0 =_A x_1} \arrow[dr, "x_1 = \vec{t}"'] & & \Gamma^{\pi_2}_{x_1 =_A x_2} \arrow[dl, "x_1 = \vec{s}"] & \cdots & \Gamma^{\pi_p}_{x_{p-1} =_A x_p} \arrow[dl, "\vec{s}"] \\ & A & & \cdots & \end{tikzcd}$$

- Recall from lecture A pp 25-26 that we have a weak factorisation system $(\mathcal{I}, \mathcal{P})$ such that $r_i \in \mathcal{I}$ and $q_i \in \mathcal{P}$
  - Composing the $r_i$'s we get $\Gamma^0 \to \Gamma^i \in \mathcal{I}$
  - Using a lemma proved p.19 + exercise 2 above, we get maps $\Gamma^0 \to \Gamma^\pi$
  - All these maps commute with all $\downarrow\downarrow$

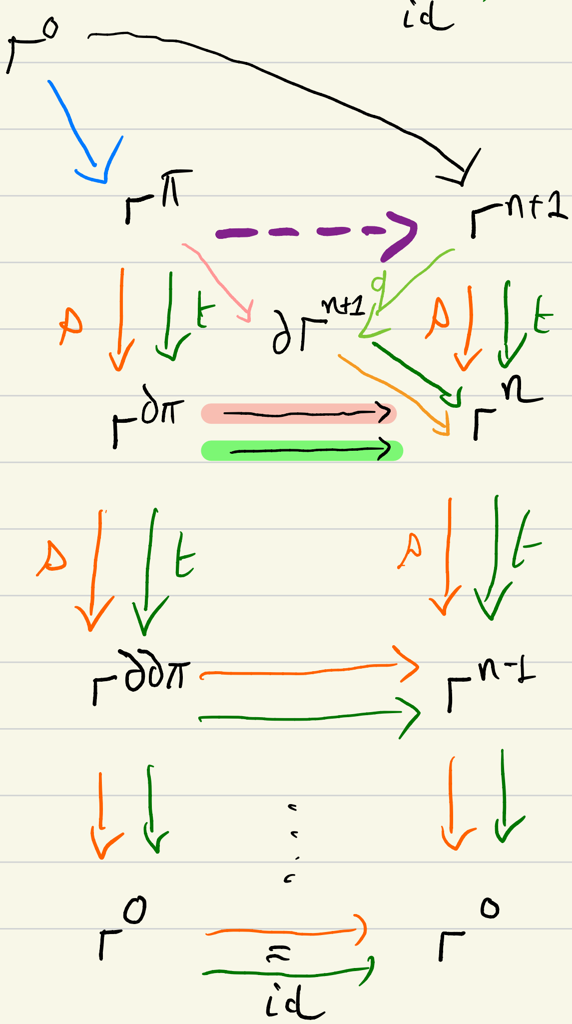
# Proving the theorem (cf p. 15)

We require by definition that our ladders are under $\Gamma^0$, more precisely: $\Gamma^0$

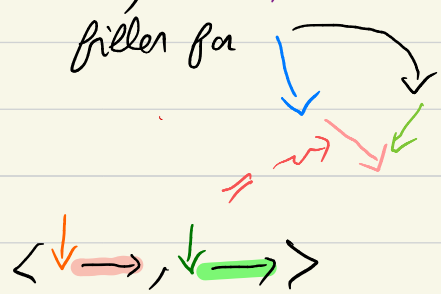



where all blue arrows are those defined on p. 16

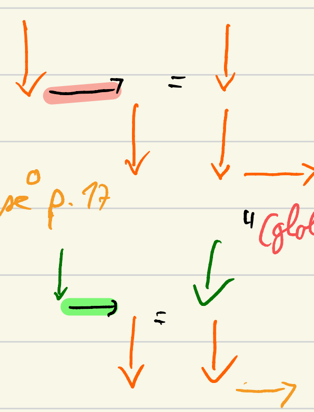
Proof




i.e., $\dashrightarrow$ is the diagonal filler for

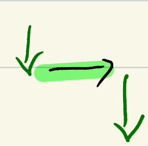


(indeed




cf. Exercise° p. 17 "(globularity)"

and likewise for



)

# Encapsulating the needed properties

Here is what we have used, for $\underset{\text{category of contexts}}{C = \mathrm{Ctx}}$

## Identity type categories

It is a category $\mathbf{C}$ endowed with two classes of morphisms $\mathcal{I}, \mathcal{P} \subseteq \overset{\mathrm{Mor}(C)}{\mathbf{C}_1}$ such that :

1. $\mathbf{C}$ has a terminal object 1, and $! : A \to 1 \ \in \ \mathcal{P}$ for all $A \in \mathbf{C}_0$
2. The classes $\mathcal{I}, \mathcal{P}$ contain the identities and are closed under composition
3. Pullbacks of $\mathcal{P}$-maps along arbitrary maps exist, and are again $\mathcal{P}$-maps
4. The pullback of an $\mathcal{I}$-map along a $\mathcal{P}$-map is an $\mathcal{I}$-map
5. Every commutative square with an $\mathcal{I}$-map on the West and a $\mathcal{P}$-map on the East has a diagonal filler $\quad \mathcal{I} \boxslash \mathcal{P}$
6. For every $\mathcal{P}$-map $p : C \to D$, the diagonal map $\Delta : C \to C \times_D C$ has a factorisation $\Delta = e \circ r$, with $e \in \mathcal{P}$ and $r \in \mathcal{I}$ :

$$C \xrightarrow{r} C^I \xrightarrow{e} C \times_D C$$

(Think of $\mathcal{I}$-maps as trivial cofibrations, and of $\mathcal{P}$-maps as fibrations.).

<u>Exercise</u> Prove that Ctx is an identity type category, with $\mathcal{I}, \mathcal{P}$ as defined pp. 25-26 of Lecture A.

**Lemma** *Suppose that*

$$
\begin{tikzcd}
 & A \arrow[dl, "i"'] \arrow[dr, "j"] \arrow[dd, "\mathrm{id}_A"] & \\
B \arrow[dr, "p"'] & & C \arrow[dl, "q"] \\
 & A &
\end{tikzcd}
$$

*is a commutative diagram in an identity type category $\mathcal{C}$. Suppose further that $i$ and $j$ are $\mathcal{I}$-maps, and $p$ and $q$ are $\mathcal{P}$-maps. Then the induced map $\langle i, j\rangle \colon A \to B \times_A C$ is also an $\mathcal{I}$-map.*

Proof Consider

$$
\begin{tikzcd}
 & & A \arrow[d, "i"] \arrow[dr, bend left, "id"] & \\
 & & B \arrow[dl, "j'"'] \arrow[dr, "p"] \arrow[dll, bend right, "id"'] & \\
 & B\times_A C \arrow[dl, "q'"'] \arrow[dr, "p'"] & & A \arrow[dl, "j"] \arrow[dll, bend left, "id"] \\
B \arrow[dr, "p"'] & & C \arrow[dl, "q"] & \\
 & A & &
\end{tikzcd}
$$

with $j' = \langle id, jp\rangle$

Since the outer rectangle is a pull-back, so is the upper square

We then observe that

$\underbrace{q'j'}_{id} i = i$ and $\underbrace{p'j'}_{jp} i = j$

$\Downarrow$

$j'i = \langle i, j\rangle$

(axioms 3+4) in $\mathcal{I}$ in $\mathcal{I}$ in $\mathcal{I}$ (axiom 2)

- An instantiation of this Lemma in Ctx was instrumental in the proof of the coherence theorem pp. 15-17

The next slides place this theorem in the context of Batanin's definition of weak $\omega$-categories.

# Plan of additional slide

- Abstract from Ctx and lift the definition of $\Gamma^{\pi}$, $\Gamma^{n}$ ladders to the abstract setting of identity type categories. Prove the coherence theorem at this level of abstraction pp 21-22.

- Strict $\omega$-categories are algebras over a monad $T$ on $\mathbf{G}$set for which $T1 = pd$.
  pasting diagrams cf. p. 13 pp. 23-25

pp. 26-27

- A globular operad consists of operations of shape $\pi$ for every $n$ and $\pi \in pd(n)$ + composition of these operations.
  A globular operad with contraction abstracts coherence
    - Ladders of p. 15 (amended p. 17) provide the material for a globular operad.

- A weak $\omega$-category is an algebra for a globular operad with contraction (+ a mild property called normality).
    - Fixing a context $\Delta$, morphisms from $\Delta$ to $\Gamma^{n}$ provide the data for a globular set, which is a weak $\omega$-category.
      varying $n$

The following abstracts the properties needed from the $\Gamma^n$ and the $\downarrow\downarrow$ between them,

## Reflexive globular contexts in an identity type category $\mathbf{C}$

A globular context is :

- a globular object in $\mathbf{C}$, given by a collection of objects $A(0), \ldots, A(n), \ldots$ equipped with source and target morphisms $s, t : A(n+1) \to A(n)$ satisfying $s \circ s = s \circ t$ and $t \circ s = t \circ t$ (i.e, a functor $A : \mathbf{G}^{op} \to \mathbf{C}$
- such that all the induced maps $q_n$ from $A(n)$ to its boundary $\partial A(n)$ are $\mathcal{P}$-maps

  (definition of boundaries : $\partial A(0) = 1$, $\partial A(n+1) = A(n) \times_{\partial A(n)} A(n)$ with $q$ the universal morphism)

A globular context is *reflexive* when it is moreover equipped

- with morphisms $r_n : A(n) \to A(n+1)$ such that $s \circ r = t \circ r = Id$
- with all these morphisms being $\mathcal{I}$ maps.

Exercise. Synthesise the construction of an object $A^{\Pi}$ in an arbitrary identity type category (Hint : imitate exercise 2 p.16).

Exercise. Formulate and prove the coherence theorem in an arbitrary identity type category.

- The following abstracts the construction of the $T^n$ and the $\downarrow, \downarrow$ between them.

<u>Construction</u> For each $A \in \mathrm{Ob}\,\mathbf{C}$, the following inductive process yields a reflexive globular context in $\mathbf{C}$ (we show only the first three steps)

- We set
  - $A(0) = A$ (and hence, following the def., $\partial A(0) = 1$).
  - $q_0 : A(0) \to \partial A(0)$ = unique map onto $1$
    $\Leftarrow \in P$ by axiom 1

- Following the definition, we set $\partial A(1)$ as below, and contemplate

$$\begin{tikzcd}
A(1) \arrow[dr, "q_1"] & & A(0) \arrow[ll, "r_0"'] \arrow[dl] \arrow[ddll, "id"'] \arrow[dd, "id"] \\
& \partial A(1) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & \\
A(0) \arrow[dr, "q_0"'] & & A(0) \arrow[dl, "q_0"] \\
& 1 = \partial A(0) &
\end{tikzcd}$$

where $\underbrace{\xrightarrow{r_0}\xrightarrow{q_1}}_{=\ \longrightarrow}$ is obtained by axiom 6

and we set

$p_1 = \searrow\swarrow$ , $t_1 = \downarrow\searrow$

(as displayed below)

- We continue in the same way:

$$\begin{tikzcd}
A(2) \arrow[dr, "q_2"] \arrow[dd, "p_2"'] & A(1) \arrow[l, "r_1"'] \arrow[r, "r_1"] \arrow[d] & A(2) \arrow[dl, "q_2"'] \arrow[dd, "t_2"] \\
& \partial A(2) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & \\
A(1) \arrow[dr, "q_1"] \arrow[dd, "p_1"'] & A(0) \arrow[l, "r_0"'] \arrow[r, "r_0"] & A(1) \arrow[dl, "q_1"'] \arrow[dd, "t_1"] \\
& \partial A(1) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner"] & \\
A(0) \arrow[dr] & & A(0) \arrow[dl] \\
& \partial A(0) &
\end{tikzcd}$$

The globularity conditions are checked by diagram chasing

# Strict $\omega$-categories

- With a small 2-category $\underline{C}$ (cf. lecture 3 p.9) we can associate a (truncated) globular set $C_0 \leftleftarrows C_1 \leftleftarrows C_2$ $(\leftleftarrows \emptyset \leftleftarrows \cdots \emptyset \cdots)$

  $C_2(x,y;f,g) = C_2(x,y)(f,g)$ — category

  Thus a 2-category is a globular set with extra structure

- The same for categories $C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1$ $(f \in C_1(x,y))$

- A strict $\omega$-category is a globular set

$$C = C_0 \leftleftarrows C_1 \leftleftarrows C_2 \cdots C_{n-1} \leftleftarrows C_n \cdots$$

such that, for all $m < l < k$ a 2-category structure is given on $C_k \leftleftarrows C_l \leftleftarrows C_m$

cf. exercise p.19

in such a way that the category structures on $C_k \leftleftarrows C_l$, $C_l \leftleftarrows C_m$ and $C_k \leftleftarrows C_m$ do not depend on $m$, $k$, $l$, respectively.

**Exercise**: Spell out all the operations involved and all the equations that need to be satisfied.

(Hints: • for all $k$ and $l < k$, there exists a composition

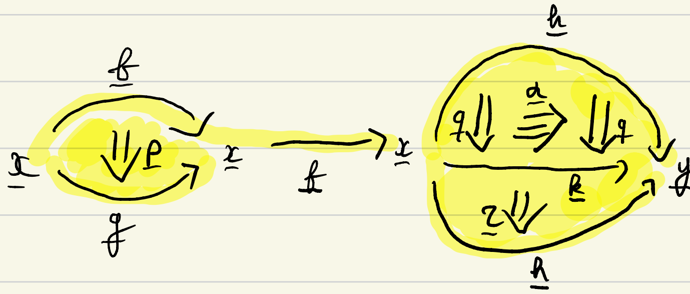
$$\circ_l : C_k \times_{C_l} C_k \to C_k$$

- there are identities
- compatibilities of the operations with source and target
- associativities + unit laws
- exchange laws.)

# The free ω-category monad

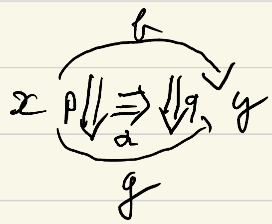
- We define $$(TX)(m) = \sum_{\pi \in pd(m)} \mathbf{Gset}[\hat{\pi}, X]$$ hence $T1 = pd$

An element of $[TX](m)$ is a pasting diagram with decorations in $X$: for example

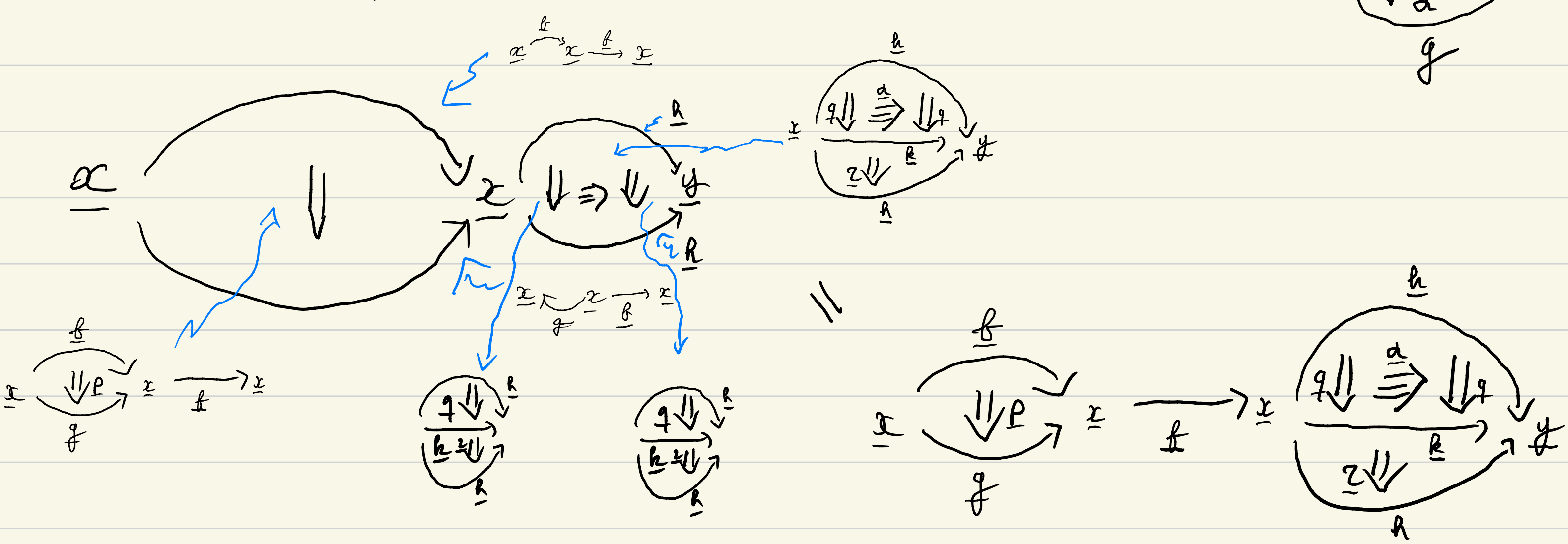


where now $x, y, f, g, \rho, q, z, \alpha$ are not anymore $\alpha$-convertible names, but are elts of $X_i$ (for the relevant dimension $i$) (possibly repeated)

- $T$ is a monad on $\mathbf{Gset}$:
  - $\eta$ is defined by, say, for $\alpha \in X_3(x, y; f, g; \rho, q)$:



  - $\mu$ is defined by zooming/substitution in place:



# Defining the multiplication of the monad formally

**Exercise** • Check the details of the following inductive definition of $\mu_1 : T(\mathrm{pd}) \to \mathrm{pd}$ (recall that $T1 = \mathrm{pd}$):

If $X$ is a globular set, define $\mathrm{s}X$ by $(\mathrm{s}X)(n) = X(n+1)$.

A formal definition of the multiplication $\mu$ of the monad (we write $\pi[G] = \mu(G : \widehat{\pi} \to \mathrm{pd})$) :

1. If $\pi = (\pi_1, \ldots, \pi_n)$, then every $G : \widehat{\pi} \to X$ induces $G_i : \widehat{\pi}_i \to (\mathrm{s}X)$ $(i = 1, \ldots, n)$.
2. For every pasting diagram $\rho$, every $H : \widehat{\rho} \to \mathrm{s}(\mathrm{pd})$ can be written $H = (H_1, \ldots, H_k)$ (with $H_1, \ldots, H_k : \widehat{\rho} \to \mathrm{pd}$), where $k$ is the common length of $H(i)$ ($i$ being any 0-cell of $\widehat{\rho}$). One proves this using :
   – that $H$ is a morphism of globular sets
   – and that in pd, the source and target functions coincide
3. Let $\pi = (\pi_1, \ldots, \pi_n)$, $G : \widehat{\pi} \to \mathrm{pd}$, inducing $G_{i,1}, \ldots, G_{i,k_i} : \widehat{\pi}_i \to \mathrm{pd}$, by (1) and (2). Then we set (by induction) :

$$\pi[G] = (\pi_1[G_{i,1}], \ldots, \pi_1[G_{1,k_1}], \pi_2[G_{2,1}], \ldots, \pi_n[G_{n,k_n}])$$

• One then defines $\mu_X$ by pullback

$$\begin{tikzcd} TTX \arrow[r, "\mu_X"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "T!"] \\ TT1 \arrow[r, "\mu_1"'] & T1 \end{tikzcd}$$

Spell out this pull-back and show that $\downarrow$ is $(TT!)$.

A natural transformation whose naturality squares are pull-backs is called cartesian

**Exercise**. Show that strict $\omega$-categories = $T$-algebras

Hint: take inspiration from

$$\xrightarrow{\underline{f}} \xrightarrow{\underline{g}} \quad \xrightarrow{\underline{h}} \qquad \wr \qquad \wr \qquad \longmapsto \qquad \xrightarrow{\underline{g \circ f}} \; \xrightarrow{\underline{h}}$$
$$\to \; \to$$

$$\begin{tikzcd} TTA \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

$$\underline{h \circ (g \circ f)}$$

$$\xrightarrow{\underline{f}} \xrightarrow{\underline{g}} \xrightarrow{\underline{h}} \longmapsto \xrightarrow{\underline{h \circ g \circ f}}$$

(Similarly, $\underline{h \circ g \circ f} = (\underline{h \circ g}) \circ \underline{f}$ )

unbiased composition

# Weak $\omega$-categories, after Batanin

- A <u>globular operad</u> is a monad $P$ on $\mathbf{Gset}$ together with a cartesian $\underbrace{\text{monad morphism}}$ $\rho : P \to T$

  ← natural transformation commuting with $\eta, \mu$

$$\begin{tikzcd}
PX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P1 \arrow[d] \\
TX \arrow[r] & T1
\end{tikzcd}$$

An element of $(PX)_n$ is thus [by the formula for $TX$ p. 24]

a tuple $(\pi, g, a_0)$ where
- $\pi \in pd(n)$,
- $g \in \mathbf{Gset}(\hat{\pi}, X)$ and
- $a_0 \in P_\pi \stackrel{\Delta}{=} \{a \in P1(n) \mid \rho(a) = \pi\}$

Thus $\boxed{(PX)(n) = \sum_{\pi \in pd(n)} P_\pi \times \mathbf{Gset}(\hat{\pi}, X)}$ ← $P$ as a fat version of $T$

If we think of a $T$-algebra structure on $X$ as providing a unique way of composing each $X$-labelled pasting diagram of shape $\pi$, then a $P$-algebra structure provides a set of possible ways of composing such diagrams, indexed by the elements of $P_\pi$.

- A globular operad is
  - <u>normalised</u> if $P_\bullet$ is a singleton for $\bullet$ (the unique element of $pd(0)$), and
  - <u>contractible</u>, i.e.

- Given $\pi \in pd(1)$ and $\theta_1, \theta_2 \in P_\bullet$, there exists an element $\phi \in P_\pi$ with $s(\phi) = \theta_1$ and $t(\phi) = \theta_2$; (in particular, $P \neq \emptyset$, allowing to compose $f$ $g$)
- Given $\pi \in pd(n)$ (for $n > 1$) and $\theta_1, \theta_2 \in P_{\partial\pi}$ satisfying $s(\theta_1) = s(\theta_2)$ and $t(\theta_1) = t(\theta_2)$, there exists an element $\phi \in P_\pi$ such that $s(\phi) = \theta_1$ and $t(\phi) = \theta_2$.

- **Definition.** A *weak $\omega$-category* is an algebra for a contractible, normalised, globular operad.

$\alpha : PX \to X \quad \alpha(1) : (PX)(1) \to X(1) \quad \alpha(1)[p, f\ g] = h$, $p \in P$

("a composite of $f, g$")

# Rounding it up

We come back to Ctx (or to an identity type category)

We set $\boxed{P_\pi = \{ \text{ladder diagrams whose top layer is } \Gamma^\pi \to \Gamma^n \}}$

Exercise. Show that these provide the data for a globular operad $\rho : P \to T$.

This operad is normalised (basis of the inductive construction of ladders).

The coherence theorem says that it is contractible.

Finally, we get a $P$-algebra, i.e, a weak $\omega$-category, by setting, for a fixed context $\Delta$:

$$\boxed{X(n) = \{ \Delta \vdash p : \Gamma^n \}}$$

The algebra structure $\xi : PX \to X$ is defined as follows

Given $n$, $\pi \in pd(n)$, $L$ a ladder in $P_\pi$, and $U \in \mathbf{G}set(\vec{\pi}, X)$,

- we extract from $L$ its top layer $\Gamma^\pi \to \Gamma^n$;
- we read $U$ as $\pi$ decorated in $X$ : this information can be arranged as a morphism $\Delta \to \Gamma^\pi$;
- We set

$$\boxed{\xi(\pi, L, U) = \Delta \xrightarrow{\text{"}U\text{"}} \Gamma^\pi \xrightarrow{\text{"}L\text{"}} \Gamma^n}$$

There is more to it: $X$ is in fact a weak $\omega$-groupoid.

# The weak $\omega$-groupoid structure

One of the possible definitions of weak $\omega$-groupoids, due to Eugenia Cheng, is that we require (on top of a weak $\omega$-category structure) that every $n$-cell $f : x \to y$ (i.e. $sf = x$, $tf = y$) has a dual, i.e. there exists $f^* : y \to x$, $\eta : id \to f^* \circ f$ and $\varepsilon : f \circ f^* \to id$.

In the following exercises, we indicate the steps to show that $X$ (as defined p. 27) is a weak $\omega$-groupoid.

<u>Exercise</u> Show that for all $n$, there exists a canonical morphism $* : T^n \to T^n$ st.

$$\begin{tikzcd} T^n \arrow[rr, "*"] \arrow[dr, "{(s,t)}"'] & & T^n \arrow[dl, "{(t,s)}"] \\ & \partial T^n & \end{tikzcd}$$

Hint: proceed as for the coherence theorem.

<u>Exercise</u> for $f : \Delta \to T^n$, set $f^* := * \circ f$.
Prove the existence of $\eta$ and $\varepsilon$ by coherence
(hint: $id$ and $f^* \circ f$ are parallel)

**THE END !**