INTRODUCTION TO INFINITY-CATEGORIES

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Abstract. These are lecture notes for the lecture courses "Introduction to ∞-categories" and "Infinity-categories" taught at the university of Regensburg in the winter term 2018/2019 and the summer term 2019. Most of the material is not original and in many occasions I follow the arguments of Rezk. Further source of inspiration are the standard notes: Lurie's Higher Topos theory, Cisinski's book on higher categories and homotopical algebra, and Joyal's paper about quasicategories. Further sources I recommend are Haugseng, Groth.

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Motivation and overview

Lecture 1 – 15.10.2018. This lecture should be called "introduction to ∞-categories" as opposed to "introduction to higher categories": Let me briefly explain what kind of higher category we will be talking about:

Informally, an n-category consists of

(1) a set of objects (these will be called 0-morphisms)
(2) a set of morphisms between objects (these will be called 1-morphisms), and
(3) a set of morphisms between $(k-1)$-morphisms, (these will be called k-morphisms), for every $k \leq n$.

For every k-object, there should be an identity $k+1$-morphism, and there should be composition maps that "compose" two composable n-morphisms to a new one.

The question is what the axioms are that are supposed to be satisfied. Informally, one definition could be that we define a 1-category to be an ordinary category, and define inductively an n-category to be a category enriched in $(n-1)$-categories:

For $n = 2$, we obtain the following examples:

Example 0.1. Consider the category of categories Cat. It is canonically enriched in Cat itself, because for any two categories $\mathcal{C}$ and $\mathcal{D}$, not only is there a set of functors from $\mathcal{C}$ to $\mathcal{D}$, in fact there is a category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ from $\mathcal{C}$ to $\mathcal{D}$: The 2-morphisms are precisely natural transformations of functors.

Example 0.2. Consider the category of groups and group homomorphisms: This is also canonically a 2-category, namely we say that a 2-morphism from $\varphi \to \psi \colon G' \to G$ is an element $g \in G$ such that $\psi = g^{-1}\varphi g$. Notice that every 2-morphism is automatically an isomorphism: the inverse being given by g^{-1}.

An (n,k)-category is then an n-category in which all ℓ-morphisms are invertible if $\ell > k$ (this can be read as saying that an n-category determines an (n,n)-category).

Example 0.3. Cat is really only a $(2,2)$-category, whereas the 2-category of groups described above is in fact a $(2,1)$-category.

What we will consider in our lecture is the notion of $(\infty,1)$-categories, which will be called ∞-categories.

There are motivating examples:

Example 0.4. A category is a $(1,1)$-category and should certainly give rise to an $(\infty,1)$-category.

Example 0.5. Let X be a topological space. Then consider the following ∞-category:

(1) objects are the points of X,
(2) 1-morphisms are paths from x to y,
(3) 2-morphisms are homotopies (rel. endpoint) between paths from x to y,
(4) 3-morphisms are homotopies between homotopies between paths, and so on...

There is also another informal construction we can do: Associated to any (n,k)-category $\mathcal{C}$, and two objects x and y in $\mathcal{C}$, we obtain an $(n-1,k-1)$-category of morphisms between them. Thus an $(\infty, 1)$-category should have $(\infty, 0)$-categories as hom-objects between any to spaces.

$(\infty, 0)$-categories deserve to be called ∞-groupoids. Observe now that the ∞-category associated to a topological space X is in fact an ∞-groupoid.

One of the guiding principles of higher category theory is the **Homotopy hypotheses**, which states that topological spaces provide *all* ∞-groupoids.

Thus in some sense, ∞-categories should give a theory which contains ordinary categories and topological spaces as special cases. One can make such a formalism very precise, and the idea is to perform category theory, where one has replaced the category of sets (which is the basic building block for category theory) with a (to be defined) ∞-category of spaces. What should come out of such a construction is just the theory of ∞-categories (this approach is for instance something Cisinski, among others, likes to advertise).

Let us just spend a little time on thinking about the (to be defined) ∞-category of spaces, and how it should behave. We first consider the following: Let CW be the category whose objects are CW-complexes and whose maps are simply maps of topological spaces. Associated to this category we can define new category: the homotopy category of CW-complexes, whose objects are still CW complexes and whose morphisms are homotopy classes of morphisms of topological spaces.

There is a canonical functor

$$\mathrm{CW} \to h(\mathrm{CW})$$

which is the initial functor from CW to any other category which sends homotopy equivalences to isomorphisms. Any "homotopy invariant" functor descends through this functor and thus one might be tempted to replace CW by $h(\mathrm{CW})$ when dealing with homotopy invariant notions of spaces.

Here is a problem with this approach: One great feature of topological spaces is that they are very flexible, i.e. we can manipulate them in many ways: We can build new spaces from old, by passing to subsets and forming quotients for instance. These constructions are special cases of limits and colimits of spaces. A (co)limit is a "universal space" associated to a diagram $I \to \mathrm{CW}$ of spaces indexed over a small category I (e.g. pushouts and pullbacks, or directed colimits, inverse limits). Now it may well happen that there is a natural transformation between two diagrams, which is a homotopy equivalence on each $i \in I$, but where the associated map on colimits is not a homotopy equivalence. So, a priori, our flexibility is now less: what are homotopy theoretic meaningful colimits? There is something like this, and these are called homotopy colimits.

Unfortunately, homotopy colimits are (a priori) only a method of constructing something that should behave like a colimit *and* is invariant under the above transformations, and in fact they do not satisfy a universal property in the category CW. Maybe they do however in the category $h(\mathrm{CW})$? This is also not true, and even worse, although any diagram $I \to \mathrm{CW}$ admits a homotopy colimit, the category $h(\mathrm{CW})$ does not admit all colimits (this is a fun exercise in homotopy theory, I recommend to do it!)

Here is all motivating fact for now that I want to give: The ∞-category of spaces, denoted by An receives a functor

$$\mathrm{CW} \to \mathrm{An} \to h(\mathrm{CW})$$

which has the following properties:

(1) the functor $\mathrm{CW} \to \mathrm{An}$ is now initial among all functors $\mathrm{CW} \to \mathcal{D}$, where $\mathcal{D}$ is an ∞-category, which map homotopy equivalences to equivalences,

(2) any ∞-category has a homotopy category, and the homotopy category of An is $h(\mathrm{CW})$,
(3) there are ∞-categorical notions of colimits and limits, which have universal properties as one would like, and An has all of them,
(4) a homotopy (co)limit in CW is mapped to a (co)limit in An.

Informally, this says that if we are interested in studying topological spaces up to homotopy invariant notions, we are free to replace CW by An by (1), but we do not loose the flexibility to construct new spaces from old ones, this is part (3), and moreover, the ad hoc construction of homotopy colimits now actually have a universal property (4). Furthermore, if one liked the category $h(\mathrm{CW})$ a lot, we still know about this as much as before.

The fact that homotopy categories are typically not well behaved was observed a very long time ago: This is where model categories are useful: They help us getting a concrete grasp on the homotopy category. But model categories are still not quite optimal in some aspects:

For instance it is in general quite hopeless, for two model categories $\mathcal{M}$ and $\mathcal{M}'$, to define a model category structure on the category of functors $\mathrm{Fun}(\mathcal{M}, \mathcal{M}')$. In the context of ∞-categories, defining this is not too hard (although it is *not* tautological!).

Replacing model categories with their associated ∞-categories (there is such a construction), has many technical advantages: There is an ∞-category of ∞-categories, which for instance has all small (co)limits. This allows to study to which extend certain functors with values in a category of categories satisfy nice "glueing properties" (are sheaves for some Grothendieck topology). In principle, one can do the same thing with ordinary categories, but it just turns out that in many situations the passage to ∞-categories has the advantage of

(1) allowing very clean formulations of what it means to satisfy nice "glueing properties" (being a sheaf), and
(2) that sometimes the ∞-categorical statements hold true, whereas the corresponding 1- or 2-categorical statement is simply not correct.

I suggest, we leave it at that for the moment and just dive into the mathematics!

1. Categories and Simplicial Sets

We work in a model $\mathbb{V}$ of ZFC-set theory and assume the large cardinal axiom. This implies that there exists an inaccessible cardinal κ larger than $\aleph_0$, which we can use to define its associated Grothendieck universe $\mathbb{U} \in \mathbb{V}$. $\mathbb{U}$ itself satisfies all axioms of ZFC-set theory and it is this model of ZFC that we will work in generally. We refer to the sets of $\mathbb{U}$ simply as small sets, and call the elements of $\mathbb{V}$ (large) sets.

Definition 1.1. A category $\mathcal{C}$ consists of a (possibly large) set of objects $\mathrm{ob}(\mathcal{C})$, and for any two objects x and y a (also possibly large) set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of morphisms between them, equipped with composition maps

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \times \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

and identities $* \to \mathrm{Hom}_{\mathcal{C}}(x, x)$ for all objects x, satisfying associativity and unitality.

A category is called locally small if all hom sets are small, it is called small if it is locally small and the set of objects is also small. It is called essentially small if it is locally small and the set of isomorphism classes of objects are small.

Remark. Think about this set theory business just as saying that a set corresponds to a small set, and a class corresponds to a set. It is part of the above axiomatic that, for instance the collection of all vector spaces (which consist of small sets) is itself not small, but an element of $\mathbb{V}$.

If you do this, then what usually is called a category is now what we call a locally small category.

Definition 1.2. A partially ordered set is a set P equipped with a reflexive, antisymmetric and transitive relation $\leq$. That is, $a \leq a$, if $a \leq b$ and $b \leq a$ then $a = b$, and if $a \leq b$ and $b \leq c$, then $a \leq c$. A map of partially ordered sets is a map of sets $f\colon P \to Q$ such that $x \leq y$ implies that $f(x) \leq f(y)$. This defines a category PoSet whose objects are posets an whose morphisms are maps of posets.

Example 1.3. Finite linearly ordered sets: The set $\{0, 1, \ldots, n\}$ is linearly ordered: $0 \leq 1 \leq \cdots \leq n$ and written as $[n]$. A general finite poset S is called linearly ordered if it is isomorphic to one of the $[n]$'s. Morphisms of linearly ordered sets are just morphisms of the underlying poset. We obtain a category LinOrdSet.

Example 1.4. The subset poset: Let S be a set. Consider its set $\mathcal{P}(S)$ of subsets: $\mathcal{P}(S) = \{I \subseteq S\}$. This is partially ordered by inclusion: $I \leq J \Leftrightarrow I \subseteq J$.

Definition 1.5. The category Δ is the full subcategory of the category PoSet of posets consisting of the linearly ordered set $[n]$ for all $n \geq 0$. Notice that a morphism from $[n]$ to $[m]$ is thus simply a weakly monotonic map.

Example 1.6. There are special maps in Δ: the face and degeneracy maps: For every $n \geq 0$ and $0 \leq i \leq n$, there are maps

$$d_i\colon [n-1] \to [n]$$

uniquely determined by the property that $i \notin \mathrm{Im}(d_i)$ and that d_i is injective. Furthermore, for $n \geq 1$ and $0 \leq i \leq n-1$, there are maps

$$s_i\colon [n] \to [n-1]$$

uniquely determined by the property $|s_i^{-1}(i)| = 2$ and that s_i is surjective. I.e. we have $s_i(i) = s_i(i+1) = i$.

Definition 1.7. Let $\mathcal{C}$ be a category. We denote the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by $\mathcal{P}(\mathcal{C})$ and call it the category of presheaves on $\mathcal{C}$. An object $x \in \mathcal{C}$ determines a *representable* presheaf, namely the presheaf $\mathrm{Hom}_{\mathcal{C}}(-, x)$ which sends $y \in \mathcal{C}$ to the set of morphisms from y to x. This determines a functor $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ which is called the *Yoneda embedding.*

Definition 1.8. A simplicial set is a presheaf on Δ, i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. Given a simplicial set X, its set of n-simplices is given by $X([n])$ and will be written X_n. An n-simplex x is called degenerate if there exists a surjection $\alpha\colon [n] \to [m]$ with $m \neq n$, and an n-simplex y such that $x = \alpha^*(y)$. Equivalently, x is degenerate if $x = s_i^*(y)$ for some $y \in X_{n-1}$ and some $0 \leq i \leq n-1$. An n-simplex is called non-degenerate if it is not degenerate.

Definition 1.9. We let Δ^n be the simplicial set represented by $[n] \in \Delta$. Concretely, we have $(\Delta^n)_m = \mathrm{Hom}_\Delta([m], [n])$.

Lecture 2 – 18.10.2018.

Definition 1.10. Let X be a simplicial set. We define $\pi_0^\Delta(X)$ to be the following set. We consider the equivalence relation $\sim$ on the set of 0-simplices X_0 which is generated by the relation $x \sim y$ if and only if there exists a 1-simplex $f \in X_1$ such that $d_0(f) = x$ and $d_1(f) = y$. (This relation is reflexive but in general neither transitive nor symmetric). Then we let $\pi_0^\Delta(X) = X_0/\sim$.

Lemma 1.11. *The Yoneda lemma: Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor and $x \in \mathcal{C}$ an object. Then the map*
$$\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-,x), F) \to F(x)$$
given by sending η to $\eta(\mathrm{id}_x)$ is a bijection.

Proof. The inverse is given by sending an element $s \in F(x)$ to the function $\mathrm{Hom}_{\mathcal{C}}(y,x) \to F(y)$ sending f to $f^*(s)$. It is an explicit check to see that this is a natural transformation and an inverse the the above described map. $\square$

Lemma 1.12. *The Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ is fully faithful.*

Proof. This follows immediately from the Yoneda Lemma: The effect of the Yoneda embedding on morphisms is the map
$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-,x), \mathrm{Hom}_{\mathcal{C}}(-,y))$$
given by sending f to
$$\mathrm{Hom}_{\mathcal{C}}(z,x) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(z,y).$$
We claim that this map is inverse to the map described in the Yoneda lemma, which is given by sending a map $f\colon \mathrm{Hom}_{\mathcal{C}}(x,y)$ to the function $\mathrm{Hom}_{\mathcal{C}}(z,x) \to \mathrm{Hom}_{\mathcal{C}}(z,y)$ given by sending φ to $\varphi^*(f) = f_*\varphi$. $\square$

Corollary 1.13. *For a simplicial set, there is a canonical bijection*
$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \cong X_n.$$

Definition 1.14. A (co)limit of a functor $F\colon I \to \mathcal{C}$ is an object of $\mathcal{C}$, written $\mathrm{colim}_I\, F$, equipped with maps $F(i) \to \mathrm{colim}_I\, F$ for every i which are compatible in the sense that for every morphism $i \to j$ in I, the diagram

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d] & \mathrm{colim}_I\, F \\ F(j) \arrow[ur] & \end{tikzcd}$$

commutes. This datum is required to satisfy the following universal property: Whenever given a further object $X \in \mathcal{C}$, also equipped with maps $F(i) \to X$ which are compatible in the above way, then there exists a unique morphism $\mathrm{colim}_I\, F \to X$ making the diagrams

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d] & X \\ \mathrm{colim}_I\, F \arrow[ur] & \end{tikzcd}$$

commute.

Dually, a limit of F is an object $\lim_I F$, equipped with maps $\lim_I F \to F(i)$, which are again compatible, satisfying the dual universal property: Whenever we are given an object X equipped with compatible morphisms $X \to F(i)$ for all $i \in I$, there exists a unique morphism $X \to \lim_I F$ making the obvious diagram commute.

Remark. Notice that such a universal property specifies an object up to unique isomorphism. Notice also that the universal property refers to more than just the object $\operatorname{colim}_I F$. The reference maps are part of the data, and this is what makes the object unique up to unique isomorphism.

Example 1.15. (1) A colimit of the empty diagram $\emptyset \to \mathcal{C}$ is an initial object: It is an object which admits a unique morphism to any other object. Dually, A limit of the empty diagram $\emptyset \to \mathcal{C}$ is a terminal object: It is an object which admits a unique morphism from any other object.
(2) A colimit of the diagram $\bullet \leftarrow \bullet \to \bullet$ is called a pushout.
(3) A limit of the diagram $\bullet \to \bullet \leftarrow \bullet$ is called a pullback.

Example 1.16. The quotient vector space V/U is a pushout of the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ 0 \arrow[r] & V/U \end{tikzcd}$$

Observation 1.17. One can phrase general (co)limits via inital and terminal objects. This point of view will be used later when we discuss limits and colimits in ∞-categories. Given a functor $F\colon I \to \mathcal{C}$ we can consider the category of (co)cones of this functor. Given a category I we consider a new category $I^{\triangleleft}$ and $I^{\triangleright}$, which are constructed from I by adding an initial respectively a terminal object. There is an obvious functor $I \to I^{\triangleleft}$ and $I \to I^{\triangleright}$. We can thus consider the functor categories

$$\mathrm{Fun}_F(I^{\triangleleft}, \mathcal{C}) \text{ and } \mathrm{Fun}_F(I^{\triangleright}, \mathcal{C})$$

of functors which restrict to F along the above mentioned inclusion. These are called the categories of cones and cocones over F, respectively.

The following lemma is immediate from the definition of (co)limits, and the fact established in Exercise 1.5 that the category Set is bicomplete (else the statement does not make sense).

Lemma 1.18. *Let $\mathcal{C}$ be a category and let $F\colon I \to \mathcal{C}$ be an I-shaped diagram in $\mathcal{C}$. Then, for every object $x \in \mathcal{C}$, there are canonical bijections*

(1) $\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}_I F, x) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(F(i), x)$, *and*
(2) $\mathrm{Hom}_{\mathcal{C}}(x, \lim_I F) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(x, F(i))$.

Moreover, this property characterizes (co)limits uniquely.

Definition 1.19. An adjunction consists of a pair of functors $(F\colon \mathcal{C} \to \mathcal{D}, G\colon \mathcal{D} \to \mathcal{C})$ together with a natural isomorphism between the two functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$ given by

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \text{ and } \mathrm{Hom}_{\mathcal{C}}(-, G(-)).$$

Lemma 1.20. *Left adjoints preserve colimits, right adjoints preserve limits.*

Proof. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits a right adjoint, say G. Let $X\colon I \to \mathcal{C}$ be a diagram which has a colimit $\operatorname{colim}_I X \in \mathcal{C}$. We claim that F sends that colimit to a colimit of the diagram $I \to \mathcal{C} \to \mathcal{D}$. In formulas, we claim that the canonical map $\operatorname{colim}_I F(X(i)) \to F(\operatorname{colim}_I X(i))$ induced from the compatible maps $F(X(i)) \to F(\operatorname{colim}_I X(i))$ that are part of the datum of the colimit (and then applying F) is an isomorphism. To see this, it suffices to show that it induces a bijection on hom sets for all other objects $y \in \mathcal{D}$:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}}(F(\operatorname*{colim}_I X(i)), y) &\cong \operatorname{Hom}_{\mathcal{C}}(\operatorname*{colim}_I X(i), Gy) \\
&\cong \lim_I \operatorname{Hom}_{\mathcal{C}}(X(i), Gy) \\
&\cong \lim_I \operatorname{Hom}_{\mathcal{D}}(F(X(i)), y) \\
&\cong \operatorname{Hom}_{\mathcal{D}}(\operatorname*{colim}_I F(X(i)), y)
\end{aligned}$$

so we are done by the Yoneda lemma. The argument for the claim that right adjoints preserve limits is similar. □

Lemma 1.21. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which admits right adjoints G and G'. Then there is a specified natural isomorphism between G and G'. (Adjoints, if they exist, are unique up to unique isomorphism).*

Proof. Consider the following two natural bijections

$$\operatorname{Hom}_{\mathcal{C}}(Gx, G'x) \cong \operatorname{Hom}_{\mathcal{D}}(FGx, x) \cong \operatorname{Hom}_{\mathcal{C}}(Gx, Gx).$$

Then the identity of Gx corresponds to a natural transformation $G \to G'$. Applying the same trick for $\operatorname{Hom}_{\mathcal{C}}(G'x, Gx)$ shows that this must be a natural isomorphism. □

Definition 1.22. A category is called (co)complete, if it admits (co)limits indexed over arbitrary small (co)limits. It is called bicomplete if it is both complete and cocomplete.

Lemma 1.23. *If $\mathcal{C}$ is bicomplete, then (co)lim is left/right adjoint to the constant diagram functor. In particular, forming (co)limits determines a functor*

$$\operatorname{Fun}(I, \mathcal{C}) \to \mathcal{C}.$$

Proof. Let's spell out the colimit case. Consider the constant functor $\mathrm{const}\colon \mathcal{C} \to \operatorname{Fun}(I, \mathcal{C})$. Now we specify, for each functor $F\colon I \to \mathcal{C}$ an object, namely $\operatorname{colim}_I F$. Part of the datum of a colimit are compatible maps $\{F(i) \to \operatorname{colim}_I F\}_{\{i \in I\}}$ which are easily seen to assemble into a natural transformation

$$F \to \mathrm{const}(\operatorname*{colim}_I F).$$

Then we consider the composite

$$\operatorname{Hom}_{\mathcal{C}}(\operatorname*{colim}_I F, X) \to \operatorname{Hom}_{\operatorname{Fun}(I,\mathcal{C})}(\mathrm{const}(\operatorname*{colim}_I F), \mathrm{const}X) \to \operatorname{Hom}_{\operatorname{Fun}(I,\mathcal{C})}(F, \mathrm{const}X)$$

which is a bijection by the universal property of a colimit. The lemma thus follows from Exercise 1.9. The case of limits is completely analogous. □

Lecture 3 – 22.10.2018.

Lemma 1.24. *Given an adjunction with $F\colon \mathcal{C} \to \mathcal{D}$ being left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$, and given a further auxiliary small category I, then the functors*

$$F_*\colon \operatorname{Fun}(I, \mathcal{C}) \rightleftarrows \operatorname{Fun}(I, \mathcal{D})\colon G_*$$

again form an adjoint pair.

Proof. The adjunction is determined by a counit map $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ and a unit map $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ that satisfy the snake identities of Exercise 1.8. We now use these to construct counit and unit maps for the pair of functors (F_*, G_*) as follows: Let $\varphi \in \operatorname{Fun}(I, \mathcal{D})$. We need to specify a natural map $\varepsilon_*\colon F_*(G_*(\varphi)) \to \varphi$ of functors $I \to \mathcal{D}$, so let $x \in \mathcal{E}$. We define the new counit ε_* to be the map

$$F(G(\varphi(x))) \xrightarrow{\varepsilon_{\varphi(x)}} \varphi(x).$$

It is easy to see that this is natural in φ, since ε itself is a natural transformation. Similarly we define a natural transformation $\eta_*\colon \psi \to G_*F_*(\psi)$ to be given by

$$\psi(y) \xrightarrow{\eta_{\psi(y)}} G(F(\psi(y))).$$

It is then easy to see that the snake identities are satisfied, because (ε, η) satisfy the snake identities. $\square$

Proposition 1.25. *Let $\mathcal{C}$ be a bicomplete category, then $\operatorname{Fun}(I, \mathcal{C})$ is bicomplete as well. A (co)limit of a diagram $X\colon J \to \operatorname{Fun}(I, \mathcal{C})$ is given by the functor sending $i \in I$ to $\operatorname{colim}_J X(j)(i)$.*

Proof. Let us argue that $\operatorname{Fun}(I, \mathcal{C})$ is cocomplete. The completeness argument is similar (or can be formally deduced from this case by applying op correctly). We claim that the composite

$$\operatorname{Fun}(J, \operatorname{Fun}(I, \mathcal{C})) \cong \operatorname{Fun}(I, \operatorname{Fun}(J, \mathcal{C})) \xrightarrow{\operatorname{colim}_J} \operatorname{Fun}(I, \mathcal{C})$$

is a colimit functor we wish to show exists. By Lemma 1.24 this functor has a right given by

$$\operatorname{const}_*\colon \operatorname{Fun}(I, \mathcal{C}) \to \operatorname{Fun}(I, \operatorname{Fun}(J, \mathcal{C})) \cong \operatorname{Fun}(J, \operatorname{Fun}(I, \mathcal{C}))$$

it the proposition is shown once we convince ourselves that this is itself the constant functor (which is immediate from the definition), as then we allude to Lemma 1.23. $\square$

Corollary 1.26. *The category of simplicial sets* sSet *is bicomplete.*

Definition 1.27. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and let $x \in \mathcal{D}$ be an object. Then the slice category F/x has as objects pairs $(c \in \mathcal{C}, \alpha\colon F(c) \to x)$. For x/F the arrow goes the other direction. Morphisms from (c, α) to $(c'\alpha')$ are given by a morphism $\beta\colon c \to c'$ in $\mathcal{C}$ making the obvious diagram commute.

If F is a subcategory, we also simply write $\mathcal{C}/x$ and $x/\mathcal{C}$. For an object $(c, \alpha) \in \mathcal{C}/x$ we sometimes simply write $c \to x$.

Lemma 1.28. *Every presheaf is a colimit of representables. More precisely, every presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ satisfies that the tautological map*

$$\operatorname*{colim}_{X \to F} \operatorname{Hom}_{\mathcal{C}}(-, X) \to F$$

is an isomorphism.

Proof. We prove this again by the Yoneda lemma, i.e. we show that this map induces a bijection on maps to an auxiliary presheaf G. We calculate

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\operatorname*{colim}_{X\to F}\mathrm{Hom}_{\mathcal{C}}(-,X),G) &\cong \lim_{X\to F}\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(-,X),G)\\ &\cong \lim_{X\to F} G(X)\end{aligned}$$

and it is not hard to see that the latter is in fact the set of natural transformations from F to G. □

Let $i\colon \mathcal{C}_0 \subseteq \mathcal{C}$ be a small subcategory of a category, and let $\mathcal{D}$ be a bicomplete category.

Fact 1.29. The restriction functor

$$i^*\colon \mathrm{Fun}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_0,\mathcal{D})$$

has a left adjoint denoted by $i_!$ and a right adjoint denoted by i_*. They are given as follows

$$i_!(F)(x) = \operatorname*{colim}_{c\in \mathcal{C}_0/x} F(c)$$

and

$$i_*(F)(y) = \lim_{c\in x/\mathcal{C}_0} F(x).$$

Notice that the slices are small by assumption, so that the colimits and limits exist. I will now not spell out why this formula produces adjoints for the restriction functor i^*.

Definition 1.30. In the situation above, we call $i_!(F)$ the left Kan extension of F along i and $i_*(F)$ the right Kan extension of F along i.

Observation 1.31. The statement that the tautological map is an isomorphism shows that the identity of $\mathcal{P}(\mathcal{C})$ is left Kan extended from the Yoneda embedding (along the Yoneda embedding).

Corollary 1.32. *If $\mathcal{D}$ is a cocomplete category and $\mathcal{C}$ is a small category, then the canonical functor*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathcal{P}(\mathcal{C}),\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

obtained by restriction along the Yoneda embedding is an equivalence.

Proof. Given a functor $f\colon \mathcal{C}\to\mathcal{D}$, we want to construct a colimit preserving functor $\hat{f}\colon \mathcal{P}(\mathcal{C}) \to \mathcal{D}$, such that $\hat{f}(X) = f(X)$ for $X \in \mathcal{C}$. By Lemma 1.28, given an object $F \in \mathcal{P}(\mathcal{C})$, we are forced to define

$$\hat{F}(F) = \operatorname*{colim}_{X\to F} f(X).$$

One can check that this is in fact a functor: For $F \to G$ a morphism in $\mathcal{P}(\mathcal{C})$, there is an induced functor of the category of $X \to F$, to $X \to G$ given by postcomposition with the given morphism. Then it is not hard to see that taking colimits produces a map

$$\operatorname*{colim}_{X\to F} f(X) \to \operatorname*{colim}_{X\to G} f(X).$$

Also, it is not hard to see that this is in fact a functor.

To see that it is colimit preserving, we observe that $\hat{f}$ admits a right adjoint G given by the following formula:

$$G(d)(X) = \mathrm{Hom}_{\mathcal{D}}(f(X),d).$$

In the proof of Lemma 1.28 we saw that the set of natural transformations between F and F' is given by

$$\mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(F, F') = \lim_{X \to F} F'(X).$$

We thus see that

$$\mathrm{Hom}_{\mathcal{D}}(\hat{f}(F), d) \cong \lim_{X \to F} \mathrm{Hom}_{\mathcal{D}}(f(X), d) \cong \lim_{X \to F} G(d)(X).$$

Hence $\hat{f}$ is left adjoint to G and thus preserves colimits. □

Corollary 1.33. *Let X be a simplicial set. Then*

$$X \cong \operatorname*{colim}_{[n] \in \Delta/X} \Delta^n.$$

Lemma 1.34. *Let X be a fixed simplicial set. Then the functor* sSet $\to$ sSet *sending Y to $X \times Y$ admits a right adjoint $\underline{\mathrm{Hom}}(X, -)$ determined by the formula*

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \underline{\mathrm{Hom}}(X, Z)) = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times X, Z).$$

Sometimes we will also write Z^X for $\underline{\mathrm{Hom}}(X, Z)$.

Proof. Mapping $[n]$ to the set on the right hand side clearly determines a simplicial set which we call $\underline{\mathrm{Hom}}(X, Z)$. It satisfies the adjunction property on representable simplicial sets, so we can extend the adjunction to all simplicial sets because every simplicial set is a colimit of representables. Notice that we use here that the functor $X \times -\colon \mathrm{sSet} \to \mathrm{sSet}$ preserves colimits: This is certainly true in Set because again the hom set provides a right adjoint. □

Definition 1.35. Let $n \geq 0$ be a natural number. Then the topological n-simplex Δ^n_{top} is the subspace of $\mathbb{R}^{n+1}_{\geq 0}$ consisting of those points whose coordinates add up to 1. The topological simplices form a cosimplicial space $[n] \mapsto \Delta^n_{\mathrm{top}}$: The induced maps are the unique affine linear maps that do what they should on vertices: Precisely given $\alpha\colon [n] \to [m]$, the induced map $\alpha_*\colon \Delta^n_{\mathrm{top}} \to \Delta^m_{\mathrm{top}}$ is given by

$$\alpha_*(t_0, \ldots, t_n) = (v_0, \ldots, v_m)$$

where $v_i = \sum_{j \mapsto i} t_j$.

Definition 1.36. The singular simplicial set of a topological space X is the simplicial set

$$\mathcal{S}(X) = \big([n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, X)\big).$$

Definition 1.37. The geometric realization is the unique colimit preserving functor sSet $\to$ Top which sends Δ^n to Δ^n_{top}. Concretely, the geometric realization of a simplicial set X is the following topological space

$$|X| = \operatorname*{colim}_{\Delta^n \to X} \Delta^n_{\mathrm{top}}.$$

An even more concrete formula is given by

$$|X| = \Big(\coprod_{n \in \Delta} X_n \times \Delta^n_{\mathrm{top}}\Big)/((f^*(x), t) \sim (x, f_*(t))$$

for $x \in X_n$, $t \in \Delta^m_{\mathrm{top}}$ and $f\colon [m] \to [n]$ a morphism in Δ.

Proposition 1.38. *The singular complex is right adjoint to geometric realization.*

Proof. By definition of adjunctions we need to specify a natural isomorphism of functors $\mathrm{sSet}^{\mathrm{op}} \times \mathrm{Top} \to \mathrm{Set}$ between

$$\mathrm{Hom}_{\mathrm{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathrm{sSet}}(X, \mathcal{S}(Y))$$

But by the previous work we know that these functors are equivalent to

$$\lim_{\Delta^n \to X} \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, Y) \text{ and } \lim_{\Delta^n \to X} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathcal{S}(Y))$$

and the latter two are already isomorphic (by definition of $\mathcal{S}(Y)$) before forming the limit. □

Definition 1.39.
(1) The boundary $\partial\Delta^n$ is the subsimplicial set of Δ^n whose k-simplices consist of the non-surjective maps $[k] \to [n]$.
(2) For any subset $S \subseteq [n]$ the S-horn $\Lambda^n_S \subseteq \Delta^n$ consists of those k-simplices $f\colon [k] \to [n]$ where there exists a $i \in [n] \setminus S$ such that i is not in the image of f. A horn Λ^n_j is called inner if $0 < j < n$ and it is called outer if $j = 0, n$ (left, right horn).
(3) The spine $I^n \subseteq \Delta^n$ is given by those k-simplices $f\colon [k] \to [n]$ whose image is either of the form $\{j\}$ or $\{j, j+1\}$.

Lecture 4 – 25.10.2018.

Definition 1.40. The n-skeleton $\mathrm{sk}_n(X)$ of a simplicial set X is given by the simplicial set $i_!i^*(X)$, where $i\colon \Delta_{\leq n} \subseteq \Delta$ is the inclusion of the full subcategory on objects of cardinality $\leq n+1$. Dually, the n-coskeleton $\mathrm{cosk}_n(X)$ of a simplicial set is given by the simplicial set $i_*i^*(X)$. This implies that the k-simplices of $\mathrm{cosk}_n(X)$ are given by

$$\mathrm{cosk}_n(X)_k = \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{sk}_n(\Delta^k), X).$$

Dually, $\mathrm{cosk}_n(X)$ is the largest simplicial set which contains X and whose k-simplices are the same as those of X for $k \leq n$.

Lemma 1.41.
(1) *The skeleton* $\mathrm{sk}_n(X)$ *is the smallest sub simplicial set of* X *whose set of* k*-simplices coincides with the ones of* X *for* $k \leq n$.
(2) *The functors* sk_n *and* cosk_n *are left and right adjoint to each other.*
(3) *There is the formula* $\mathrm{cosk}_n(X) = \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{sk}_n(\Delta^k), X)$.

Proof.
(1) It is easy to see that $\mathrm{sk}_n(X)$ is a sub simplicial set of X, and that $\mathrm{sk}_n(X)_k = X_k$ for $k \leq n$. Given any other sub simplicial set Z with this property we have $\mathrm{sk}_n(X) = \mathrm{sk}_n(Z) \subseteq Z$, so the claim follows.
(2) Obvious, since adjoints compose.
(3) Obvious, by (2).

□

Lemma 1.42. *The geometric realization of the horn is a horn, the geometric realization of the spine is a spine.*

Proof. This follows from the fact that $|\Delta^n| = \Delta^n_{\mathrm{top}}$ and the following observations.
(1) $I^n = I^{n-2} \amalg_{\Delta^0} \Delta^1$, and
(2) there is a coequalizer

$$\coprod_{0 \leq i \leq j \leq n} \Delta^{[n]\setminus i} \times_{\Delta^n} \Delta^{[n]\setminus j} \to \coprod_{0 \leq i \leq n, i \neq k} \Delta^{[n]\setminus i} \to \Lambda^n_i.$$

how one obtains horns and spines as pushouts of Δ^k's. □

Corollary 1.43. *The geometric realization of a simplicial set is a CW complex.*

Proof. Given as simplicial set X, define a filtration on $|X|$ through $|sk_n(X)| \subseteq |X|$. Since geometric realization commutes with colimits, we see that this is in fact a filtration of $|X|$ and the the pushouts of above give pushouts of geometric realizations. Then use that $|\partial\Delta^n| \cong S^{n-1}$ and $|\Delta^n| \cong D^n$. □

Observation 1.44. A poset determines a category in the following way: Objects are the elements of the posets P and for each pair of elements $x, y \in P$ we have

$$\operatorname{Hom}(x, y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{else.} \end{cases}$$

Furthermore, a functor between categories associated to posets is the same thing as a map of posets, i.e. a map of sets respecting the partial ordering. This determines a fully faithful functor PoSet $\to$ Cat. It follows that we can view $[n]$ a category. Sending $[n]$ to this category produces a cosimplicial small category.

Definition 1.45. The nerve of a category $\mathcal{C}$ is the simplicial set given by

$$[n] \mapsto \operatorname{Fun}([n], \mathcal{C})$$

i.e. given by taking functors out of the previous cosimplicial category to the given one.

Lemma 1.46. *Δ^n is isomorphic to the nerve of the category $[n]$.*

Proof. Unravelling the definitions we find that

$$(\Delta^n)_m = \operatorname{Hom}_\Delta([m], [n])$$

whereas

$$\mathrm{N}([n])_m = \operatorname{Fun}([m], [n]).$$

It thus suffices to recall that the functor Posets $\to$ Cat is fully faithful. □

Definition 1.47. The classifying space BG of a group G is the geometric realization of the nerve of the group considered as groupoid with one object.

Definition 1.48. A Kan complex is a simplicial set which has the extension property for horn inclusions $\Lambda^n_j \to \Delta^n$ for $0 \leq j \leq n$.

Lemma 1.49. *The singular complex of a topological space is a Kan complex.*

Proof. By adjuction, there is an equivalence of lifting problems

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & \mathcal{S}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} {|\Lambda^n_j|} \arrow[r] \arrow[d] & X \\ {|\Delta^n|} \arrow[ur, dashed] & \end{tikzcd}$$

Then we recall that the topological horn inclusion has a retract, so that the right lifting problem can be solved. □

Fact 1.50. A Kan complex satisfies the extension property for any monomorphism of simplicial sets $K \to L$ which induces a weak equivalence on geometric realizations (these are called anodyne maps).

Lemma 1.51. *If the nerve of a category is a Kan complex then the category is a groupoid.*

Proof. Because we can lift outer 2-horns, one can easily show that every morphism in $\mathcal{C}$ has a right and a left inverse, so is itself invertible. $\square$

Definition 1.52. Let $f, g\colon X \to Y$ be a map of simplicial sets. We say that f and g are homotopic if there exists $H\colon X \times \Delta^1 \to Y$ such that H restricts to f and g. Given a pointed Kan complex (X, x) we define its simplicial homotopy groups to be the

$$\pi_n^\Delta(X, x) = [(\Delta^n, \partial\Delta^n), (X, x)]_*$$

Fact 1.53. The homotopy relation is in fact an equivalence relation if Y is a Kan complex. The simplicial homotopy groups of a Kan complex agree with the ordinary homotopy groups of the geometric realization. In particular, they are groups for $n \geq 1$ and abelian groups for $n \geq 2$. For more about simplicial homotopies and simplicial homotopy groups see the book of Goerss–Jardine, I.6 and I.7.

Lecture 5 – 29.10.2018.

Theorem 1.54. *For a simplicial set X, the following three conditions are equivalent.*
1. *X has unique extensions for $\Lambda^n_j \to \Delta^n$ if $0 < j < n$.*
2. *X has unique extensions for $I^n \to \Delta^n$ for $n \geq 2$.*
3. *X is isomorphic to the nerve of a category.*

Proof. We will show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To show (3) $\Rightarrow$ (2), we consider a category $\mathcal{C}$ and its nerve $\mathrm{N}(\mathcal{C})$. Recall that its n-simplices are given by $\mathrm{Fun}([n], \mathcal{C})$, and thus are given by chains of composable morphisms. Face and degeneracies are given by composition and inserting identities. In particular, the restriction along the spine inclusion I^n, picks out precisely the morphisms, so that restriction along the spine induces an bijection between $\mathrm{Fun}([n], \mathcal{C})$ and $\mathrm{Hom}_{\mathrm{sSet}}(I^n, \mathrm{N}(\mathcal{C}))$.

To show (2) $\Rightarrow$ (3), consider a simplicial set X which has unique liftings against spines. We define a category $\mathcal{C}$ as follows: The objects are given by X_0, the 0-simplices of X. The morphisms from x to y are given by all 1-simplices $f \in X_1$ such that $d_1(f) = x$ and $d_0(f) = y$. Identities are given by $s_0(x)$.

We need to explain how to compose morphisms: Two composable morphisms determine a map $I^2 = \Lambda^2_1 \to X$ which we can extend over Δ^2 and restrict to the new edge. We claim that this indeed is a category: We have to check that identities are such and that composition is associative (both follow from uniqueness).

(a) $s_0(f)\colon \Delta^2 \to X$ has $d_0 s_0(f) = f = d_1 s_0(f)$ and $d_2 s_0(f) = s_0 d_1(f) = s_0(x)$. Similarly, $s_1(f)\colon \Delta^2 \to X$ is a 2-simplex whitnessing that $\mathrm{id}_y \circ f = f$.

(b) Let f, g, h be composable 1-simplices. Consider the associated map $I^3 \to X$. It can be uniquely filled to a map $\Phi\colon \Delta^3 \to X$. The restriction of this map to $\Delta^{\{0,2\}}$ is gf. The 2-simplex $d_2(\Phi)$ is thus a composition of gf and h, i.e. $\Phi_{|\Delta^{\{0,3\}}} = h \circ gf$. On the other hand, the 2-simplex $d_0(\Phi)$ gives hg, and thus the 2-simplex $d_1(\Phi)$ gives $hg \circ f$. Hence

$$h \circ gf = \Phi_{\Delta^{\{0,3\}}} = hg \circ f$$

and associativity of composition is shown.

We claim that there is a preferred map $X \to \mathrm{N}(\mathcal{C})$ given by the following construction: A map $\Delta^n \to X$ can be restricted along the spine and thus determines a sequence of composable morphisms of $\mathcal{C}$, so that we obtain a map $I^n \to \mathrm{N}(\mathcal{C})$. This map can be (uniquely) extended over Δ^n and thus gives an association mapping n-simplices of X to n-simplices of $\mathrm{N}(\mathcal{C})$. Using that an n-simplex of X is determined by its restriction to the spine we now easily see that the map $X \to \mathrm{N}(\mathcal{C})$ satisfies:

(a) It is an isomorphism on 0- and 1-simplices,
(b) There is a commutative diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C})_n \arrow[d] \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[r] & \mathrm{N}(\mathcal{C})_1 \times_{\mathrm{N}(\mathcal{C})_0} \cdots \times_{\mathrm{N}(\mathcal{C})_0} \mathrm{N}(\mathcal{C})_1 \end{tikzcd}$$

where the vertical maps are bijections and where the lower fibre product is over the source and target maps and has n-many factors on both sides.

By (a) the lower map is a bijection, so the upper horizontal map is a bijection as well. This shows that the map $X \to \mathrm{N}(\mathcal{C})$ is an isomorphism of simplicial sets, which shows (2) $\Rightarrow$ (3).

We now show (1) $\Rightarrow$ (2). We prove this via induction over n. For $n = 2$ this is clear, as the 2-spine is the inner 2-horn. So we may assume that one can uniquely lift maps $I^k \to X$ to Δ^k for all k strictly smaller than n and consider a map $I^n \to X$ which we wish to show extends uniquely to Δ^n. We will show that it extends uniquely to Λ^n_j for some $0 \leq j \leq n$, and then use (1) to deduce the claim. We first observe that $I^n \cap \Delta^{n \setminus \{n\}}$ is the spine I^{n-1} of this simplex, and likewise that $I^n \cap \Delta^{n\setminus\{0\}}$ is also the spine. Thus, by the inductive hypothesis, there is are unique maps $\Delta^{n\setminus\{\varepsilon\}} \to X$ extending the map from the spine to X for $\varepsilon = 0, n$. Since the intersection of these two faces is given by $\Delta^{n\setminus\{0,n\}}$, which intersects the spine again in a smaller spine, these two extensions agree on this intersection, by the inductive hypothesis. We hence obtain a map

$$I^n \cup \Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}} = \Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}} \to X$$

where the union is in Δ^n. We claim that there exists a unique extension to the union

$$\Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}} \cup \Delta^{n\setminus\{1\}}.$$

For this we claim that $\Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}}$ contains the spine of $\Delta^{n\setminus\{1\}}$: The edges from $i \to i+1$ for $2 \leq i \leq n-1$ all lie in $\Delta^{n\setminus\{0\}}$, and the edge from $0 \to 2$ lies in $\Delta^{n\setminus\{n\}}$ because $n \geq 3$. Hence there is a unique map from $\Delta^{n\setminus\{1\}} \to X$ extending this map on the spine. We need to argue that it agrees with the given one on

$$\big(\Delta^{n\setminus\{0\}} \cup \Delta^{n\setminus\{n\}}\big) \cap \Delta^{n\setminus\{1\}} = \Delta^{n\setminus\{0,1\}} \cup \Delta^{n\setminus\{1,n\}}.$$

On both of these simplices, the map is determined by its restriction to the spine which shows the claim. Inductively, we find that there exists a unique extension of the map in question to a map $\Lambda^n_{n-1} \to X$. This can now uniquely be extended to Δ^n by assumption.

To see that (2) $\Rightarrow$ (1), we consider an extension problem $\beta\colon \Lambda^n_i \to X$ which we want to show to uniquely extend to Δ^n. Clearly we may assume that $n \geq 3$, because $I^2 = \Lambda^2_1$. By assumption (and an exercise) there is an inclusion $I^n \to \Lambda^n_i$ and we can consider the restricted extension problem. This can be solved uniquely by assumption, so that we obtain a map $\alpha\colon \Delta^n \to X$. This map can in turn be restricted to Λ^n_i, and we want to show that this map is given by β. To do so, we may restrict to the faces of Λ^n_i, i.e. to the union of $\Delta^{n\setminus\{j\}}$

for $j \neq i$. It is easy to see that

$$\alpha_{|\Delta^{n\setminus\{0\}}} = \beta_{|\Delta^{n\setminus\{0\}}}$$

because the spine of that simplex is given by a subset of the big spine, and by its very definition $\alpha_{|I^n} = \beta_{|I^n}$. The same holds for

$$\alpha_{|\Delta^{n\setminus\{n\}}} = \beta_{|\Delta^{n\setminus\{n\}}}$$

provided β is defined there. We need to show that

$$\alpha_{|\Delta^{n\setminus\{j\}}} = \beta_{|\Delta^{n\setminus\{j\}}}$$

and may assume that $j \neq 0, n$. For this, we show that $\alpha_{|\Delta^{\{j-1,j+1\}}} = \beta_{|\Delta^{\{j-1,j+1\}}}$ (again, all the other edges of the spine are contained in the big spine already). Since $n \geq 3$ this edge is contained in $\Delta^{n\setminus\{\varepsilon\}}$ for ε either 0 or 1.

Then we induct and see that this determines the map from Λ^n_i and thus that we are done. □

One can in fact say slightly more:

Lemma 1.55. *If $n \geq 4$ and $0 \leq j \leq n$ and $\mathcal{C}$ is an ordinary category, then every lifting problem*

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

can be solved uniquely.

Proof. It is an exercise to show that nerves of categories are 2-coskeletal, and we give another argument in Corollary 2.21. It hence suffices to recall that $\mathrm{sk}_2(\Lambda^n_j) \to \mathrm{sk}_2(\Delta^2)$ is an isomorphism for $n \geq 4$ and all $0 \leq j \leq n$. □

Lemma 1.56. *The nerve of a category $\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.*

Proof. We have seen already the direction that $\mathrm{N}(\mathcal{C})$ being a Kan complex implies that $\mathcal{C}$ is a groupoid, Lemma 1.51. So we need to show the other direction. Let us thus assume that $\mathcal{C}$ is a groupoid, and let us show that $\mathrm{N}(\mathcal{C})$ is a Kan complex. By Theorem 1.54 we know already that we can (uniquely) lift all inner horns and all horns of dimension greater or equal to 4. So we need to prove that we can lift outer 2-horns and outer 3-horns. By passing to opposite categories it suffices to show that every extension problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a (unique) solution for $n = 2, 3$. For $n = 2$, such a map is given by two maps $f\colon x \to y$ and $g\colon x \to z$. We can then choose $f \circ g^{-1}$ for the other edge. To show the claim for the left outer 3-horn, we consider the restriction along the spine and obtain three composable maps f, g, and h. We find an extension to Δ^3 precisely if the edge $\Delta^{\{1,3\}} \to \Lambda^3_0 \to \mathcal{C}$ is given by the composite hg. Considering the 2-simplex $\Delta^{\{0,1,3\}} \to \Lambda^3_0 \to \mathcal{C}$, we find that this edge satisfies that precomposition with f is given by hgf. Since f is an isomorphism, the claim follows. □

2. ∞-CATEGORIES

Definition 2.1. A *composer* is a simplicial set which has the extension property for spine inclusions $I^n \to \Delta^n$.

Definition 2.2. In a composer (in fact in a general simplicial set) we call 0-simplices objects, and a 1-simplex f is a called a morphism from $d_1(f)$ (the source) to $d_0(f)$ (the target). We define the identity morphism of an object x to be $s_0(x)$. For composers, we define a composition of n-composable morphisms to be a choice of an extension to a Δ^n, sometimes also just the restriction to the edge $\Delta^{\{0,n\}} \subseteq \Delta^n$.

The name composer thus comes from the fact that one *can* compose morphisms. To avoid associativity questions, a composer is equipped with an "n-ary" composition law.

Example 2.3. The singular set of a topological space is a composer: Objects are the points, morphisms from x to y are paths. A composition of morphisms is any path which is homotopic relative endpoints to the concatenation of the paths.

Definition 2.4. Let X be a simplicial set. We call two 1-simplices f and g from x to y equivalent if there exists a 2-simplex $\sigma\colon \Delta^2 \to X$ which satisfies the following:

(1) $\sigma_{|\Delta^{\{0,1\}}} = f$,
(2) $\sigma_{|\Delta^{\{0,2\}}} = g$, and
(3) $\sigma_{|\Delta^{\{1,2\}}} = \mathrm{id}_y$

Observation 2.5. This relation is obviously reflexive, but again a priori neither transitive nor symmetric. Can you find a further lifting criterion for a composer so that this becomes in fact an equivalence relation? The inner 3-horn lifting condition.

Lecture 6 – 05.11.2018.

Definition 2.6. Let X be a simplicial set. We define a category hX by means of generators and relations: The objects are given by X_0. Morphisms are generated by X_1, i.e. for every 1-simplex $f\colon \Delta^1 \to X$ there is a morphism from $d_1(f)$ to $d_0(f)$. The free composites will be denoted by $f \star g$. Now we start imposing relations:

(1) The 1-simplex $s_0(x)$ is the identity of x,
(2) for every 2-simplex $\sigma\colon \Delta^2 \to X$ with boundary given by a triple (f, g, h) we impose the relation that $h = g \star f$,
(3) if $f \sim f'$, then $f \star g \sim f' \star g$ and $g' \star f \sim g' \star f'$.

The category hX is called the *homotopy category* of X.

Remark. This construction is obviously functorial, i.e. we have a functor $h\colon \mathrm{sSet} \to \mathrm{Cat}$.

Lemma 2.7. *Let $f\colon X \to Y$ be a map of simplicial sets which induces an isomorphism $\mathrm{sk}_2(X) \to \mathrm{sk}_2(Y)$. Then the induced map $hX \to hY$ is an isomorphism of categories.*

Proof. The whole construction only referred to the 2-skeleton of X. In other words, the evident map $h(\mathrm{sk}_2(X)) \to hX$ is an isomorphism. Then we use that the diagram

$$\begin{tikzcd}
h(\mathrm{sk}_2 X) \arrow[r, "\cong"] \arrow[d, "\cong"] & h(\mathrm{sk}_2 Y) \arrow[d, "\cong"] \\
hX \arrow[r] & hY
\end{tikzcd}$$

commutes. □

Observation 2.8. In general, the morphisms in hX are formal composites of 1-simplices (with correct source and target). If X is a composer, we see that the set of morphisms of hX is a quotient of the set of 1-simplices, and that any two composites of the same two morphisms will be identified. In particular, equivalent morphisms are identified: If we have a 2-simplex

$$\begin{tikzcd}
& y \arrow[dr, "\mathrm{id}_y"] & \\
x \arrow[ur, "f"] \arrow[rr, "g"] & & y
\end{tikzcd}$$

we find that $g \sim \mathrm{id}_y \star f \sim f$.

Lemma 2.9. *Suppose that a composer X has in addition the lifting property with respect to inner 3-horn inclusions. Let f and g be composable 1-simplices in X. Then*

(1) *There exists a composite of f and g,*
(2) *The relation "equivalence" of morphisms in the sense of Definition 2.4 is an equivalence relation,*
(3) *Any two composites of f and g are equivalent in the sense of Definition 2.4, and*
(4) *given a 2-simplex σ with $\sigma_{\Delta^{\{0,1\}}} = \mathrm{id}_x$, $\sigma_{\Delta^{\{1,2\}}} = h$ and $\sigma_{\Delta^{\{0,2\}}} = h'$, then $h' \sim h$.*

Proof. (1) follows from the definition of a composer. For (2), we need to prove symmetry and transitivity. Let us first prove symmetry. So let $f, g \colon x \to y$ be morphisms with $f \sim g$. Pick a 2-simplex σ with $\sigma_{\Delta^{\{0,1\}}} = f$, $\sigma_{\Delta^{\{0,2\}}} = g$, and $\sigma_{\Delta^{\{1,2\}}} = \mathrm{id}_y$. Together with $s_0(f)$ and $s_0(\mathrm{id}_y)$, this determines a map $\Lambda^3_1 \to X$:

$$\begin{tikzcd}
& y & \\
& & y \arrow[ul, "\mathrm{id}_y"'] \arrow[d, "\mathrm{id}_y"] \\
x \arrow[uur, "f"] \arrow[urr, "f"] \arrow[rr, "g"'] & & y \arrow[uul, "\mathrm{id}_y"]
\end{tikzcd}$$

Since X has the extension property for inner 3-horns, there exists an extension to Δ^3, which can be restricted to the face $\Delta^{\{0,2,3\}}$. This 2-simplex witnesses that $g \sim f$.

To show transitivity, suppose that $f \sim g \sim h$. Pick 2-simplices σ and σ' witnessing these relations. Together with $s_0(\mathrm{id}_y)$, these define a map $\Lambda^3_2 \to X$:

$$\begin{tikzcd}
& y & \\
& & y \arrow[ul, "\mathrm{id}_y"'] \arrow[d, "\mathrm{id}_y"] \\
x \arrow[uur, "h"] \arrow[urr, "f"] \arrow[rr, "g"'] & & y \arrow[uul, "\mathrm{id}_y"]
\end{tikzcd}$$

Extending to Δ^3 and restricting then to $\Delta^{\{0,1,3\}}$ shows that $f \sim h$. To prove (3), let $f\colon x \to y$ and $g\colon y \to z$ be composable morphisms. Choosing compositions h and h', together with $s_0(g)$ determines a map $\Lambda^3_1 \to X$:

$$\begin{tikzcd}
& z & \\
& & y \arrow[ul, "g"'] \arrow[d, "g"] \\
x \arrow[uur, "h'"] \arrow[urr, "f"] \arrow[rr, "h"'] & & z \arrow[uul, "\mathrm{id}_y"]
\end{tikzcd}$$

Extending to Δ^3 and then restricting to $\Delta^{\{0,1,3\}}$ shows that $h \sim h'$. To prove (4), we consider the map $\Lambda^3_2 \to X$ given by the diagram

$$\begin{tikzcd}
& y & \\
& & x \arrow[ul, "h"'] \arrow[d, "h"] \\
x \arrow[uur, "h"] \arrow[urr, "\mathrm{id}_x"] \arrow[rr, "h'"'] & & y \arrow[uul, "\mathrm{id}_y"]
\end{tikzcd}$$

and extend to Δ^3. Restricting the result to $\Delta^{\{0,2,3\}}$ shows that $h' \sim h$. $\square$

Lemma 2.10. *Suppose X is a composer with the inner 3-horn extension property. Then there is a category $\pi(X)$ with objects given by 0-simplices of X and morphisms given by equivalence classes (in the sense of Definition 2.4) of 1-simplices in X. Composition is defined via lifting along $I^2 \to \Delta^2$. The uniqueness of composition (up to equivalence) shows that composition in $\pi(X)$ is associative.*

Proof. It remains only to prove that composition is associative: For this, it suffices to show that if f, g, h are composable morphisms, then $h \circ (gf)$ is a composition of hg and f. Consider the map $\Lambda^3_1 \to X$ given by

$$\begin{tikzcd}
 & z & \\
 & & y \arrow[ul, "hg"'] \arrow[d, "g"] \\
x \arrow[uur, "(hg)f"] \arrow[urr, "f"] \arrow[rr, "gf"'] & & z \arrow[uul, "h"']
\end{tikzcd}$$

Extending this to Δ^3 and then restricting to $\Delta^{\{0,2,3\}}$ shows that $(hg)f$ is a composition of gf and h. Since composition in X is unique up to equivalence, associativity of composition in $\pi(X)$ follows. $\square$

Corollary 2.11. *Let X be a composer which furthermore has the inner 3-horn lifting property. Then hX is isomorphic to $\pi(X)$. In particular, for composers with the additional inner 3-horn lifting condition, there is a very explicit description of the homotopy category of X.*

Proof. There is a canonical functor $hX \to \pi(X)$ constructed as follows: It is the identity on objects and induced by the identity on 1-simplices. Since all relations imposed in hX are fulfilled in $\pi(X)$ this in fact descends to a functor as needed. It suffices to prove now that for any two objects $x, y \in X$, the canonical map

$$\mathrm{Hom}_{hX}(x,y) \to \mathrm{Hom}_{\pi(X)}(x,y)$$

is a bijection. To show this, we observe that there is a commutative diagram

$$\begin{tikzcd}
X_1(x,y) \arrow[d, two heads] \arrow[dr, two heads] & \\
\mathrm{Hom}_{hX}(x,y) \arrow[r] & \mathrm{Hom}_{\pi(X)}(x,y)
\end{tikzcd}$$

where $X_1(x,y)$ denotes the set of 1-simplices f with $d_1(f) = x$ and $d_0(f) = y$. Both maps from $X_1(x,y)$ are surjective, thanks to the definition of π and Observation 2.8. Thus the horizontal map is surjective. It thus remains to show that it is injective as well. Hence assume given two morphisms $f, g \in X_1(x,y)$ with the same image in $\pi(X)$. This means that they are equivalent in the sense of Definition 2.4. But again by Observation 2.8, we know that then $f \sim g$. $\square$

Remark. We observe that in the above corollary, X needed not really be a composer: Only a composition of 2 composable morphisms was needed, we will call a simplicial set which has the extension property for the 2-spine (which is the inner 2-horn) a weak composer. To compose many morphisms at the same time, we compose inductively and obtain a well-defined "n-fold composition" up to equivalence (provided the weak composer satsifies the extension property for inner 3-horns as well). Now notice that $I^2 = \Lambda^2_1$. Thus we can reformulate the above as

Corollary 2.12. *Let X be a simplicial set which admits liftings for inner 2- and 3-horns. Then hX is isomorphic to $\pi(X)$.*

Corollary 2.13. *Let $\mathcal{C}$ be a category. Then $h(\mathrm{N}(\mathcal{C}))$ is canonically isomorphic to $\mathcal{C}$.*

Proof. We have seen that $\mathrm{N}(\mathcal{C})$ admits (unique) lifts for many horns, including the ones described in the previous corollary. It hence suffices to prove that $\pi(\mathrm{N}(\mathcal{C})) \cong \mathcal{C}$. But we recall that the relation of "equivalence" for morphisms in $\mathrm{N}(\mathcal{C})$ ist the relation of "being equal". $\square$

Definition 2.14. A simplicial set is called an ∞-category if it has the extension property for *all* inner horn inclusions $\Lambda^n_j \to \Delta^n$, $n \geq 2$, $0 < j < n$.

Definition 2.15. A functor between two ∞-categories is just a map of simplicial sets. I.e. the category of ∞-categories is the full subcategory of sSet on objects which are ∞-categories.

It will take some time, but we will see that, informally, an ∞-category is a composer in which the choice of a composition is unique up to a contractible space of choices: For each pair of composable morphisms in a composer X, there is a simplicial set of compositions of f and g, $\mathrm{Comp}_X(f,g)$, given by the pullback

$$\begin{tikzcd}
\mathrm{Comp}_X(f,g) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^2, X) \arrow[d] \\
\Delta^0 \arrow[r, "{(f,g)}"] & \underline{\mathrm{Hom}}(\Lambda^2_1, X)
\end{tikzcd}.$$

It will be an exercise next week that the inner 3-horn lifting condition tells us that $\mathrm{Comp}_X(f,g)$ is connected (i.e. its $\pi_0^\Delta(-)$ vanishes). If we now demand that X in fact has the extension property for inner horn inclusions for all $\Lambda^n_j \to \Delta^n$, we obtain the following two facts (which we will prove later!)

(1) The simplicial set $\mathrm{Comp}_X(f,g)$ is a Kan complex,
(2) *all* simplicial homotopy groups $\pi_n^\Delta(\mathrm{Comp}_X(f,g))$ vanish.

Thus, an ∞-category is a composer in which composition is well-defined up to a contractible space of choices.

Example 2.16. A Kan complex, and thus the singular set $\mathcal{S}(X)$ of a space X is an ∞-category. Also, the nerve of any category is an ∞-category.

To make sure that we were not just smart enough we state the following proposition. The proof will be given later, see Proposition 3.17 and Proposition 3.10.

Proposition 2.17. *Every ∞-category is a composer. However, there exist composers which are not ∞-categories.*

Proposition 2.18. *The pair of functors*

$$h\colon \mathrm{sSet} \rightleftarrows \mathrm{Cat}\colon \mathrm{N}$$

are adjoint, N *being the right adjoint to h.*

Proof. We need to specify unit and counit transformations: The counit is the isomorphism

$$h(\mathrm{N}(\mathcal{C})) \cong \mathcal{C}$$

of Corollary 2.13. To construct a natural map $X \to \mathrm{N}(hX)$ (the unit of the adjunction) we find that there are canonical maps on 0 and 1-simplices: Recall that the objects of hX are the 0-simplices of X and that there is a canonical map from X_1 to the morphisms of hX (this map is a surjection if X is a composer). To construct the map on general n-simplices, we observe that an n-simplex $\Delta^n \to \mathrm{N}(hX)$ is uniquely determined by its restriction to the spine, in other words consider the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \arrow[r, dashed] \arrow[d] & \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{N}(hX)) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathrm{sSet}}(I^n, X) \arrow[r] & \mathrm{Hom}_{\mathrm{sSet}}(I^n, \mathrm{N}(hX)) \end{tikzcd}$$

But then simply use the bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(I^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

and the corresponding one for $\mathrm{N}(hX)$ and the previous observation to obtain a map for general n-simplices. Checking that these are in fact unit and counit of an adjunction is easy.

Alternatively, you can also show that the map induced by h and the above counit

$$\mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{N}(\mathcal{C})) \to \mathrm{Fun}(hX, \mathcal{C})$$

is a bijection. This follows immediately from the definitions. $\square$

Lecture 7 – 8.11.2018.

Lemma 2.19. *Let $(F, G, \varepsilon, \eta)$ be an adjunction with $F\colon \mathcal{C} \to \mathcal{D}$ the left adjoint, G the right adjoint, $\varepsilon\colon FG \to \mathrm{id}$ the counit and $\eta\colon \mathrm{id} \to GF$ the unit. Then*

(1) *G is fully faithful if and only if ε is an isomorphism,*
(2) *F is fully faithful if and only if η is an isomorphism.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{Hom}(X,Y) \arrow[r, "G"] \arrow[rrd, bend right, "\varepsilon_X^*"'] & \mathrm{Hom}(GX,GY) \arrow[r, "F"] \arrow[rd, "\tau", "\cong"'] & \mathrm{Hom}(FGX,FGY) \arrow[d, "(\varepsilon_Y)_*"] \\ & & \mathrm{Hom}(FGX,Y) \end{tikzcd}$$

The right triangle commutes by one of the first exercises (this is how adjunctions with a binatural isomorphism τ are translated into adjunctions using unit and counits). We now claim that the big diagram also commutes: Spelling this out, we need to see that for every morphism $f\colon X \to Y$, the diagram

$$\begin{tikzcd} FGX \arrow[r, "FG(f)"] \arrow[d, "\varepsilon_X"] & FGY \arrow[d, "\varepsilon_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. This is true since $\varepsilon\colon FG \to \mathrm{id}$ is a natural transformation of functors. Hence we see that G is fully faithful if and only if for all $X, Y \in \mathcal{D}$, the map ε_X^* is an isomorphism. By the Yoneda lemma, this is the case if and only if ε_X itself is an isomorphism for all X, which shows claim (1). The argument for (2) is similar. $\square$

Corollary 2.20. *The nerve functor is in addition fully faithful. Its essential image is described by Proposition 1.54.*

Proof. We proved in Proposition 2.18 that N is a right adjoint to h by constructing explicit unit and counit maps. Thus, by Lemma 2.19 it suffices to see that the counit of this adjunction is an isomorphism. But by construction, the counit is given by the canonical isomorphism $h(\mathrm{N}(\mathcal{C})) \to \mathcal{C}$. ☐

Let us record also the following consequence, which was a previous exercise:

Corollary 2.21. *The nerve of a category is 2-coskeletal.*

Proof. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(hX, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{N}(\mathcal{C})) \arrow[d] \arrow[dr] & \\
\mathrm{Fun}(h(\mathrm{sk}_2 X), \mathcal{C}) \arrow[r] & \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{sk}_2 X, \mathrm{N}(\mathcal{C})) \arrow[r] & \mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{cosk}_2 \mathrm{N}(\mathcal{C}))
\end{tikzcd}$$

where the diagonal arrow is induced by the canonical map $\mathrm{N}(\mathcal{C}) \to \mathrm{cosk}_2\mathrm{N}(\mathcal{C})$. By Proposition 2.18, the two left horizontal maps are bijections, and by Lemma 2.7 the remaining horizontal arrow is a bijection as well. Thus the vertical map is a bijection if and only if the diagonal map is a bijection. Since X is arbitrary, the claim follows from Yoneda's lemma. ☐

Definition 2.22. A morphism in an ∞-category is called an equivalence if its image in the homotopy category is an isomorphism.

Lemma 2.23. *A morphism $f\colon x \to y$ in an ∞-category $\mathcal{C}$ is an equivalence if and only if there exist 2-simplices $\sigma^l\colon \Delta^2 \to \mathcal{C}$ and $\sigma^r\colon \Delta^2 \to \mathcal{C}$ such that*

$$\sigma^l_{|\Lambda^2_0} = (f, \mathrm{id}) \text{ and } \sigma^r_{|\Lambda^2_2} = (f, \mathrm{id}).$$

ff Draw pictures here:

Proof. If a 2-simplex σ^l exists, then $\sigma^l_{|\Delta^{\{1,2\}}}$ is a left inverse of the image of f in $h\mathcal{C}$. Similarly, $\sigma^r_{|\Delta^{\{0,1\}}}$ is a right inverse of the image of f in $h\mathcal{C}$. For the converse, suppose that the image of f is an equivalence in $h\mathcal{C}$. This means that there exists a 1-simplex $g\colon y \to x$ such that $[fg]$ and $[gf]$ are the identity in $h\mathcal{C}$, i.e. that there is a 2-simplex η which witnesses that h is a composite of f and g, and that there is a further 2-simplex η' which witnesses that h is equivalent to the identity. We can use these two 2-simplices (plus an degenerate 2-simplex on g) to obtain a map $\Lambda^3_2 \to \mathcal{C}$ which can be filled because $\mathcal{C}$ is an ∞-category:

$$\begin{tikzcd}
& x & \\
& & y \arrow[ul, "g"'] \arrow[d, "g"] \\
x \arrow[uur, "\mathrm{id}_x"] \arrow[urr, "f"] \arrow[rr, "h"'] & & x \arrow[uul, "\mathrm{id}_x"']
\end{tikzcd}$$

Restricting the resulting 3-simplex to the 2-simplex $\Delta^{\{0,1,3\}}$ produces a 2-simplex σ^l. The argument for σ^r is analogous. ☐

Definition 2.24. An ∞-category is called ∞-groupoid if every morphism is an equivalence.

Definition 2.25. The maximal sub-groupoid of an ordinary category is the subcategory consisting of all isomorphisms. We denote this by $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$. For an ∞-category $\mathcal{C}$, we define the maximal sub-∞-groupoid to be the pullback

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{N}(h\mathcal{C}^{\simeq}) \arrow[r] & \mathrm{N}(h\mathcal{C}) \end{tikzcd}$$

Remark. For an ordinary category $\mathcal{C}$, we have that $\mathrm{N}(\mathcal{C}^{\simeq}) = \mathrm{N}(\mathcal{C})^{\simeq}$. This is because, by definition there is a pullback

$$\begin{tikzcd} \mathrm{N}(\mathcal{C})^{\simeq} \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \arrow[d] \\ \mathrm{N}(h(\mathrm{N}(\mathcal{C}))^{\simeq}) \arrow[r] \arrow[d, "\cong"] & \mathrm{N}(h(\mathrm{N}(\mathcal{C}))) \arrow[d, "\cong"] \\ \mathrm{N}(\mathcal{C}^{\simeq}) \arrow[r] & \mathrm{N}(\mathcal{C}) \end{tikzcd}$$

where the right vertical map composite is the identity, and the lower vertical maps are induced by the canonical isomorphism $h(\mathrm{N}(\mathcal{C})) \cong \mathcal{C}$.

Lemma 2.26. *An n-simplex x of an ∞-category $\mathcal{C}$ belongs to the maximal sub-∞-groupoid if and only if all edges are equivalences.*

Proof. It suffices to observe that an n-simplex in $\mathrm{N}(h\mathcal{C})$ is determined by its restriction to all edges, and that this n-simplex lies in the $\mathrm{N}(h\mathcal{C}^{\simeq})$ if and only if all edges are isomorphisms in $h\mathcal{C}$. $\square$

Corollary 2.27. *The maximal sub-∞-groupoid of an ∞-category is in fact an ∞-groupoid, and it is the largest such which sits inside the given ∞-category.*

Proof. Let us first prove that $\mathcal{C}^{\simeq}$ is an ∞-category. For this, consider a lifting problem

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \subseteq \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

for $0 < j < n$. Since $\mathcal{C}$ is an ∞-category, this problem can be solved in $\mathcal{C}$. The claim follows if we can prove that if we are given a map $\Delta^n \to \mathcal{C}$ which induces a map

$$\Lambda^n_j \to \Delta^n \to \mathcal{C} \to \mathrm{N}(h\mathcal{C})$$

having image contained in $\mathrm{N}(h\mathcal{C}^{\simeq})$, then already the map $\Delta^n \to \mathrm{N}(h\mathcal{C})$ factors through $\mathrm{N}(h\mathcal{C}^{\simeq})$. For this we again recall that an n-simplex of the nerve of a category is determined by its restriction to the spine. Since the spine includes in the inner horns Λ^n_j for all n, the claim follows. This formalizes that $\mathcal{C}^{\simeq}$ is the sub-∞-category consisting of the equivalences of $\mathcal{C}$. In particular, it follows that $h(\mathcal{C}^{\simeq}) = (h\mathcal{C})^{\simeq}$, so that $\mathcal{C}^{\simeq}$ is an ∞-groupoid. It is then clear that $\mathcal{C}^{\simeq}$ is the largest such sitting inside $\mathcal{C}$. $\square$

Lemma 2.28. *A Kan complex X is an ∞-groupoid.*

Proof. It suffices to show that for every morphism in X, there is a left and a right inverse σ^l and σ^r as in Lemma 2.23. But since X is a Kan complex, the maps $(f, \mathrm{id})\colon \Lambda^2_0 \to X$ and $(f, \mathrm{id})\colon \Lambda^2_2 \to X$ can be filled to 2-simplices σ^l and σ^r. $\square$

The converse is also true – this is a non-trivial and very important theorem in higher categories. Proving it will be what we aim at next. But before that, we will spend some time on more examples of ∞-categories, i.e. how can we produce ∞-categories.

Definition 2.29. Let V be a category. A monoidal structure on V consists of the following data:

(1) A functor $- \otimes - \colon V \times V \to V$, called the monoidal product,
(2) a unit object $\mathbb{1} \in V$, together with natural isomorphisms $\eta_l\colon X \to \mathbb{1} \otimes X$, and $\eta_r\colon X \to X \otimes \mathbb{1}$, called *left unit* and *right unit*, and
(3) natural associativity isomorphisms $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$.

These data are required to satisfy some axioms...

Example 2.30. Given a category $\mathcal{C}$ which admits finite products, then there is a cartesian monoidal structure given by the product bifunctor $(X, Y) \mapsto X \times Y$. The unit is given by the terminal object (a product over the empty set). Dually, a category with finite coproducts admits a cocartesian monoidal structure with $(X, Y) \mapsto X \amalg Y$. The unit is given by the initial object (the coproduct over the empty set).

Explicit examples we will care about are

(1) The category $(\mathrm{Set}, \times, *)$,
(2) the category $(\mathrm{Cat}, \times, [0])$, and
(3) the category $(\mathrm{sSet}, \times, \Delta^0)$.

Definition 2.31. Let $(V, \otimes_V, \mathbb{1}_V)$ and $(W, \otimes_W, \mathbb{1}_W)$ be monoidal categories. A lax monoidal functor consists of the following data

(1) a functor $F\colon V \to W$,
(2) a natural map $\mathbb{1}_W \to F(\mathbb{1}_V)$, and
(3) a natural map $FX \otimes_W FY \to F(X \otimes_V Y)$.

Dually, an oplax monoidal functor consists of the following data

(1) a functor $F\colon V \to W$,
(2) a natural map $F(\mathbb{1}_V) \to \mathbb{1}_W$, and
(3) a natural map $F(X \otimes_V Y) \to FX \otimes_W FY$.

The structure morphisms have to satisfy compatibility with respect to the associativity isomorphisms und left/right unit isomorphisms. Precisely, the following diagrams are required to commute:

$$\begin{tikzcd}
(FX \otimes FY) \otimes FZ \arrow[r] \arrow[d, "\alpha_W"'] & F(X \otimes Y) \otimes FZ \arrow[d] \\
FX \otimes (FY \otimes FZ) \arrow[d] & F((X \otimes Y) \otimes Z) \arrow[d, "F(\alpha_V)"] \\
FX \otimes F(Y \otimes Z) \arrow[r] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

where α_W and α_V are the associativity isomorphism in $(W, \otimes, \mathbb{1})$ and α_V and $(V, \otimes, \mathbb{1})$. Likewise for the units:

$$\begin{tikzcd} FX \arrow[r, "F(\eta_r^V)"] \arrow[d, "\eta_r^W"] & F(X \otimes \mathbb{1}) \\ FX \otimes \mathbb{1} \arrow[r] & FX \otimes F(\mathbb{1}) \arrow[u] \end{tikzcd}$$

where η_r^V and η_r^W are the right unit isomorphisms of V and W. Similarly, the diagram involving left units is required to commute as well.

A lax monoidal (or oplax monoidal) functor is called monoidal if the natural maps of (2) and (3) are isomorphisms.

Lecture 8 – 12.11.2018.

Definition 2.32. Let F and G be lax monoidal functors between monoidal categories V and W. A natural transformation $\tau\colon F \to G$ is called lax monoidal if for all $X, Y \in V$, the diagrams

$$\begin{tikzcd} FX \otimes FY \arrow[r] \arrow[d] & F(X \otimes Y) \arrow[d] \\ GX \otimes GY \arrow[r] & G(X \otimes Y) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{1} \arrow[d] \arrow[dr] & \\ F(\mathbb{1}) \arrow[r] & G(\mathbb{1}) \end{tikzcd}$$

commute. We let $\mathrm{Fun}^{\mathrm{lax}}(V, W)$ be the category whose objects are the lax monoidal functors and whose morphisms are lax monoidal transformations. We let

$$\mathrm{Fun}^{\otimes}(V, W) \subseteq \mathrm{Fun}^{\mathrm{lax}}(V, W)$$

be the full subcategory on monoidal functors.

Remark. We thus see that the category MonCat of monoidal categories with monoidal functors is canonically a 2-category: The Hom-category between V and W is given by $\mathrm{Fun}^{\mathrm{lax}}(V, W)$. Of course, in order for this to make sense, we need to observe that the identity of a monoidal category is canonically lax monoidal (in fact monoidal) and that the composition of two lax monoidal functors is canonically lax monoidal.

Definition 2.33. Let $(V, \otimes, \mathbb{1})$ be a monoidal category. Then a V-enriched category $\mathcal{C}$ consists of a set of objects, and for any two object $x, y \in \mathcal{C}$ an object $\mathrm{Hom}_{\mathcal{C}}(x, y) \in V$, together with "composition functors"

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \otimes \mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

and furthermore for every object an "identity" id_x

$$\mathbb{1} \to \mathrm{Hom}_{\mathcal{C}}(x, x)$$

satisfying the obvious associativity and unitality conditions, namely that the following diagrams are required to commute:

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{C}}(x,y)\otimes \mathrm{Hom}_{\mathcal{C}}(y,z))\otimes \mathrm{Hom}_{\mathcal{C}}(z,w) \arrow[r] \arrow[dd, "\alpha"] & \mathrm{Hom}_{\mathcal{C}}(x,z)\otimes \mathrm{Hom}_{\mathcal{C}}(z,w) \arrow[d] \\
& \mathrm{Hom}_{\mathcal{C}}(x,w) \\
\mathrm{Hom}_{\mathcal{C}}(x,y)\otimes (\mathrm{Hom}_{\mathcal{C}}(y,z)\otimes \mathrm{Hom}_{\mathcal{C}}(z,w)) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(x,y)\otimes \mathrm{Hom}_{\mathcal{C}}(y,w) \arrow[u]
\end{tikzcd}$$

where α denotes the associativity isomorphism of $\mathcal{V}$. Furthermore

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r, "\eta_r"] \arrow[dr, "\mathrm{id}"] & \mathrm{Hom}_{\mathcal{C}}(x,y)\otimes \mathbb{1} \arrow[d] \\
& \mathrm{Hom}_{\mathcal{C}}(x,y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[r, "\eta_l"] \arrow[dr] & \mathbb{1}\otimes \mathrm{Hom}_{\mathcal{C}}(x,y) \arrow[d] \\
& \mathrm{Hom}_{\mathcal{C}}(x,y)
\end{tikzcd}$$

Definition 2.34. A V-enriched functor between V-enriched categories $f\colon \mathcal{C} \to \mathcal{D}$ consists of a map on objects $x \mapsto f(x)$, and for each two objects a morphism $f_{x,y}\colon \mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(fx,fy)$ in V such that the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(x,y)\otimes \mathrm{Hom}_{\mathcal{C}}(y,z) \arrow[r] \arrow[d, "f_{x,y}\otimes f_{y,z}"'] & \mathrm{Hom}_{\mathcal{C}}(x,z) \arrow[d, "f_{x,z}"] \\
\mathrm{Hom}_{\mathcal{D}}(fx,fy)\otimes \mathrm{Hom}_{\mathcal{D}}(fy,fz) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(fx,fz)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_{fx}"'] & \mathrm{Hom}_{\mathcal{C}}(x,x) \arrow[d, "f_{x,x}"] \\
& \mathrm{Hom}_{\mathcal{D}}(fx,fx)
\end{tikzcd}$$

commute.

Definition 2.35. Let V be a monoidal category. Then we let Cat_V be the category of V-enriched categories, that is: objects are V-enriched categories and morphisms are V-enriched functors.

Example 2.36. A Set-enriched category is just an ordinary category. A Cat-enriched category is a 2-category.

Example 2.37. A Cat-enriched category with one object is the same datum as a monoidal category which is strictly associative (i.e. where the associativity isomorphisms are the identity). An enriched functor between two such categories is the same datum as a monoidal functor.

Lemma 2.38. *If $\Phi\colon V \to V'$ is a lax monoidal functor between monoidal categories, then applying this functor to each hom-object produces a functor $\Phi_*\colon \mathrm{Cat}_V \to \mathrm{Cat}_{V'}$. This construction in fact determines a 2-functor*

$$\mathrm{MonCat} \to \mathrm{Cat}$$

from the 2-category of monoidal categories to the 2-category of categories. In particular, a monoidal adjunction between V and V' determines an adjunction on the level of enriched categories.

Proof. To construct this 2-functor, we consider the map on the level of objects first: It takes a monoidal category V to the category Cat_V of V-enriched categories. To make this a 2-functor we need to construct for every pair V, W of monoidal categories a *functor*

$$\mathrm{Fun}^{\mathrm{lax}}(V, W) \to \mathrm{Fun}(\mathrm{Cat}_V, \mathrm{Cat}_W)$$

and then show that this construction is compatible with composition. To construct this functor, again we first consider its effect on objects: Given a lax monoidal functor $\Phi\colon V \to W$, and a V-enriched category $\mathcal{C}$, we consider the following W-enriched category $\Phi_*(\mathcal{C})$: The objects are the same as the objects of $\mathcal{C}$, and for $X, Y \in \mathrm{ob}(\mathcal{C})$ we define

$$\mathrm{Hom}_{\Phi_*(\mathcal{C})}(X, Y) = \Phi(\mathrm{Hom}_{\mathcal{C}}(X, Y)).$$

It is straight forward to show that $\Phi_*(\mathcal{C})$ is a W-enriched category: For instance, to show that composition satisfies the associativity constraint, one uses the compatibility of Φ with the associativity isomorphisms of V and W. Next, we need to explain the effect on morphisms. So let $\tau\colon \Phi \to \Psi$ be a monoidal transformation. We wish to construct a natural transformation between Φ_* and Ψ_*. Concretely, we need to construct natural maps $\Phi_*(\mathcal{C}) \to \Psi_*(\mathcal{C})$ in Cat_W, i.e. natural W-enriched functors $\tau_*\colon \Phi_*(\mathcal{C}) \to \Psi_*(\mathcal{C})$. On objects this functor is defined to be the identity, and on morphisms between X and Y we have to construct a map

$$\tau_*\colon \Phi(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \to \Psi(\mathrm{Hom}_{\mathcal{C}}(X, Y))$$

and we simply use the natural map $\tau_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$ given by the natural transformation $\tau\colon \Phi \to \Psi$. To see that this is compatible with composition, we use that τ is a monoidal transformation, so that the diagram

$$\begin{tikzcd}
\Phi(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \otimes \Phi(\mathrm{Hom}_{\mathcal{C}}(Y,Z)) \arrow[r] \arrow[d] & \Phi(\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z)) \arrow[d] \\
\Psi(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \otimes \Psi(\mathrm{Hom}_{\mathcal{C}}(Y,Z)) \arrow[r] & \Psi(\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z))
\end{tikzcd}$$

commutes. Then we use naturality of τ to see that also the diagram

$$\begin{tikzcd}
\Phi(\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z)) \arrow[r] \arrow[d] & \Phi(\mathrm{Hom}_{\mathcal{C}}(X,Z)) \arrow[d] \\
\Psi(\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z)) \arrow[r] & \Psi(\mathrm{Hom}_{\mathcal{C}}(X,Z))
\end{tikzcd}$$

commutes. Glueing these two diagrams together shows that the map τ_* is compatible with composition, and thus is in fact a W-enriched functor $\Phi_*(\mathcal{C}) \to \Psi_*(\mathcal{C})$ as needed.

It is then also clear by definition that for two composable lax monoidal functors Ψ and Φ, we have $\Psi_*(\Phi_*(\mathcal{C})) = (\Psi \circ \Phi)_*(\mathcal{C})$, so that compatibility with composition is immediate.

To see the in particular is now easy: An monoidal adjunction consists of lax monoidal functors Φ and Ψ and unit and counit transformations which are itself monoidal transformations. By the previously established parts, these are sent to functors Φ_* and Ψ_* equipped with candidates for the unit and counit. The only thing to check is the snake identities, but they follow from the fact that they hold for Φ and Ψ, and that the constructed functor preserves identities. $\square$

Definition 2.39. Let V be a monoidal category and $\mathcal{C}$ a V-enriched category. Then Its *underlying category* $u\mathcal{C}$ is obtained via the lax monoidal functor $\mathrm{Hom}(\mathbb{1}, -)\colon V \to \mathrm{Set}$. In formulas, we have

$$u\mathcal{C} = \mathrm{Hom}_V(\mathbb{1}, -)_*(\mathcal{C}) \in \mathrm{Cat}_{\mathrm{Set}} = \mathrm{Cat}.$$

Definition 2.40. We will call a category enriched in simplicial sets simply a simplicial category. Here sSet is viewed as a monoidal category via the cartesian product. We will write Cat_Δ for $\mathrm{Cat}_{\mathrm{sSet}}$.

Remark. There is some ambiguity in the above definition: Usually a simplicial category would rather refer to a simplicial object in categories, i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{Cat}$. Luckily we have the following lemma about this:

Lemma 2.41. *There is a canonical fully faithful embedding* $\mathrm{Cat}_\Delta \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ *determined by the family of functors* $(\mathrm{ev}_n)_*\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}$*, and the essential image can be characterized as those simplicial objects in categories, whose underlying simplicial set of objects is constant.*

Lemma 2.42. *Suppose that* $F\colon \mathcal{C} \to \mathcal{D}$ *is left adjoint to* $G\colon \mathcal{D} \to \mathcal{C}$*. Suppose furthermore that*

(1) $\mathcal{C}$ *is cocomplete, and*
(2) G *is fully faithful.*

Then $\mathcal{D}$ *is cocomplete as well.*

Dually, if $\mathcal{D}$ *is complete, and* F *is fully faithful, then* $\mathcal{C}$ *is complete as well.*

Proof. Assume that (1) and (2) hold. Let $X\colon I \to \mathcal{D}$ be a diagram and consider the object $F(\mathrm{colim}_I\, G(X_i))$ which exists since $\mathcal{C}$ is cocomplete. We wish to prove that it satisfies the universal property of a colimit:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(F(\operatorname*{colim}_I G(X_i)), Y) &\cong \mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_I G(X_i), G(Y)) \\ &\cong \lim_I \mathrm{Hom}_{\mathcal{C}}(G(X_i), G(Y)) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(X_i, Y)\end{aligned}$$

where the last bijection holds by fully faithfulness of G.

Dually, let $X\colon I \to \mathcal{C}$ be a diagram and consider the object $G(\lim_I F(X_i))$ which exists since $\mathcal{D}$ is complete. Then we calculate that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(Y, G(\lim_I F(X_i))) &\cong \mathrm{Hom}_{\mathcal{D}}(F(Y), \lim_I F(X_i)) \\ &\cong \lim_I \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X_i)) \cong \lim_I \mathrm{Hom}_{\mathcal{C}}(Y, X_i)\end{aligned}$$

where the last bijection uses that F is fully faithful. $\square$

Lecture 9 – 15.11.2018.

Proposition 2.43. *The category* Cat *is bicomplete, and the functor* $\mathrm{ob}(-)\colon \mathrm{Cat} \to \mathrm{Set}$ *preserves limits and colimits.*

Proof. The case of limits can be done by hand: Given a diagram $\mathcal{C}\colon I \to \mathrm{Cat}$, sending $i \in I$ to $\mathcal{C}_i$, we define its limit $\lim_I \mathcal{C}_i$ as follows:

(1) $\mathrm{ob}(\lim_I \mathcal{C}_i) = \lim_I \mathrm{ob}(\mathcal{C}_i)$, and
(2) for any two objects $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ we have
$$\mathrm{Hom}_{\lim_I \mathcal{C}_i}(\{X_i\},\{Y_i\}) = \lim_I \mathrm{Hom}_{\mathcal{C}_i}(X_i, Y_i).$$

To see that this is in fact a category, one uses that limits commute with products (when defining composition) and that one has a canonical map from a limit indexed over $I \times I$ to the limit indexed over the restriction to I along the diagonal. This is a reason why the situation is more complicated with colimits. By definition then, the functor $\mathrm{ob}(-)\colon \mathrm{Cat} \to \mathrm{Set}$ commutes with limits (as it must, since it is right adjoint to the discrete category functor $d\colon \mathrm{Set} \to \mathrm{Cat}$).

To prove existence of colimits we make use of Lemma 2.42: We consider the adjunction (h, N) of functors between sSet and Cat of Proposition 2.18. We have seen that

(1) sSet is cocomplete (in fact bicomplete by Corollary 1.26), and
(2) $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful, Corollary 2.20.

It follows that Cat is cocomplete and that a colimit of a diagram $\mathcal{C}\colon I \to \mathrm{Cat}$ is given by $h(\mathrm{colim}_I \mathrm{N}(\mathcal{C}_i))$. With this we calculate its objects as follows:
$$\begin{aligned}\mathrm{ob}(\operatorname*{colim}_I \mathcal{C}_i) &\cong \mathrm{ob}(h(\operatorname*{colim}_I \mathrm{N}(\mathcal{C}_i)))\\ &\cong (\operatorname*{colim}_I \mathrm{N}(\mathcal{C}_i))_0 \cong \operatorname*{colim}_I(\mathrm{N}(\mathcal{C}_i)_0)\\ &\cong \operatorname*{colim}_I \mathrm{ob}(\mathcal{C}_i)\end{aligned}$$
as needed. $\square$

Lemma 2.44. *Let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory of a category $\mathcal{C}$ and let $X\colon I \to \mathcal{C}'$ be a diagram. If X has a (co)limit in $\mathcal{C}$ which happens to lie in $\mathcal{C}'$, then this is also a (co)limit in $\mathcal{C}'$.*

Proof. Obvious from the universal property and the fact that the inclusion is full. $\square$

Corollary 2.45. *The category* Cat_Δ *is bicomplete.*

Proof. Consider an I-shaped diagram of simplicial categories $\mathcal{C}_i$. By Lemma 2.41 this gives rise to an I-shaped diagram in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Cat})$, whose associated I-shaped diagram of simplicial sets of objects in constant, i.e. where for all $i \in I$, we have that $\mathrm{ob}(\mathcal{C}_i)$ is a constant simplicial set. We wish to show that then also $\mathrm{ob}(\mathrm{colim}_I \mathcal{C}_i)$ is a constant simplicial set. By Proposition 2.43, we have
$$\mathrm{ob}(\operatorname*{colim}_I \mathcal{C}_i) \cong \operatorname*{colim}_I \mathrm{ob}(\mathcal{C}_i)$$
and the latter is a colimit in simplicial sets, over constant simplicial sets. Since the constant functor $c\colon \mathrm{Set} \to \mathrm{sSet}$ admits a right adjoint (ev_0) it preserves colimits. Thus the colimit over constant simplicial sets is itself constant (on the colimit of the sets involved). The argument for limits works the same. $\square$

Remark. Nothing is special about Δ here. In fact the same argument goes through to prove that for any small category $\mathcal{C}$, there is a fully faithful inclusion $\mathrm{Cat}_{\mathcal{P}(\mathcal{C})} \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$ with essential image given by those functors whose presheaf of objects is constant. It follows completely analogously that $\mathrm{Cat}_{\mathcal{P}(\mathcal{C})}$ is bicomplete. Here, we always use the pointwise (cartesian) monoidal structure on $\mathcal{P}(\mathcal{C})$.

Example 2.46. The nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ is monoidal: It preserves products since it is a right adjoint. Hence, given a 2-category, we obtain a simplicial category by applying the nerve functor to all hom-categories. Furthermore, for a 2-category $\mathcal{C}$ we have $u(\mathrm{N}_*(\mathcal{C})) = u\mathcal{C}$: This is simply because

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, -)_* \circ \mathrm{N}_* = (\mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, -) \circ \mathrm{N}) = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, \mathrm{N}(-)) = \mathrm{Hom}_{\mathrm{Cat}}(h\Delta^0, -)$$

and the $h\Delta^0$ is the unit of the monoidal structure on Cat given by cartesian product.

Lemma 2.47. *The functors $c\colon \mathrm{Set} \to \mathrm{sSet}$ and $\pi_0, \mathrm{ev}_0\colon \mathrm{sSet} \to \mathrm{Set}$ are canonically monoidal.*

Proof. We claim that every functor $F\colon (V, \times, *) \to (W, \times, *)$ is canonically oplax monoidal. The oplax monoidal structure maps are given by the maps

(1) $F(*) \to *$, the unique map to the terminal object of W, and
(2) $F(X \times Y) \to F(X) \times F(Y)$ given by the effect of F on the two projections

$$X \leftarrow X \times Y \to Y.$$

It hence suffices to see that the canonical oplax structure maps in our examples are isomorphisms. For c and ev_0 this follows directly from the definitions. Only the functor π_0 requires an actual argument. We want to check that the map

$$\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$$

is a bijection. By definition of π_0 of a simplicial set, Definition 1.10, we have a commutative square

$$\begin{tikzcd}
(X \times Y)_0 \arrow[r, "\cong"] \arrow[d, two heads] & X_0 \times Y_0 \arrow[d, two heads] \\
\pi_0(X \times Y) \arrow[r] & \pi_0(X) \times \pi_0(Y)
\end{tikzcd}$$

in which the vertical maps are bijections. Since the top horizontal map is a bijection (this is the monoidality of ev_0), it follows that the oplax monoidal structure map is surjective. To see that it is injective, it suffices to check that generators of the relations can be lifted. Since these are given by 1-simplices, and ev_1 is also monoidal, the claim follows. Also it follows directly that the map $\pi_0(*) \to *$ is an isomorphism. $\square$

Definition 2.48. We obtain a functors

(1) $c = c_*\colon \mathrm{Cat} \to \mathrm{Cat}_\Delta$, which sends a category to the simplicially enriched category with constant simplicial enrichment,
(2) $\pi = (\pi_0)_*\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}$, called the homotopy category of a simplicial category, and
(3) $u = (\mathrm{ev}_0)_*\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}$, called the underlying category.

Notice that $\mathrm{ev}_0 = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^0, -)$ so that this is the same underlying category as in Definition 2.39.

Definition 2.49. A simplicial functor $f\colon \mathcal{C} \to \mathcal{D}$ between simplicial categories is called a weak equivalence if it induces

(1) a weak equivalence on all hom simplicial sets, (weakly fully faithful) and
(2) an essentially surjective functor $\pi(\mathcal{C}) \to \pi(\mathcal{D})$ (weakly essentially surjective).

Lemma 2.50. *Every functor between cartesian monoidal categories is canonically oplax monoidal. Every natural transformation between two such functors is also canonically oplax monoidal. In particular, the adjunctions* (π_0, c) *and* (c, ev_0) *are monoidal adjunctions.*

Proof. We have seen that every functor $F\colon (V, \times, *) \to (W, \times, *)$ is canonically oplax monoidal in the proof of Lemma 2.47. So let $\tau\colon F \to G$ be a natural transformation. We wish to show that the diagram

$$\begin{tikzcd} F(X\times Y) \arrow[r] \arrow[d, "\tau_{X\times Y}"] & F(X)\times F(Y) \arrow[d, "\tau_X\times\tau_Y"] \\ G(X\times Y) \arrow[r] & G(X)\times G(Y) \end{tikzcd}$$

commutes. For this it suffices to see that each of the two following squares commutes

$$\begin{tikzcd} F(X) \arrow[d, "\tau_X"] & F(X\times Y) \arrow[l] \arrow[r] \arrow[d, "\tau_{X\times Y}"] & F(Y) \arrow[d, "\tau_Y"] \\ G(X) & G(X\times Y) \arrow[l] \arrow[r] & G(Y) \end{tikzcd}$$

which follows from naturality of τ.

It now suffices to show that an oplax monoidal transformation between monoidal functors is also a monoidal transformation. This follows from the general fact that if a square commutes, in which both vertical maps are isomorphisms, then the square with the inverse vertical maps also commutes. □

Corollary 2.51. *The two functors* $c\colon \mathrm{Cat} \to \mathrm{Cat}_\Delta$ *and* $\pi\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}$ *form an adjoint pair, with* π *being left adjoint to the constant functor* c*. Similarly, the two functors* $u\colon \mathrm{Cat}_\Delta \to \mathrm{Cat}$ *and* $c\colon \mathrm{Cat} \to \mathrm{Cat}_\Delta$ *form an adjoint pair with* c *being left adjoint to the underlying functor.*

Proof. This follows from Lemma 2.38 and Lemma 2.50. □

Observation 2.52. The adjunction gives rise to a canonical functor $\mathcal{C} \to c\pi(\mathcal{C})$ of simplicial categories.

Lecture 10 – 19.11.2018.

Definition 2.53. Given a simplicial category $\mathcal{C}$, and two objects $x, y \in \mathcal{C}$, we say that a morphism from x to y is a morphism in the underlying category $u\mathcal{C}$. In other words, it is a 0-simplex of $\mathrm{Hom}_{\mathcal{C}}(x, y)$. Such a morphism is called an equivalence if its image in $\pi(\mathcal{C})$ is an isomorphism.

We wish to extend the notion of the nerve of a category to simplicially enriched categories. For this we need a version of the category $[n]$ which is well suited for simplicially enriched categories.

Definition 2.54. Let J be a finite non-empty linearly ordered set, and let $i, j \in J$ be elements. We let $P_{i,j}$ be the following set of subsets of J:

$$P_{i,j} = \{I \subseteq J : i, j \in I \text{ and } k \in I \Rightarrow i \leq k \leq j\}$$

In words, $P_{i,j}$ consists of all subsets of $[i, j] \subseteq J$ which contain i and j.

$P_{i,j}$ is partially ordered by inclusion: $I \leq I' \Leftrightarrow I \subseteq I'$. Notice that $P_{i,j}$ is only non-empty if $i \leq j$.

Observation 2.55. Given a triple $i \leq j \leq k$ in J, there is a canonical map of partially ordered sets

$$P_{i,j} \times P_{j,k} \to P_{i,k}$$

given by sending (I, I') to $I \cup I'$. This clearly defines an associative binary operation.

As a consequence we obtain the following:

Definition 2.56. Let J be a non-empty linearly ordered set. Then the following defines a simplicially enriched category $\mathfrak{C}[\Delta^J] \in \mathrm{Cat}_\Delta$. Objects of $\mathfrak{C}[J]$ are given by the elements of J. Furthermore

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i, j) = \begin{cases} \emptyset & \text{if } j < i \\ \mathrm{N}(P_{i,j}) & \text{if } i \leq j. \end{cases}$$

Composition is defined via the previous observation.

Lemma 2.57. *For every $n \geq 1$, we have that $\mathrm{N}(P_{0,n})$ is isomorphic to $(\Delta^1)^{n-1}$. Furthermore, $P_{i,j} \cong P_{0,j-i}$.*

Proof. The furthermore simply follows by choosing an isomorphism $J \cong [n]$ for some n, and then considering the unique order preserving isomorphism of $[i, j]$ with $[0, j - i]$. To show that $\mathrm{N}(P_{0,n}) \cong (\Delta^1)^{n-1}$ it suffices to find an isomorphism of posets

$$P_{0,n} \cong [1] \times \cdots \times [1]$$

where the latter product is has $n - 1$ many factors. This simply comes from the following construction: A subset $I \subseteq [n]$ containing $\{0, n\}$ is determined by which of the elements $1, \ldots, n - 1$ is contained in I. We label an element contained in I with a 1 and elements not contained in I with a 0. This constructs a map of posets $P_{0,n} \to [1] \times \cdots \times [1]$, which is clearly an isomorphism. □

Lemma 2.58. *Let $\mathcal{C}$ be a category with initial or terminal object. Then $\mathrm{N}(\mathcal{C})$ is contractible, i.e. its identity map is homotopic to the constant map at the initial or terminal object.*

Proof. Let us consider the case where $\mathcal{C}$ has an initial object $\emptyset$. The other case follows from the fact that a simplicial set X is contractible if and only if X^{op} is contractible. We claim that the identity functor of $\mathcal{C}$ admits a natural transformation from the constant functor with value $\emptyset$, simply given on an object $X \in \mathcal{C}$ by the unique map $\emptyset \to X$. The relevant diagrams commute by the uniqueness of the maps. We obtain a functor $\mathcal{C} \times [1] \to \mathcal{C}$ whose restriction to 0 is the constant map at $\emptyset$ and whose restriction to 1 is the identity of $\mathcal{C}$. Applying the nerve functor N, we obtain a simplicial homotopy

$$\mathrm{N}(\mathcal{C}) \times \Delta^1 \to \mathrm{N}(\mathcal{C})$$

from the constant map at the vertex given by $\emptyset$ to the identity of $\mathrm{N}(\mathcal{C})$. □

Corollary 2.59. *For $i \leq j$, the simplicial set $\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j)$ is contractible.*

Proof. We have just seen that $\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \cong \Delta^{j-i-1} = \mathrm{N}([1] \times \cdots \times [1])$. The category $[1] \times \cdots \times [1]$ has both an initial and a terminal object, so we can appeal to Lemma 2.58. □

Remark. The functor $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ preserves products. To see this, it suffices to see that $|\Delta^n \times \Delta^m| \cong \Delta^n_{\mathrm{top}} \times \Delta^m_{\mathrm{top}}$. This is a concrete calculation. Hence, a simplicial homotopy induces a homotopy of geometric realizations. It follows that every contractible simplicial set is weakly contractible (i.e. its geometric realization is contractible).

Lemma 2.60. *There is a unique isomorphism $\pi(\mathfrak{C}[\Delta^n]) \cong [n]$ which is the identity on objects. By adjunction we obtain a canonical functor $\mathfrak{C}[\Delta^n] \to c[n]$, and this functor is a weak equivalence of simplicial categories.*

Proof. Everything follows from Corollary 2.59: It calculates $\pi(\mathfrak{C}[\Delta^n])$ to be $[n]$. It follows that the induced functor $\mathfrak{C}[\Delta^n] \to c[n]$ is bijective on objects (and thus weakly essentially surjective) and a weak equivalence on hom simplicial sets. □

Remark. ADJUST THIS REMARK. In fact it is a *cofibrant replacement* in a suitable model structure on simplicial categories. In the former category we have dropped the strict associativity for composition $i \to j \to k$ in $[n]$, now have instead a (contractible) space of compositions.

Lemma 2.61. *The association $J \mapsto \mathfrak{C}[\Delta^J]$ extends to a functor*

$$\mathrm{Lin.or.Set} \to \mathrm{Cat}_\Delta.$$

In particular, we obtain a functor

$$\Delta \to \mathrm{Cat}_\Delta \ [n] \mapsto \mathfrak{C}[\Delta^n]$$

which is a cosimplicial object in simplicially enriched categories.

Proof. We need to show that every map $J \to J'$ of linearly ordered sets induces a simplicially enriched functor $\mathfrak{C}[\Delta^J] \to \mathfrak{C}[\Delta^{J'}]$. For this it suffices to explain how such a map $J \to J'$ induces, for every $i \leq j$ in J a map of posets $P_{i,j} \to P_{f(i),f(j)}$. This is simply given by sending a subset I to its image $f(I)$. It is easy to see that this in fact produces a functor. □

Definition 2.62. Let $\mathcal{C}$ be a simplicially enriched category. Then we define its simplicial nerve (or the homotopy coherent nerve) as the following simplicial set

$$\mathrm{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Lemma 2.63. *If $\mathcal{C}$ is an ordinary category, viewed as a simplicially enriched category $c\mathcal{C}$ via the constant functor, then there is an isomorphism*

$$\mathrm{N}(\mathcal{C}) \cong \mathrm{N}(c\mathcal{C}).$$

Proof. This is because:

$$\begin{aligned} \mathrm{N}(c\mathcal{C})_n &= \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], c\mathcal{C}) \\ &\cong \mathrm{Hom}_{\mathrm{Cat}}(\pi(\mathfrak{C}[\Delta^n]), \mathcal{C}) \\ &\cong \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) = \mathrm{N}(\mathcal{C})_n \end{aligned}$$

The only thing that needs justification is the isomorphism $\pi(\mathfrak{C}[\Delta^n]) \cong [n]$ which we proved in Lemma 2.60. □

Remark. The analog of such a statement with $\pi(\mathcal{C})$ and $u\mathcal{C}$ is wrong!!

Lecture 11 – 22.11.2018.

Observation 2.64. Let $\mathcal{C}$ be a simplicially enriched category. Let us investigate the simplicial set $\mathrm{N}(\mathcal{C})$ in more detail: We recall that

$$\mathrm{N}(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

Unravelling the simplicial categories $\mathfrak{C}[\Delta^0]$ and $\mathfrak{C}[\Delta^1]$ we find the following

(1) $\mathfrak{C}[\Delta^0]$ is a simplicial category with a single object, and its endomorphism simplicial set is given by $\mathrm{N}(P_{0,0})$ which is also given by Δ^0.
(2) $\mathfrak{C}[\Delta^1]$ is a simplicial category with two objects, 0, and 1, and all morphism simplicial sets are given by Δ^0.

In other words, both $\mathfrak{C}[\Delta^0]$ and $\mathfrak{C}[\Delta^1]$ are given by $c[0]$ and $c[1]$, where we view [0] and [1] as categories, and then apply the constant functor $c\colon \mathrm{Cat} \to \mathrm{Cat}_\Delta$. As a consequence we obtain

(1) $\mathrm{N}(\mathcal{C})_0 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^0], \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}([0], u\mathcal{C})$, and
(2) $\mathrm{N}(\mathcal{C})_1 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^1], \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}([1], u\mathcal{C})$,

by of Corollary 2.51. In words, objects of $\mathrm{N}(\mathcal{C})$ are given by the objects of $\mathcal{C}$, and morphisms of $\mathrm{N}(\mathcal{C})$ are given by the morphisms (i.e. the 0-simplices of the mapping simplicial sets) of $\mathcal{C}$.

Let us go one step further and analyze the simplicial category $\mathfrak{C}[\Delta^2]$. Its objects are given by $\{0, 1, 2\}$. All endomorphism simplicial sets are given by Δ^0. Furthermore

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^2]}(0, 1) = \Delta^0 = \mathrm{Hom}_{\mathfrak{C}[\Delta^2]}(1, 2).$$

However, to analyze the simplicial set of morphism from 0 to 2 we have to investigate $\mathrm{N}(P_{0,2})$. By definition, $P_{0,2}$ is the partially ordered set of subsets of $\{0 < 1 < 2\}$ which contain 0 and 2. There are precisely two such subsets, so that we obtain $P_{0,2} = [1]$. In particular,

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^2]}(0, 2) = \mathrm{N}(P_{0,2}) = \Delta^1.$$

Unravelling, we find that a 2-simplex in $\mathrm{N}(\mathcal{C})$ consists of the following data: Objects X, Y, and Z (associated to the three objects 0, 1, and 2). A morphism $f\colon X \to Y$ (associated to the unique morphism from 0 to 1 in $\mathfrak{C}[\Delta^2]$), and a morphism $g\colon Y \to Z$ (associated to the unique morphism from 1 to 2 in $\mathfrak{C}[\Delta^2]$), a 1-simplex $\Delta^1 \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ (associated to the mapping simplicial set between 0 and 2 in $\mathfrak{C}[\Delta^2]$) whose restriction to 0 is given by gf and whose restriction to 1 is given by some other morphism.

Informally, a 2-simplex thus consists of the data of two composable morphisms $X \to Y$ and $Y \to Z$, a further morphism $X \to Z$ and a homotopy between the composite of the first two to the last morphism.

Lemma 2.65. *The category* Cat_Δ *of simplicially enriched categories admits all small colimits. Hence there exists a unique colimit preserving functor*

$$\mathfrak{C}[-]\colon \mathrm{sSet} \to \mathrm{Cat}_\Delta$$

which sends Δ^n *to* $\mathfrak{C}[\Delta^n]$. *This functor is automatically left adjoint to the simplicial nerve functor.*

Proof. The existence of colimits was done in Corollary 2.45. The rest is formal, we have seen the argument several times. ∎

Fact 2.66. Given a simplicial set X, consider subsimplicial sets $A_i \subseteq X$. Then also the union $A = \bigcup A_i$ is a sub simplicial set of X. In this situation we have that $\mathfrak{C}[A]$ is the sub simplicial category of $\mathfrak{C}[X]$ generated by the $\mathfrak{C}[A_i]$. Recall that a sub simplicial category of a simplicial category $\mathcal{C}$ is determined by a subset of the objects, and for any two such, a sub simplicial set of the hom simplicial set in $\mathcal{C}$.

This fact is for instance shown in [Joy07, Corollary 1.15]. It builds fundamentally on the fact that the functor $\mathfrak{C}[-]\colon \mathrm{sSet} \to \mathrm{Cat}_\Delta$ sends monomorphisms of simplicial sets to monomorphisms of simplicial categories.

Lemma 2.67. *Let $0 < j < n$ and consider the horn Λ^n_j. We have that $\mathfrak{C}[\Lambda^n_j]$ is the sub simplicial category of $\mathfrak{C}[\Delta^n]$ with:*

1. *The objects of $\mathfrak{C}[\Lambda^n_j]$ are given by the vertices of Λ^n_j, thus by all objects of $\mathfrak{C}[\Delta^n]$.*
2. *The morphism simplicial sets are given as follows:*
$$\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(i,k) = \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,k)$$
unless $(i,k) = (0,n)$, and
$$\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \subseteq \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n) = \mathrm{N}(P_{0,n})$$
is given by the sub simplicial set of $(\Delta^1)^{n-1}$ obtained by deleting the interior and the bottom j-face.

Proof. Let us first show that the candidate for $\mathfrak{C}[\Lambda^n_j]$ is in fact a sub simplicial category of $\mathfrak{C}[\Delta^n]$. We know that $\Lambda^n_j = \bigcup\limits_{i\neq j} \Delta^{[n]\setminus\{i\}}$. So let us first describe the sub simplicial category
$$\mathfrak{C}[\Delta^{[n]\setminus\{i\}}] \subseteq \mathfrak{C}[\Delta^n].$$
The objects of this subcategory are given by all objects except for the object corresponding to $i \in [n]$. Now assume $k,l \in [n] \setminus \{i\}$ with $k \leq l$. If $l < i$, then the there is an obvious isomorphism
$$P^{[n]\setminus\{i\}}_{k,l} \cong P^{[n]}_{k,l}$$
where the superscript indicates in which linearly ordered set to perform the construction $P_{k,l}$ of Definition 2.54. Thus we obtain an equality of simplicial hom sets
$$\mathrm{Hom}_{\mathfrak{C}[\Delta^{[n]\setminus\{i\}}]}(k,l) = \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(k,l).$$
Likewise there is such an isomorphism if $k > i$.

Now let $(k,l) \neq (0,n)$, i.e. either $k \neq 0$ or $l \neq n$. Assume first that $k \neq 0$. Then we have the following chain on inclusions of simplicial sets
$$\mathrm{Hom}_{\mathfrak{C}[\Delta^{[n]\setminus\{0\}}]}(k,l) \subseteq \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(k,l) \subseteq \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(k,l)$$
in which the composition is an equality by the previous arguments. We thus see that in fact both inclusions must be equalities. If $l \neq n$, the same argument works. We summarize that
$$\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(k,l) = \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(k,l)$$
unless $(k,l) = (0,n)$.

Let us next determine $\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n)$. By Fact 2.66 we have for every $i \neq j$ and $i \neq 0,n$ inclusions as follows:
$$\mathrm{Hom}_{\mathfrak{C}[\Delta^{[n]\setminus\{i\}}]}(0,n) \subseteq \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \subseteq \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n).$$

For $0 < i < n$ we find that

$$P_{0,n}^{[n]\setminus\{i\}} \subseteq P_{0,n}^{[n]}$$

is a sub poset, consisting precisely of those $I \in P_{k,l}^{[n]}$ for which $i \notin I$. In other words, it is given by the map of posets

$$[1]^{\times(n-2)} \to [1]^{\times(n-1)}$$

which includes a 0 at the i^{th} spot of $[1]^{\times(n-1)}$. Upon taking the nerve, this gives the map $(\Delta^1)^{n-2} \subseteq (\Delta^1)^{n-1}$ which is the zero vertex of Δ^1 in the i^{th} coordinate.

It follows that for every $i \neq j$, the face $(\Delta^1)^{n-2} \subseteq (\Delta^1)^{n-1}$ where the i^{th} entry is 0 is contained in $\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n)$. We call the face where the i^{th} entry is 0 the *bottom i-face*. We find that a priori, the bottom j-face is not contained in $\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n)$ as promised in the statement of the lemma. It remains to show that the top k-face is contained in $\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n)$ for all $0 < k < n$. For this we consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,k) \times \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(k,n) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \arrow[d] \\
\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,k) \times \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(k,n) \arrow[r] & \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n)
\end{tikzcd}$$

which encodes the composition in the respective categories. This diagram must commute, as $\mathfrak{C}[\Lambda^n_j]$ is a sub simplicial category of $\mathfrak{C}[\Delta^n]$. Since the left vertical map is an isomorphism it hence suffices to show that the top i-face are contained in the image of the lower horizontal map. This map is induced by the map of posets

$$P_{0,k} \times P_{k,n} \to P_{0,n}$$

which is induced by sending (I, I') to $I \cup I'$. Since I and I' contain the element k, we see that after the identification of these posets with cubes as in Lemma 2.57, we obtain the map

$$(\Delta^1)^{n-2} \cong (\Delta^1)^{k-1} \times (\Delta^1)^{n-k-1} \to (\Delta^1)^{n-1}$$

which is induced by inserting a 1 in the k^{th} slot. This shows that the top k-face is contained in $\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n)$. Since we already know that this punctured cube gives rise to a sub simplicial category, and $\mathfrak{C}[\Lambda^n_j]$ is contained in it, and itself contains the sub simplicial categories determined by the i-faces of Δ^n for $i \neq j$, the lemma is shown. $\square$

Lecture 12 – 26.11.2018.

Lemma 2.68. *The coherent nerve of a simplicial category $\mathcal{C}$ is a composer. Furthermore, it is an ∞-category if all hom-simplicial sets are Kan complexes, i.e. if it is in fact enriched in the symmetric monoidal category of Kan complexes.*

Proof. We consider an extension problem

$$\begin{tikzcd}
I^n \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

which, by adjunction is equivalent to the extension problem

$$\begin{tikzcd} \mathfrak{C}[I^n] \arrow[r] \arrow[d] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

and to show that the latter can be solved it suffices to show that the map $\mathfrak{C}[I^n] \to \mathfrak{C}[\Delta^n]$ has a retraction. To see this, we first claim that $\mathfrak{C}[I^n] \cong c[n]$. This follows simply from the fact that it is true for $n = 1$, and induction, using that $I^n \cong I^{n-1} \amalg_{\Delta^0} I^1$, and the fact that the functor $c\colon \mathrm{Cat} \to \mathrm{Cat}_\Delta$ preserves colimits. In Lemma 2.60 we have seen that there is a unique simplicial functor $\mathfrak{C}[\Delta^n] \to c[n]$ which is the identity on objects. It follows that the composition

$$c[n] \cong \mathfrak{C}[I^n] \to \mathfrak{C}[\Delta^n] \to c[n]$$

is a functor which is the identity on objects, and therefore is an isomorphism.

To show that, in the case of a Kan enrichment, $\mathrm{N}(\mathcal{C})$ is an ∞-category, we need to solve an extension problem

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

for $0 < j < n$.

By adjunction, we need to argue why every simplicially enriched functor $\mathfrak{C}[\Lambda^n_j] \to \mathcal{C}$ extends to a simplicially enriched functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$ provided $0 < j < n$. For this we use our analysis of $\mathfrak{C}[\Lambda^n_j]$ of Lemma 2.67.

To solve the extension problem we are interested in, it thus suffices to prove that there exists an extension in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \arrow[r, "f"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(f(0), f(n)) \\ \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

It follows from the description of Lemma 2.67 that the vertical map is an anodyne map of simplicial sets, i.e. it induces a weak equivalence on geometric realizations, so by Fact 1.50 the dotted arrow exists. It now suffices to show that this construction in fact produces a simplicial functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$. To see this, one uses that $\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,0) = \Delta^0 = \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(n,n)$ so that no new relations for functoriality are to check: Since the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,k) \times \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(k,n) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \arrow[d] \\ \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,k) \times \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(k,n) \arrow[r] & \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n) \end{tikzcd}$$

commutes, any composition which factors through the object k with $0 < k < n$, is already contained in the sub simplicial category $\mathfrak{C}[\Lambda^n_j]$. $\square$

Observation 2.69. We find that a 2-category gives rise to a simplicial category, so that its coherent nerve is a composer. Moreover, a $(2,1)$-category gives rise to a simplicial category where all hom simplicial sets are nerves of groupoids and thus Kan complexes by Lemma 1.56. In particular, a $(2,1)$-category gives rise to an ∞-category in our sense.

Definition 2.70. Consider the simplicial category with objects CW-complexes and mapping simplicial sets given by the singular set of the mapping space. (Notice that the singular set commutes with products). The simplicial nerve of this category gives an ∞-category which we will call the ∞-category of spaces and denote by An.

Observation 2.71. Objects of An are CW complexes, and morphism are given by points in the space $\mathrm{map}(X, Y)$, i.e. by a continuous map from X to Y. The homotopy category is what one would expect: Morphisms are homotopy classes of maps.

Remark. We would like to have a "purely simplicial" model of the ∞-category of spaces (what a perverse thing to say – but it comes from the fact that we wish to think of Kan complexes/spaces as ∞-groupoids and later want to have spaces and ∞-categories on equal footing), i.e. where we directly construct a simplicial category whose objects are Kan complexes. For this we will need to show that for a simplicial set K and a Kan complex X, the simplicial set of maps $\underline{\mathrm{Hom}}(K, X)$ is again a Kan complex. The things needed to show this are also needed on the way of showing that ∞-groupoids are Kan complexes, and we will start to develop these tools in the next section.

Lemma 2.72. *The product of ∞-categories is an ∞-category. The coproduct of ∞-categories is an ∞-category.*

Proof. For products, one can solve the extension problem in every ∞-category individually. This provides an extension for the product. For coproducts, we observe that both Λ^n_j and Δ^n are connected. Thus an extension problem for a coproduct of ∞-categories is in fact an extension problem for a single one. $\square$

Definition 2.73. A sub-∞-category $\mathcal{C}'$ of an ∞-category $\mathcal{C}$ is a sub-simplicial set determined by a subset $X \subseteq \mathcal{C}_0$ of 0-simplices and a subset $S \subseteq \mathcal{C}_1$ of 1-simplices between objects lying in X, such that S contains identities and is closed under compositions and equivalences. Then an n-simplex of $\mathcal{C}$ belongs to $\mathcal{C}'$ if and only its restriction to the spine I^n has its edges contained in S. A subcategory is called full if $S = \mathcal{C}_1$.

Lecture 13 – 29.11.2018.

Lemma 2.74. *A sub-∞-category of an ∞-category is itself an ∞-category. Its homotopy category is the subcategory of $h\mathcal{C}$ on the image of the morphisms lying in S. The diagram*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathrm{N}(h\mathcal{C}') \arrow[r] & \mathrm{N}(h\mathcal{C}) \end{tikzcd}$$

is a pullback. Furthermore, for any subcategory $\mathcal{C}_0 \subseteq h\mathcal{C}$ of the homotopy category, this pullback defines a sub-∞-category of $\mathcal{C}$ with X and S given by the preimage of objects and morphisms along the canonical map $\mathcal{C} \to \mathrm{N}(h\mathcal{C})$.

Proof. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D} \subseteq h\mathcal{C}$ be a subcategory of its homotopy category. Let $\mathcal{C}'$ be the pullback $\mathrm{N}(\mathcal{D}) \times_{\mathrm{N}(h\mathcal{C})} \mathcal{C}$. As pullbacks preserve monomorphisms, $\mathcal{C}'$ is a subsimplicial set of $\mathcal{C}$. Let us spell out what it means for an n-simplex of $\mathcal{C}$ to lie in $\mathcal{C}'$: It means precisely that the induced n-simplex of $\mathrm{N}(h\mathcal{C})$ lies in $\mathrm{N}(\mathcal{D})$. This is the case if and only if the spine, and thus every edge of the simplex lies in $\mathcal{D}$. Now observe, that a subset $S \subseteq \mathcal{C}_1$ which contains all identities of objects in X and is closed under compositions is in fact the preimage of the 1-morphisms of the nerve of a subcategory of $h\mathcal{C}$. Notice that in the above definition, we have that $(\mathcal{C}')_1 = S$. $\square$

Corollary 2.75. *There is a bijective correspondence between sub-∞-categories of $\mathcal{C}$ and subcategories of $h\mathcal{C}$ induced by taking the homotopy category. Full sub-∞-categories of $\mathcal{C}$ correspond precisely to full subcategories of $h\mathcal{C}$.*

Definition 2.76. A natural transformation between two functors $f, g\colon \mathcal{C} \to \mathcal{D}$ is a simplicial map $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ which restricts to the given functors appropriately.

Observation 2.77. We observe that functors and natural transformations of functors between X and Y are precisely the 0- and 1-simplices of the hom-simplicial set $\underline{\mathrm{Hom}}(X, Y)$. As in ordinary category theory we would like to have an ∞-category of functors between two ∞-categories. Also, one would expect that one has $\mathrm{N}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) = \mathrm{Fun}(\mathrm{N}\mathcal{C}, \mathrm{N}\mathcal{D})$. Since one also has $\mathrm{N}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) = \underline{\mathrm{Hom}}(\mathrm{N}\mathcal{C}, \mathrm{N}\mathcal{D})$ (use that the nerve functor is fully faithful) one might hope that the hom-simplicial set is again an ∞-category. This will turn out to be true, and like in the case of Kan complexes is not a triviality. It will be the objective of the next lectures to prove (amongst other things) this fact.

3. Anodyne maps and fibrations

Definition 3.1. A map of simplicial sets $X \to Y$ is an (inner, left, right) fibration if it satisfies the right lifting property (RLP) with respect to (inner, left, right) horn inclusions

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Definition 3.2. A map of simplicial sets $A \to B$ is an (inner, left, right) anodyne map, if it satisfies the left lifting property (LLP) with respect to (inner, left, right) fibrations

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Definition 3.3. Let S be a set of morphisms in a category $\mathcal{C}$. We let $\chi_R(S)$ be the set of morphisms having the RLP wrt to S and $\chi_L(S)$ be the set of morphisms having the LLP wrt S. We let $\chi(S) = \chi_L(\chi_R(S))$, i.e. $\chi(S)$ is the set of morphisms which have the LLP wrt to morphisms having the RLP wrt S.

Example 3.4. Let S be the set of (inner, left, right) horn inclusions. Then the (inner, left, right) fibrations are given by $\chi_R(S)$, and the (inner, left, right) anodyne maps are given by $\chi(S)$.

Definition 3.5. A saturated set of morphisms is a set of morphisms T which is closed under taking pushouts (along arbitrary maps), arbitrary coproducts, countable compositions (i.e. colimits along $\mathbb{N}$), and retracts. Given an arbitrary set of morphisms S, we call the smallest saturated set containing S the *saturated closure* of S and denote it by $\overline{S}$.

Remark. Notice that the intersection of saturated sets is again saturated. Thus to see that the saturated closure exists, it suffices to show that there is a saturated set containing S. One can for instance take simply all morphisms: It is obviously saturated and contains S.

Definition 3.6. A morphism $f\colon A \to B$ is a retract of a morphism $f'\colon A' \to B'$ if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & B \end{tikzcd}$$

Remark. Let us make more precise what the conditions of Definition 3.5 mean: Being closed under arbitrary coproducts means that given a family $\{f_i\}_{i\in I}$ such that each f_i is an element of T, then also $\coprod\limits_{i\in I} f_i$ is an element of T. Being closed under pushouts means that given a map $f\colon A \to B$ which belongs to T and any other map $\varphi\colon A \to A'$, then in the following pushout diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

also the map f' belongs to T. Being closed under countable compositions means the following: Consider the category $\mathbb{N}$ (the category associated to the poset $\mathbb{N}$) and consider a functor $X\colon \mathbb{N} \to \mathrm{sSet}$. A set S is closed under countable compositions if the following condition holds. Suppose that for every $i \in \mathbb{N}$, the canonical map

$$X(i) \to X(i+1)$$

is contained is S, then also the map $X(0) \to \operatorname{colim}_{\mathbb{N}} X$ is contained in S. Being closed under retracts means that for f and f' as in Definition 3.6, if f' is an element of T, the so is f.

Example 3.7. The set of monomorphisms in sSet is a saturated set. A further description of this set will be worked out in **??**, namely that the monomorphisms are given by $\chi(\{\partial\Delta^n \to \Delta^n\}_{n\geq 0})$.

Lemma 3.8. *Given a set of morphisms S, we have that $\chi_L(S)$ is a saturated set. In particular $\chi(S)$ is a saturated set.*

Proof. Let $\alpha\colon A \to B$ be a morphism in $\chi_L(S)$ and let $\varphi\colon A \to A'$ be an arbitrary morphism. Consider the pushout $B' = A' \amalg_A B$. We wish to show that the canonical map $A' \to B'$ is contained in $\chi_L(S)$. So let $f\colon X \to Y$ be a map in S and consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[urr, dotted] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where we wish to construct the dashed map. We certainly have the dotted map, and hence obtain the dashed map by the universal property of a pushout. Likewise, suppose that $A \to B$ is a retract of $A' \to B'$ and that $A' \to B'$ is contained in $\chi_L(S)$. To show that then also $A \to B$ is contained in $\chi_L(S)$ we consider a map $f\colon X \to Y$ in S and a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & B' \arrow[r] \arrow[urr, dotted] & B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

and we wish to show that the dashed arrow exists. Again, the dotted arrow exists, which we may restrict to B along the map $B \to B'$.

Suppose now given a family $\{f_i\colon A_i \to B_i\}_{i\in I}$ of elements of $\chi_L(S)$. We then want to show that also $\coprod\limits_{i\in I} f_i \in \chi_L(S)$. So consider a lifting problem

$$\begin{tikzcd} \coprod\limits_{i\in I} A_i \arrow[r] \arrow[d, "\coprod\limits_{i\in I} f_i"'] & X \arrow[d, "\varphi"] \\ \coprod\limits_{i\in I} B_i \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with $\varphi\colon X \to Y$ being an element of S. Then the dashed arrow exists simply by the universal property of coproducts and the assumption that each f_i is an element of $\chi_L(S)$.

It remains to show that for any diagram $A\colon \mathbb{N} \to \mathcal{C}$ where each map $A_i \to A_{i+1}$ is contained in $\chi_L(S)$, then also the map $A_0 \to A = \operatorname{colim}_i X_i$ is contained in $\chi_L(S)$. This follows simply from the universal property of colimits: We consider again $f\colon X \to Y$ in S and a diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & X \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where we need to show existence of the dashed arrow. This is a map out of a colimit, so it suffices to show that there exists compatible maps $A_i \to X$ making everything commute. This can be done inductively using that each map $A_i \to A_{i+1}$ is contained in $\chi_L(S)$. $\square$

The following is known as the small object argument and is a very useful tool to construct non-trivial factorizations of maps.

Proposition 3.9. *Let S be a set of morphisms $\{A_i \to B_i\}_{i\in I}$ such that for every $i \in I$, the simplicial set A_i has only finitely many non-degenerate simplices. If f is an arbitrary morphism, then f can be factored as first a map contained in $\overline{S}$ followed by a map in $\chi_R(S)$, i.e. one which has the RLP wrt S.*

Proof. Consider first the set Θ_S which consists of triples (α_i, u_i, v_i) where $\alpha_i \colon A_i \to B_i$ is an element of S, and where $u_i \colon A_i \to X$ and $v_i \colon B_i \to Y$ are maps such that the diagram

$$\begin{tikzcd} A_i \arrow[r, "u_i"] \arrow[d, "\alpha_i"] & X \arrow[d, "f"] \\ B_i \arrow[r, "v_i"] & Y \end{tikzcd}$$

commutes. We obtain a commutative diagram

$$\begin{tikzcd} \coprod_{\Theta_S} A_i \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\ \coprod_{\Theta_S} B_i \arrow[r] \arrow[drr, bend right] & E^1(f) \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

where $E^1(f)$ is defined to be the pushout. Since $\overline{S}$ is closed under arbitrary coproducts and pushouts, we deduce that the map $X \to E^1(f)$ is contained in $\overline{S}$.

Doing the same with the map $f \colon X \to Y$ replaced by the map $E^1(f) \to Y$, we obtain a sequence

$$X \to E^1(f) \to E^2(f) \to \cdots \to Y.$$

Let us define $E^\omega(f) = \operatorname{colim}_k E^k(f)$ so that we obtain a factorization

$$X \to E^\omega(f) \to Y.$$

By construction every map $E^k(f) \to E^{k+1}(f)$ is contained in $\overline{S}$ because $\overline{S}$ is closed under pushouts and coproducts. Hence, since $\overline{S}$ is saturated, also the map $X \to E^\omega(f)$ is contained in $\overline{S}$. It remains to show that the map $E^\omega(f) \to Y$ is contained in $\chi_R(S)$. So let us consider a map $A_i \to B_i$ in S and a diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & E^\omega(f) \arrow[d] \\ B_i \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where we wish to show the existence of the dashed arrow, making both triangles commute. We now claim that the canonical map

$$\operatorname*{colim}_{k \in \mathbb{N}} \operatorname{Hom}_{\mathrm{sSet}}(A_i, E^k(f)) \to \operatorname{Hom}_{\mathrm{sSet}}(A_i, E^\omega(f))$$

is a bijection (one calls such A_i *compact*). This follows from the fact that there are only finitely many non-degenerate simplices in A_i, and that any simplicial map is determined on the non-degenerate simplices. Hence, we find a $k \in \mathbb{N}$, such that the given map $A_i \to E^\omega(f)$ factors as a composition

$$A_i \to E^k(f) \to E^\omega(f).$$

Now the diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & E^k(f) \arrow[r] & E^\omega(f) \arrow[d] \\ B_i \arrow[rr] & & Y \end{tikzcd}$$

commutes, and thus by the very definition of $E^{k+1}(f)$, there exists a commutative diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & E^k(f) \arrow[r] \arrow[d] & E^\omega(f) \arrow[d] \\ B_i \arrow[r] & E^{k+1}(f) \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

which solves our lifting problem. □

Remark. In all of our situations, the simplicial sets A_i for a small object argument will have only finitely many non-degenerate simplices. In general, we would have to find a regular cardinal κ which is larger that the cardinality of any of the A_i's appearing in S (this ensures that all A_i are κ-compact). Then we could continue the above inductive process: for successor ordinals consider pushouts as before, and for limit ordinals take a colimit as before. At some point one has defined $E^\kappa(f)$ which sits in a factorization $X \to E^\kappa(f) \to Y$, and a similar argument as before will show that the first map is contained in $\overline{S}$ (if one defines saturated sets as being closed under colimits over arbitrary ordinals (as opposed to ω) and that the latter map is contained in $\chi_R(S)$. We chose to not deal with colimits over ordinals in this text since we will not need it, but it is of course useful to know that the small object argument does not depend on such size issues.

Lecture 14 – 03.12.2018.

Remark. The factorization described above is functorial: Whenever given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

The small object argument in fact provides a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E^\omega(f) \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & E^\omega(f') \arrow[r] & Y' \end{tikzcd}$$

With the small object argument we can now give a brut-force proof of Proposition 2.17.

Proposition 3.10. *The saturated set generated by spine inclusions is not equal to the inner anodyne maps. Moreover, there exists a composer which is not an ∞-category.*

Proof. Consider the inclusion $\Lambda^3_1 \to \Delta^3$. By the small object argument we can factor this map as a compostion

$$\Lambda^3_1 \to X \to \Delta^3$$

such that the first map is in the saturated class generated by the spine inclusions, and the last map satisfies RLP wrt spine inclusions, and hence wrt the saturated set generated by the spine inclusions. It follows that X is a composer because also the map $\Delta^3 \to \Delta^0$ has the RLP wrt spine inclusions. If the saturated set generated by the spine inclusions equals the inner anodyne maps, it would follow that the map $X \to \Delta^3$ is an inner fibration (as it has the RLP wrt inner anodyne maps) and we will show that this is not the case.

We claim that the lifting problem

$$\begin{tikzcd} \Lambda^3_1 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^3 \arrow[r, equal] \arrow[ur, dashed] & \Delta^3 \end{tikzcd}$$

does not have a solution. One proves this via induction over the filtration on X obtained from the small object argument using the following observation. Given a pushout

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & A \arrow[d] \\ \Delta^n \arrow[r] & B \end{tikzcd}$$

of simplicial sets with $n \geq 3$, it follows that the image of $\Lambda^n_j \subseteq \Delta^n$ in B is not contained in A.

If X is an ∞-category, then the map $X \to \Delta^3$ were an inner fibration, see Exercise 52, but we have shown it is not. □

Remark. The proposition also follows from a different argument: Combining Exercise 44 and Lemma 2.68 we see that there exists a composer X which is not an infinity category. Now the map $X \to \Delta^0$ is contained in $\chi_R(\{I^n \to \Delta^n\}_{n\in\mathbb{N}})$ but is not an inner fibration. This shows that the saturated closure of the spine inclusions cannot be given by the inner anodyne maps, as else any map that satisfies the RLP with respect to the spine inclusions were also an inner fibration.

Notice that the sheer fact that the saturated closure of the spine inclusions is not equal to the inner anodyne maps does not formally imply that there exists a composer which is not an ∞-category. However, the converse holds: The existence of a composer which is not an ∞-category shows that the saturated set generated by spine inclusions can not contain all inner anodyne maps.

Remark. We will show in Proposition 3.17 that spine inclusions are inner anodyne, so that Proposition 3.10 can be restated as saying that the saturated set generated by spine inclusions is strictly contained in the inner anodyne maps.

Lemma 3.11. *Consider a set of morphisms $S = \{A_i \to B_i\}_{i\in I}$ such that all A_i have only finitely many non-degenerate simplices. Then the saturated closure $\overline{S}$ of S is given by $\chi(S)$.*

Proof. Obviously $S \subseteq \chi(S)$. In Lemma 3.8 we have shown that $\chi(S)$ is itself saturated so $\overline{S} \subseteq \chi(S)$. To prove the converse, consider a map $f\colon x \to y$ with $f \in \chi(S)$. By the small object argument Proposition 3.9 we find a factorization of this map

$$\begin{tikzcd} x \arrow[r, "j"] \arrow[d, "f"] & z \arrow[d, "p"] \\ y \arrow[r, equal] \arrow[ur, dashed, "\alpha"] & y \end{tikzcd}$$

where $j \in \overline{S}$ and where p satisfies the RLP wrt S. Since $f \in \chi(S)$ it follows that there exists a dashed arrow α making the diagram commute. We hence have a commutative diagram

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, "f"] & x \arrow[r, equal] \arrow[d, "j"] & x \arrow[d, "f"] \\ y \arrow[r, "\alpha"] & z \arrow[r, "p"] & y \end{tikzcd}$$

which shows that f is a retract of j. Since $j \in \overline{S}$, so is f. $\square$

Corollary 3.12. *(Inner, left, right) anodyne maps precisely the saturated closure of the (inner, left, right) horn inclusions. In particular, all of these are monomorphisms.*

We have used this already, but let us again state the following fact.

Fact 3.13. A monomorphism is anodyne if and only if its geometric realization is a weak equivalence. (This is part of the existence of the Kan–Quillen model structure on simplicial sets, this is the one which is equivalent to the Quillen model structure on topological spaces).

The following corollary is the relative version of Lemma 2.68.

Corollary 3.14. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial categories. Assume that for all objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$ is a Kan fibration. Then $\mathrm{N}(F)\colon \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$ is an inner fibration.*

Corollary 3.15. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then the induced functor $\mathrm{N}(F)\colon \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$ is an inner fibration.*

Proof. Given any map between sets $A \to B$, then the induced map of constant simplicial sets $cA \to cB$ is a Kan fibration. This follows from the fact that for all $n \geq 1$ and all $0 \leq j \leq n$, the map $\pi_0(\Lambda^n_j) \to \pi_0(\Delta^n)$ is an isomorphism. Then use that $\mathrm{N}(\mathcal{C}) = \mathrm{N}(c\mathcal{C})$ and the previous corollary. $\square$

Remark. In fact more holds true: In **??** we will show that a map of simplicial sets $X \to \mathrm{N}(\mathcal{D})$, where $\mathcal{D}$ is an ordinary category, is an inner fibration if and only if X is an ∞-category.

Recall the definition of the S-horn Λ^n_S of Definition 1.39, so that we have

$$\Lambda^n_S = \bigcup_{s \notin S} \Delta^{[n]\setminus\{s\}}.$$

Lemma 3.16. *Let $S \subseteq [n]$ be a non-empty subset. The map $\Lambda^n_S \to \Delta^n$ is*

(1) *anodyne, provided $S \neq [n]$,*
(2) *left anodyne, provided $\{n\} \notin S$,*
(3) *right anodyne, provided $\{0\} \notin S$,*
(4) *inner anodyne, provided S is not the complement of an intervall, i.e. there are $a < b < c \in [n]$ with $a, c \notin S$ but $b \in S$.*

Proof. Let $S \subseteq [n]$ be a non-empty subset and let $s \in S$ and let $S' = S \setminus \{s\}$. Consider the pushout diagram

$$\begin{tikzcd} \Delta^{[n]\setminus\{s\}} \cap \Lambda^n_S \arrow[r] \arrow[d] & \Delta^{[n]\setminus\{s\}} \arrow[d] & \\ \Lambda^n_S \arrow[r] & \Lambda^n_{S'} \arrow[r] & \Delta^n \end{tikzcd}$$

Notice that the top horizontal arrow is a generalized horn inclusion $\Lambda^{[n]\setminus\{s\}}_{S'} \subseteq \Delta^{[n]\setminus\{s\}}$. It follows that the inclusion $\Lambda^n_S \to \Delta^n$ is contained in the smallest saturated set containing the inclusions

$$\Lambda^{[n]\setminus\{s\}}_{S'} \subseteq \Delta^{[n]\setminus\{s\}} \text{ and } \Lambda^n_{S'} \subseteq \Delta^n.$$

We will prove the lemma by an induction over the size of S (for arbitrary $[n]$). Let us consider (1). If S contains only one element, say $S = \{i\}$ we have that Λ^n_S is a horn inclusion, and hence anodyne. If S contains more than one element, then S' still contains at least one element and is smaller than S, so that $\Lambda^n_{S'} \subseteq \Delta^n$ and $\Lambda^{[n]\setminus\{s\}}_{S'} \subseteq \Delta^n$ is anodyne by induction.

For (2), Suppose that $S = \{i\}$, then the horn inclusion is left anodyne, because $i \neq n$. If S contains more than one element, then S' is smaller and still does not contain n. The statement for (3) is similar.

To show (4), if $S = \{i\}$ then $0 < i < n$, else its complement is an interval, so that the inclusion $\Lambda^n_S \to \Delta^n$ is inner anodyne. If S contains more than one element, then we claim that there exists an element s in S such that $S \setminus \{s\}$ is again not the complement of an interval: By assumption, there are $a < b < c$ such that $b \in S$ and $a, c \notin S$. By assumption $S \neq \{b\}$, so choose some other element $s \in S \setminus \{b\}$. Then $S' = \setminus\{s\}$ is again not the complement of an intervall (because $b \in S'$). □

Proposition 3.17. *The spine inclusions $I^n \to \Delta^n$ are inner anodyne.*

Proof. The spine inclusion $i_n \colon I^n \to \Delta^n$ can be factored as follows:

$$I^n \xrightarrow{f_n} \Delta^{[n]\setminus\{0\}} \cup I^n \xrightarrow{g_n} \Delta^n$$

We will show by induction on n that both f_n and g_n are inner anodyne. The induction start $n = 1$ and $n = 2$ is obvious. Now let $n \geq 3$ and consider the pushout diagram

$$\begin{tikzcd} I^{[n]\setminus\{0\}} \arrow[r] \arrow[d] & \Delta^{[n]\setminus\{0\}} \arrow[d] \\ I^n \arrow[r] & \Delta^{[n]\setminus\{0\}} \cup I^n \end{tikzcd}$$

in which the upper composite is inner anodyne by induction. Hence f_n is inner anodyne as a pushout of an inner anodyne map. It remains to show that g_n is also inner anodyne. In this case we consider the pushout diagram

$$\begin{tikzcd} \Delta^{[n]\setminus\{0,n\}} \cup I^{[n]\setminus\{n\}} \arrow[r, "g_{n-1}"] \arrow[d] & \Delta^{[n]\setminus\{n\}} \arrow[d] & \\ \Delta^{[n]\setminus\{0\}} \cup I^n \arrow[r] & \Delta^{[n]\setminus\{0\}} \cup \Delta^{[n]\setminus\{n\}} \arrow[r] & \Delta^n \end{tikzcd}$$

in which the top horizontal map is inner anodyne by induction. Hence so is the left lower horizontal map. The right lower horizontal map is given by $\Lambda^n_{[n]\setminus\{0,n\}} \subseteq \Delta^n$, which is anodyne by Lemma 3.16 part (4) because $\{0, n\}$ is not an interval. □

Corollary 3.18. *Every ∞-category is a composer.*

Proof. Obvious from Proposition 3.17. $\square$

Definition 3.19. A trivial fibration is a map which has the RLP wrt the boundary inclusions $\partial\Delta^n \to \Delta^n$ for $n \geq 0$.

Lecture 15 – 06.12.2018.

Definition 3.20. Let J be the nerve of the category consisting of two objects with a unique isomorphism between them.

Observation 3.21. The category with two objects and a unique isomorphism between them is a classifier for isomorphisms in a category: In other words the functor corepresented by this category is the functor which assigns to a category its set of isomorphisms.

Given a morphism f in an ∞-category X, one can thus wonder when its classifying map $\Delta^1 \to X$ extends over J. It is easy to see that if this is the case, then f is an equivalence (Exercise). The converse turns out to be true and will be yet another application of the fact that ∞-groupoids are Kan complexes: Note that the map $\Delta^1 \to J$ is anodyne.

Definition 3.22. A Joyal fibration between ∞-categories is an inner fibration which in addition has the RLP wrt the map $\Delta^0 \to J$.

Construction 3.23. Let $f\colon X \to Y$ and $i\colon A \to B$ be maps of simplicial sets. Then there is a commutative diagram

$$\begin{tikzcd} X^B \arrow[r] \arrow[d] & X^A \arrow[d] \\ Y^B \arrow[r] & Y^A \end{tikzcd}$$

and thus there is an induced map

$$\langle f, i\rangle\colon X^B \to X^A \times_{Y^A} Y^B.$$

Construction 3.24. Dually, for morphisms $i\colon A \to B$ and $g\colon S \to T$, we obtain a commutative diagram

$$\begin{tikzcd} A \times S \arrow[r] \arrow[d] & A \times T \arrow[d] \\ B \times S \arrow[r] & B \times T \end{tikzcd}$$

and thus there is an induced map

$$i \boxtimes g\colon A \times T \amalg_{A\times S} B \times S \to B \times T.$$

Lemma 3.25. *The following two lifting problems are equivalent.*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X^B \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X^A \times_{Y^A} Y^B \end{tikzcd} \qquad \begin{tikzcd} A \times T \amalg_{A\times S} B \times S \arrow[r] \arrow[d] & X \arrow[d] \\ B \times T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Proof. $\square$

The crucial technical lemma about the maps $i \boxtimes g$ is the following.

Lemma 3.26. *In the notation of above,*

(1) *$i \boxtimes g$ is inner anodyne if i or g is,*
(2) *$i \boxtimes g$ is left anodyne if i or g is,*
(3) *$i \boxtimes g$ is right anodyne if i or g is,*
(4) *$i \boxtimes g$ is anodyne if i or g is.*

In order to prove this lemma, we will need the following steps:

Lemma 3.27. *Let S and T be a sets of morphisms whose domains are all compact. Then $\overline{S} \boxtimes T \subseteq \overline{S} \boxtimes \overline{T} \subseteq \overline{S \boxtimes T}$. In particular $\overline{\overline{S} \boxtimes T} = \overline{\overline{S} \boxtimes \overline{T}} = \overline{S \boxtimes T}$.*

Proof. The very first inclusion is obvious. To see the second inclusion, we let $\mathcal{F} = \chi_R(S \boxtimes T)$. Then $\overline{S \boxtimes T} = \chi_L(\mathcal{F})$ by Lemma 3.11. Now consider the set $\mathcal{A} = \{f : f \boxtimes T \in \chi_L(\mathcal{F})\}$. By Lemma 3.25 we have that

$$\mathcal{A} = \chi_L(\langle \mathcal{F}, T \rangle)$$

and thus is a saturated set by a previous exercise. Since $S \subseteq \mathcal{A}$ by definition of $\mathcal{A}$, it follows that $\overline{S} \subseteq \mathcal{A}$. Thus $\overline{S} \boxtimes T \subseteq \overline{S \boxtimes T}$. Then consider the set $\mathcal{B} = \{f : \overline{S} \boxtimes f \in \chi_L(\mathcal{F})\}$. As before we get that

$$\mathcal{B} = \chi_L(\langle \mathcal{F}, \overline{S} \rangle)$$

so that $\mathcal{B}$ is also a saturated set. We see that $T \subseteq \mathcal{B}$ by the previous step, so that also $\overline{T} \subseteq \mathcal{B}$. This proves the first part. For the in particular, we argue as follows: Since $\overline{S \boxtimes T}$ is saturated and contains $\overline{S} \boxtimes \overline{T}$ and $\overline{S} \boxtimes T$, we find

$$\overline{\overline{S} \boxtimes T} \subseteq \overline{\overline{S} \boxtimes \overline{T}} \subseteq \overline{S \boxtimes T}.$$

On the other hand, $S \boxtimes T \subseteq \overline{S} \boxtimes T \subseteq \overline{S} \boxtimes \overline{T}$, so the other inclusion also holds. $\square$

Lemma 3.28. *For $0 < j < n$, the inclusion $\Lambda^n_j \to \Delta^n$ is a retract of the map*

$$\Lambda^n_j \times \Delta^2 \amalg_{\Lambda^n_j \times \Lambda^2_1} \Delta^n \times \Lambda^2_1 \to \Delta^n \times \Delta^2.$$

Proof. For the first part consider the maps

$$[n] \xrightarrow{s} [n] \times [2] \xrightarrow{r} [n]$$

where

$$s(i) = \begin{cases} (i,0) & \text{if } i < j, \\ (i,1) & \text{if } i = j, \\ (i,2) & \text{if } i > j \end{cases}$$

and where

$$r(i,k) = \begin{cases} i & \text{if } i < j \text{ and } k = 0 \\ i & \text{if } i > j \text{ and } k = 2 \\ j & \text{else} \end{cases}$$

We now have to show that

(1) $rs = \mathrm{id}$,
(2) $s(\Lambda^n_j) \subseteq \Lambda^n_j \times \Delta^2 \cup \Delta^n \times \Lambda^2_1$, and
(3) $r(\Lambda^n_j \times \Delta^2 \cup \Delta^n \times \Lambda^2_1) \subseteq \Lambda^n_j$.

(1) is an immediate check. To see (2) we observe that in fact $s(\Lambda^n_j) \subseteq \Lambda^n_j \times \Delta^2$: For this it suffices to see that composing s with the projection $[n] \times [2] \to [n]$ is the identity. To further see (3) we need to show two things:

(a) $r(\Lambda^n_j \times \Delta^2) \subseteq \Lambda^n_j$, and
(b) $r(\Delta^n \times \Lambda^2_1) \subseteq \Lambda^n_j$.

To prove (a), consider a k-simplex of Λ^n_j, i.e. $f\colon [k] \to [n]$ such that there exists an $m \in [n] \setminus j$ which is not in the image of f, and an arbitrary k-simplex $\alpha\colon [k] \to [2]$ of Δ^2. The composite

$$[k] \to [n] \times [2] \xrightarrow{r} [n]$$

is easily seen to send $i \in [k]$ to either $f(i)$ or j. Hence its image is contained in the image of f union $\{j\}$. In particular, m is not in the image of this composite, and thus it represents a k-simplex of Λ^n_j. To see (b), consider again a general k-simplex $\beta\colon [k] \to [n]$ of Δ^n, and a k-simplex $f\colon [k] \to [2]$ of Λ^2_1, i.e. where either 0 or 2 is not in the image of f. For definiteness, say that 2 is not in the image (the other case works similarly). We find that the composite

$$[k] \xrightarrow{(\beta,f)} [n] \times [2] \xrightarrow{r} [n]$$

sends $i \in [k]$ to $\beta(i)$ if $\beta(i) < j$ and to j else. Thus the image is contained in $\{0, \dots, j\}$. Now since $0 < j < n$, we see that n is not in the image, so that the above composite represents a k-simplex of Λ^n_j. In the case that 0 is not in the image of f, we find that 0 is not in the image by a similar argument. This proves the lemma. □

Lecture 16 – 10.12.2018. The following is [Lur09, Proposition 2.3.2.1].

Lemma 3.29. *The following sets of morphisms all generate the set of inner anodyne maps.*

(1) *The inner horn inclusions $S_1 = \{\Lambda^n_j \to \Delta^n\}$ for all $n \geq 2$,*
(2) *the maps $S_2 = \{(K \to L) \boxtimes (i\colon \Lambda^2_1 \to \Delta^2)\}$ for all monomorphisms $K \to L$, and*
(3) *the maps $S_3 = \{(\partial\Delta^n \times \Delta^n) \boxtimes (i\colon \Lambda^2_1 \to \Delta^2)\}$ for all $n \geq 0$.*
(4) *the maps $S_4 = \{(K \to L) \boxtimes (\Lambda^n_j \to \Delta^n)\}$ for all monomorphisms $K \to L$ and all inner horns.*

Proof. Let us introduce some more notation: We denote the set of monomorphisms by T_2 and the set of boundary inclusions by T_3. We hence have $S_2 = T_2 \boxtimes i$, $S_3 = T_3 \boxtimes i$, and $S_4 = T_2 \boxtimes S_1$. It is an exercise that $\overline{T_2} = \overline{T_3}$, see Exercise 60. It thus follows from Lemma 3.27 that

$$\overline{S_3} = \overline{T_3 \boxtimes i} = \overline{\overline{T_3} \boxtimes i} = \overline{\overline{T_2} \boxtimes i} = \overline{T_2 \boxtimes i} = \overline{S_2}.$$

Tautologically, we have that $\overline{S_2} \subseteq \overline{S_4}$, since $S_2 \subseteq S_4$. We also find that $\overline{S_1} \subseteq \overline{S_2} = \overline{T_2 \boxtimes i}$, as any inner horn inclusion is a retract of a map in S_2 by Lemma 3.28. Now notice that $T_2 \boxtimes T_2 = T_2$, so that we also obtain

$$\overline{S_4} = \overline{T_2 \boxtimes S_1} \subseteq \overline{T_2 \boxtimes \overline{T_2 \boxtimes i}} = \overline{T_2 \boxtimes T_2 \boxtimes i} = \overline{T_2 \boxtimes i} = \overline{S_2}$$

so that $\overline{S_2} = \overline{S_4}$. The lemma is shown once we can show that $\overline{S_3} \subseteq \overline{S_1}$, for which it suffices to show that $S_3 \subseteq \overline{S_1}$.

So let $m \geq 0$ and consider the inclusion

$$\Delta^m \times \Lambda^2_1 \cup \partial\Delta^m \times \Delta^2 \subseteq \Delta^m \times \Delta^2.$$

If $m = 0$, then this map is given by $\Lambda^2_1 \to \Delta^n$ and thus is inner anodyne. So assume $m \geq 1$. We will construct a filtration of $\Delta^m \times \Delta^2$ as follows. For $0 \leq i \leq j < m$ consider the $(m+1)$-simplices of $\Delta^m \times \Delta^2$ given by

$$\sigma_{i,j}(k) = \begin{cases} (k,0) & \text{if } 0 \leq k \leq i \\ (k-1,1) & \text{if } i+1 \leq k \leq j+1 \\ (k-1,2) & \text{if } j+2 \leq k \leq m+1 \end{cases}.$$

For $0 \leq i \leq j \leq m$ consider the $(m+2)$-simplices of $\Delta^m \times \Delta^2$ given by

$$\tau_{i,j}(k) = \begin{cases} (k,0) & \text{if } 0 \leq k \leq i \\ (k-1,1) & \text{if } i+1 \leq k \leq j+1 \\ (k-2,2) & \text{if } j+2 \leq k \leq m+2 \end{cases}.$$

We observe that the non-degenerate k-simplices of $\Delta^m \times \Delta^2$ correspond to paths of length k in the grid $[m] \times [2]$ which do not take "a break at any point", i.e. are precisely the injective maps $[k] \to [m] \times [2]$. We claim that

(1) The simplices $\tau_{i,j}$ are the non-degenerate $(m+2)$-simplices of $\Delta^m \times \Delta^2$: Necessarily, the paths corresponding to non-degenerate $(m+2)$-simplices have to start at $(0,0)$ and end at $(m,2)$ in order for there to be an injective map $[m+2] \to [m] \times [2]$.
(2) The simplices $\sigma_{i,j}$ are non-degenerate.
(3) The simplex $\sigma_{i,j}$ is a face of $\tau_{i,j}$ and of $\tau_{i,j+1}$ but of no other of the τ's just constructed.
(4) The simplices $\sigma_{i,j}$ and $\tau_{i,j}$ are not contained in $\Delta^m \times \Lambda^2_1 \cup \partial\Delta^m \times \Delta^2$.

(1) and (2) are obvious from the previous observation. (3) follows immediately from the definitions, and (4) is also an explicit check: the projection $[m] \times [2] \to [m]$ sends the simplices in question to surjections, so that they are not contained in $\partial\Delta^m \times \Delta^2$. Likewise, the projection $[m] \times [2] \to [2]$ sends the simplices in question to surjections as well, so that they are also not contained in $\Delta^m \times \Lambda^2_1$.

Let us now inductively define simplicial sets $X(j,i)$ for $0 \leq i \leq j < m$ as follows:

$$X(0,0) = \Delta^m \times \Lambda^2_1 \cup \partial\Delta^m \times \Delta^2,$$

For fixed j, we inductively define for $i \leq j < m$:

$$X(i+1,j) = X(i,j) \cup \sigma_{i,j}$$

and we set

$$X(j+1,j) = X(0,j+1).$$

Then we define $X(0,m) = Y(0,0)$ and again inductively define for $i \leq j \leq m$:

$$Y(i+1,j) = Y(i,j) \cup \tau_{i,j}$$

and set

$$Y(j+1,j) = Y(0,j+1).$$

Because the $\tau_{i,j}$ are the non-degenerate $(m+2)$-simplices of $\Delta^m \times \Delta^2$, we find that $Y(0,m+1) = \Delta^m \times \Delta^2$.

To finish the proof of the lemma we have to show the following statements:

(A) The simplicial set $X(i,j) \cap \sigma_{i,j}$ is an inner horn, and
(B) The simplicial set $Y(i,j) \cap \tau_{i,j}$ is an inner horn.

From this it follows that all maps $X(i,j) \to X(i+1,j)$ are inner anodyne, and that all maps $Y(i,j) \to Y(i+1,j)$ are inner anodyne: In either case, they are pushouts of inner horn inclusions. Thus their composite $X(0,0) \to Y(0,m+1)$ is also inner anodyne, so we can conclude the lemma.

To show (A), it we have to prove that for all $0 \leq i \leq j < m$, the map

$$[m+1] \xrightarrow{\sigma_{i,j}} [m] \times [2]$$

sends all m-dimensional faces to $X(i,j)$, unless one inner m-dimensional face. We claim that the $(i+1)$-face of $\sigma_{i,j}$ is the one which is not contained in $X(i,j)$. Notice that $0 < i+1 < m+1$, so that $\sigma_{i,j} \cap X(i,j)$ is an inner horn. There are only two faces of $\sigma_{i,j}$ which do not lie in $\partial\Delta^m \times \Delta^2 \subseteq X(0,0)$ namely the ones which compose the horizontal edges adjacent to the unique vertical edge. These are $d_i\sigma_{i,j}$ and $d_{i+1}\sigma_{i,j}$, so that it suffices to see that $d_i\sigma_{i,j}$ is contained in $X(i,j)$ and that $d_{i+1}\sigma_{i,j}$ is not contained in $X(i,j)$. To see the former, observe that $d_i(\sigma_{i-1,j}) = d_i(\sigma_{i,j})$, provided $i > 0$ and that $d_0(\sigma_{0,j})$ is contained in $\Delta^m \times \Lambda^2_1$, since after projecting to [2], 0 is not in the image. It remains to prove the latter, namely that $d_{i+1}(\sigma_{i,j})$ is not contained in $X(i,j)$. We have already seen that $d_{i+1}\sigma_{i,j}$ is not contained in $\partial\Delta^m \times \Delta^2$. We hence need to show

(1) $d_{i+1}(\sigma_{i,j})$ is not contained in $\Delta^m \times \Lambda^2_1$,
(2) $d_{i+1}(\sigma_{i,j})$ is not contained in $\sigma_{k,j}$ for $k < i$, and
(3) $d_{i+1}(\sigma_{i,j})$ is not contained in $\sigma_{k,l}$ for $k \leq l < j$.

(1) follows from the fact that $0 < i+1 < m+1$, which says that after projecting $[m]\times[2] \to [2]$, 0 and 2 are in the image of $d_{i+1}\sigma_{i,j}$). For (2) we observe that the path corresponding to $d_{i+1}(\sigma_{i,j})$ runs through the spot $(i,0)$, which is not the case for $\sigma_{k,j}$ for $k < i$, and hence also not for any face of it. Likewise, if $i < j$ then $d_{i+1}\sigma_{i,j}$ runs through the spot $(j,1)$ which is not the case for any $\sigma_{k,l}$ with $l < j$. If $i = j$, then $d_{i+1}\sigma_{i,j}$ runs through again through $(i,0)$ which is not the case for $\sigma_{k,l}$ as $k \leq l < j = i$, so that $k < i$ as in the first case.

It remains to prove (B). Again, we claim that $d_{i+1}\tau_{i,j}$ is the only face not contained in $Y(i,j)$. We first consider the case $i < j$:

Observe again that $d_\ell\tau_{i,j}$ is contained in $\partial\Delta^m \times \Delta^2$ unless $\ell \in \{i, i+1, j+1, j+2\}$, and that $d_{j+1}\tau_{i,j} = \sigma_{i,j-1}$ and $d_{j+2}\tau_{i,j} = \sigma_{i,j}$, so that they are contained in $Y(i,j)$. Likewise, if $i > 0$, then $d_i\tau_{i,j} = d_i\tau_{i-1,j}$, so that this face is also contained in $Y(i,j)$. If $i = 0$, then $d_i\tau_{i,j}$ is contained in $\Delta^m \cup \Lambda^2_1$ as its projection to [2] does not have 0 in its image. We now show that $d_{i+1}\tau_{i,j}$ is not contained in $Y(i,j)$, so that again $Y(i,j) \cap \tau_{i,j}$ is an inner horn since $0 < i+1 < m+2$. As before we need to see that

(1) $d_{i+1}\tau_{i,j}$ is not contained in $\tau_{k,j}$ for $k < i$, and
(2) $d_{i+1}\tau_{i,j}$ is not contained in $\tau_{k,l}$ for $k \leq l < j$.

As before, $d_{i+1}\tau_{i,j}$ runs through the spot $(i,0)$ which is not the case for $\tau_{k,j}$ if $k < i$. Likewise, $d_{i+1}\tau_{i,j}$ runs through the spot $(j,1)$ which is not the case for $\tau_{k,l}$ if $l < j$.

Finally, we consider the case $i = j$, and claim that again $d_{i+1}\tau_{i,i}$ is the only face not contained in $Y(i,i)$: If $i = j = 0$, then $d_0\tau_{0,0}$ is contained in $\Delta^m \times \Lambda^2_1$, because its projection to [2] misses 0. If $i > 0$, we have that $d_i\tau_{i,i} = d_i\tau_{i-1,i}$ so $d_i\tau_{i,i}$ is contained in $Y(i,i)$ in all cases. Likewise, we have that $d_{i+2}\tau_{i,i} = \sigma_{i,i}$. If $l < i$ or $l > i+2$, then the projection to $[m]$ of $d_l\tau_{i,i}$ is not surjective and thus $d_l\tau_{i,i}$ is contained in $\partial\Delta^m \times \Delta^2$. It remains to show that

(1) $d_{i+1}\tau_{i,i}$ is not contained in $\tau_{k,j}$ for $k < i$, and
(2) $d_{i+1}\tau_{i,i}$ is not contained in $\tau_{k,l}$ for $k \leq l < j$.

Note that in both cases, $k < i$ and that $d_{i+1}\tau_{i,i}$ runs through the spot $(i, 0)$. But $\tau_{k,l}$ does not run through $(i, 0)$, no matter what l is. $\square$

Lecture 17 – 17.12.2018.

Lemma 3.30. *The following sets of morphisms all generate the set of left anodyne maps*

(1) *The left horn inclusions $S_1 = \{\Lambda^n_j \to \Delta^n\}$ for all $n \geq 1$ and $0 \leq j < n$,*
(2) *the maps $S_2 = \{(K \to L) \boxtimes (i\colon \{0\} \to \Delta^1)\}$ for all monomorphisms $K \to L$, and*
(3) *the maps $S_3 = \{(\partial\Delta^n \times \Delta^n) \boxtimes (i\colon \{0\} \to \Delta^1))\}$ for all $n \geq 0$.*
(4) *the maps $S_4 = \{(K \to L) \boxtimes (\Lambda^n_j \to \Delta^n)\}$ for all monomorphisms $K \to L$, all $n \geq 1$ and $0 \leq j < n$.*

Proof. As in the proof of Lemma 3.29, the only thing which does not follow from previous considerations are the following two statements:

(1) The maps in S_3 are left anodyne, and
(2) the map $\Lambda^n_j \to \Delta^n$ is a retract of the pushout product map

$$\Delta^n \times \{0\} \amalg_{\Lambda^n_j \times \{0\}} \Lambda^n_j \times \Delta^1 \to \Delta^n \times \Delta^1.$$

Both these statements are an exercise, see Exercise 63, we give the following hints: For (1) consider a similar filtration of $\Delta^n \times \Delta^1$ starting with the domain of the pushout product by adding the missing non-degenerate $(n+1)$-simplices in $\Delta^n \times \Delta^1$ and run the same argument as in Lemma 3.29. For (2), consider the following maps:

$$[n] \xrightarrow{r} [n] \times [1] \xrightarrow{s} [n]$$

where r is the inclusion $k \mapsto (k, 1)$ and

$$s(k,i) = \begin{cases} k & \text{if } k \neq j+1 \text{ and } i = 0 \\ j & \text{if } (k,i) = (j+1, 0) \\ k & \text{if } i = 1. \end{cases}$$

$\square$

Corollary 3.31. *The following sets of morphisms all generate the set of right anodyne maps*

(1) *The right horn inclusions $S_1 = \{\Lambda^n_j \to \Delta^n\}$ for all $n \geq 1$ and $0 < j \leq n$,*
(2) *the maps $S_2 = \{(K \to L) \boxtimes (i\colon \{1\} \to \Delta^1)\}$ for all monomorphisms $K \to L$, and*
(3) *the maps $S_3 = \{(\partial\Delta^n \times \Delta^n) \boxtimes (i\colon \{1\} \to \Delta^1))\}$ for all $n \geq 0$.*
(4) *the maps $S_4 = \{(K \to L) \boxtimes (\Lambda^n_j \to \Delta^n)\}$ for all monomorphisms $K \to L$, all $n \geq 1$ and $0 < j \leq n$.*

Proof. We observes that a map is right anodyne if and only if its opposite if left anodyne, because a map is a left fibration if and only if its opposite is a right fibration according to Exercise 52. Then the first part follows from the fact that $(\Lambda^n_j)^{\mathrm{op}} \cong \Lambda^n_{n-j}$ and that all other morphisms of simplicial sets involved are "self-opposite". $\square$

Proof of Lemma 3.26. Part (1) follows from the equality $\overline{S_4} = \overline{S_1}$ of Lemma 3.29, Part (2) follows from the equality $\overline{S_4} = \overline{S_1}$ of Lemma 3.30, and Part (3) follows from the equality $\overline{S_4} = \overline{S_1}$ of Corollary 3.31. To see Part (4) we simply observe that anodyne maps are generated (as a saturated set) by left and right anodyne maps. It hence suffices to show that for i a left, respectively right, anodyne map and g a monomorphism, then $i \boxtimes g$ is anodyne. Since left, respectively right, anodyne maps are anodyne, this follows from statement (2), respectively (3).

□

Theorem 3.32. *Let $f\colon X \to Y$ be a (inner, left, right) fibration and let $i\colon A \to B$ be a monomorphism. Then*

(1) *the map $\langle f, i\rangle$ is a (inner, left, right) fibration.*
(2) *If furthermore i is (inner, left, right) anodyne, then the map $\langle f, i\rangle$ is a trivial fibration.*

Proof. We prove (1) first. Consider a lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g"] & X^B \arrow[d, "\langle f,i\rangle"] \\ T \arrow[r] \arrow[ur, dashed] & X^A \times_{Y^A} Y^B \end{tikzcd}$$

in which the map $g\colon S \to T$ is (inner, left, right) anodyne. By Lemma 3.25 solving this is equivalent to solving the lifting problem

$$\begin{tikzcd} A \times T \amalg_{A\times S} B \times S \arrow[r] \arrow[d, "i\boxtimes g"] & X \arrow[d, "f"] \\ B \times T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

By Lemma 3.26 we know that $i \boxtimes g$ is (inner, left, right) anodyne, and by assumption f is an (inner, right, left) fibration.

To prove (2), we need to consider again a lifting problem as above, where now g is a monomorphism. Since i is (inner, left, right) anodyne, we again see that by Lemma 3.26 we can solve this lifting problem. □

Corollary 3.33. *Let K and X be simplicial sets. If X is an ∞-category, then so is X^K. If X is a Kan complex, then so is X^K.*

Proof. This is the special case of Theorem 3.32 part (1) where A is empty, $B = K$ and $Y = \Delta^0$. □

Definition 3.34. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We define the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$ by $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$, which is an ∞-category thanks to Corollary 3.33.

Observation 3.35. Notice again that the 0- and 1-simplices of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are functors and natural transformations as defined in Definition 2.15 and Definition 2.76.

If moreover $\mathcal{D}$ is a Kan complex, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is itself a Kan complex and hence in particular an ∞-groupoid. This, together with the (to be proven) fact that ∞-groupoids are Kan complexes, suggests that in the functor category, a morphism which is pointwise an equivalence is in fact an equivalence. This will turn out to be true, but even more complicated then showing that Kan complexes are precisely the ∞-groupoids, see Theorem 6.1.

We are now also in position to give a new definition of the ∞-category of spaces, which is purely simplicial:

Definition 3.36. We consider the simplicial category Kan with objects given by Kan complexes X and with hom simplicial sets from X to Y given by the internal hom simplicial set $\underline{\mathrm{Hom}}(X, Y)$. We let $\widehat{\mathrm{An}} = \mathrm{N}(\mathrm{Kan})$ be its coherent nerve.

Observation 3.37. We claim that there is a canonical functor $\mathrm{An} \to \widehat{\mathrm{An}}$ which is induced by the following simplicial functor: Recall that An is the coherent nerve of the simplicial category of CW-complexes as in Definition 2.70. We claim that sending a CW complex X to its singular complex $\mathcal{S}(X)$ induces a simplicial functor as needed. For this we need to show that there is a canonical map of simplicial sets

$$\mathcal{S}(\mathrm{map}(X,Y)) \to \underline{\mathrm{Hom}}(\mathcal{S}(X), \mathcal{S}(Y))$$

which is compatible with composition. This map is constructed as follows: By adjunction it suffices to construct a map

$$\mathcal{S}(\mathrm{map}(X,Y)) \times \mathcal{S}(X) \to \mathcal{S}(Y).$$

Using that $\mathcal{S}$ (as a right adjoint) commutes with products, this is equivalently provided by a canonical map

$$\mathcal{S}(\mathrm{map}(X,Y) \times X) \to \mathcal{S}(Y)$$

and we use the continuous evaluation map $\mathrm{map}(X,Y) \times X \to Y$ and apply the functor $\mathcal{S}$ to it. It is not hard to see that the map obtained in this fashion is compatible with composition, and hence determines a functor $\mathrm{An} \to \widehat{\mathrm{An}}$ as claimed.

Furthermore, we claim that the map

$$\mathcal{S}(\mathrm{map}(X,Y)) \to \underline{\mathrm{Hom}}(\mathcal{S}(X), \mathcal{S}(Y))$$

is a weak equivalence, so that the simplicial functor from CW-complexes to Kan is a weak equivalence in the sense of Definition 2.49. We will prove later that weak equivalences of simplicial functors induce equivalences of ∞-categories upon applying the coherent nerve, so that both ∞-categories of spaces we have defined are in fact equivalent.

Corollary 3.38. *A simplicial set $\mathcal{C}$ is an ∞-category if and only if the canonical map $\mathcal{C}^{\Delta^2} \to \mathcal{C}^{\Lambda^2_1}$ is a trivial fibration. In particular, for an ∞-category $\mathcal{C}$ the fibre over a fixed diagram $\Lambda^2_1 \to \mathcal{C}$ is a contractible Kan complex.*

Proof. The only if part follows from the fact that $\Lambda^2_1 \to \Delta^2$ is inner anodyne, $\mathcal{C} \to *$ is an inner fibration, and Theorem 3.32. To see the converse, we wish to show that $\mathcal{C} \to *$ is an inner fibration if $\mathcal{C}^{\Delta^2} \to \mathcal{C}^{\Lambda^2_1}$ is a trivial fibration.

To show that $\mathcal{C} \to *$ is an inner fibration, by Lemma 3.29 it suffices to show that it admits the extension property for maps in S_2. By adjunction this holds if and only if $\mathcal{C}^{\Delta^2} \to \mathcal{C}^{\Lambda^2_1}$ satisfies the extension property for all monomorphisms, which is the case if and only if it is a trivial fibration.

For the in particular, consider two composable morphisms f and g in $\mathcal{C}$, and view them as a map $\Lambda^2_1 \to \mathcal{C}$. Then in the pullback diagram

$$\begin{tikzcd} \mathrm{Comp}_{\mathcal{C}}(f,g) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \end{tikzcd}$$

the right vertical map is a trivial fibration, so the left vertical map is also a trivial fibration. In particular, for any two composable morphisms, the simplicial set $\mathrm{Comp}_{\mathcal{C}}(f,g)$ of compositions is a contractible Kan complex. $\square$

Remark. The same argument shows that for an ∞-category $\mathcal{C}$, the map $\mathcal{C}^{\Delta^n} \to \mathcal{C}^{I^n}$ is a trivial fibration for all $n \geq 2$, because the maps $I^n \to \Delta^n$ are inner anodyne.

Lemma 3.39. *Any trivial fibration $X \to Y$ admits a section.*

Proof. Consider the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & Y \end{tikzcd}$$

and use that trivial fibrations have the RLP wrt monomorphisms, of which $\emptyset \to Y$ is an example. ☐

We can thus choose a section of the above trivial fibration and obtain the following composite

$$\mathcal{C}^{\Lambda^2_1} \to \mathcal{C}^{\Delta^2} \to \mathcal{C}^{\Delta^{\{0,2\}}}$$

as functors of ∞-categories (i.e. maps of simplicial sets). Since the first category is equivalent to $\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1}$ where the maps are target and source, we obtain a functor which encodes composition in the ∞-category $\mathcal{C}$:

$$\mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \to \mathcal{C}^{\Delta^1}.$$

Definition 3.40. Let $\mathcal{C}$ be an ∞-category and let $x, y \in \mathcal{C}$ be objects. Then we define the mapping ∞-category between x and y to be the pullback

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ * \arrow[r, "(x,y)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

where the right vertical map is source and target (i.e. evaluation at 0 and 1).

Notice that it is easy to see that $\mathrm{map}_{\mathcal{C}}(x,y)$ is an ∞-category, since the map $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ is an inner fibration by Theorem 3.32. In fact more is true, but the proof of the following proposition has to be deferred to a later point.

Proposition 3.41. *If $\mathcal{C}$ is an ∞-category, then for all objects $x, y \in \mathcal{C}$, $\mathrm{map}_{\mathcal{C}}(x,y)$ is an ∞-groupoid.*

Taking it for granted for the moment, we thus obtain a functor

$$\mathrm{map}_{\mathcal{C}}(x,y) \times \mathrm{map}_{\mathcal{C}}(y,z) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$$

which makes the diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,y) \times \mathrm{map}_{\mathcal{C}}(y,z) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ * \arrow[r, "(x,z)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

commute. We hence obtain a functor

$$\mathrm{map}_{\mathcal{C}}(x,y) \times \mathrm{map}_{\mathcal{C}}(y,z) \to \mathrm{map}_{\mathcal{C}}(x,z)$$

which we refer to as the composition in the ∞-category $\mathcal{C}$.

4. Joins and slices

Lecture 18 – 20.12.2018.

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then the join $\mathcal{C} \star \mathcal{D}$ is given by the following category:

$$\mathrm{Ob}(\mathcal{C} \star \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \amalg \mathrm{Ob}(\mathcal{D})$$

and hom sets are given by

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x,y) & \text{if } x,y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(x,y) & \text{if } x,y \in \mathcal{D} \\ * & \text{if } x \in \mathcal{C}, y \in \mathcal{D} \\ \emptyset & \text{if } x \in \mathcal{D}, y \in \mathcal{C} \end{cases}.$$

Remark. Notice that the join is not symmetric.

Definition 4.2. Let $\mathcal{C}$ be a category, and let $x \in \mathcal{C}$ be an object. Then there are categories $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$ of objects over and under x, called slice categories. Objects are given by morphisms $y \to x$ for $\mathcal{C}_{/x}$ and $x \to y$ for $\mathcal{C}_{x/}$. Morphisms between to such objects are given by commutative triangles.

Remark. If $\mathcal{C}$ is a cocomplete category and x an object of $\mathcal{C}$, then the slice $\mathcal{C}_{x/}$ of objects under x is again cocomplete. However, the forgetful functor $\mathcal{C}_{x/} \to \mathcal{C}$ does not preserve all colimits, see **??**.

Definition 4.3. Given a linearly ordered set J, we define the set of cuts of J, $\mathrm{Cut}(J)$ as decompositions of $J = J_1 \amalg J_2$ into two disjoint pieces J_1 and J_2 such that $x < y$ whenever $x \in J_1$ and $y \in J_2$. The half empty cuts $(\emptyset, J)$ and $(J, \emptyset)$ are allowed.

Lemma 4.4. *Given linearly ordered sets J and J' with a map of such $\alpha\colon J \to J'$, and given $(J_1', J_2') \in \mathrm{Cut}(J')$, there exists a unique cut $(J_1, J_2) \in \mathrm{Cut}(J)$ such that α restricts to maps $\alpha_1\colon J_1 \to J_1'$ and $\alpha_2\colon J_2 \to J_2'$.*

Proof. We need to define $J_i = \alpha^{-1}(J_i')$. The only thing to check is that this is in fact a cut of J. This follows since α is order preserving. □

Observation 4.5. This implies that $\mathrm{Cut}(-)$ is a contravariant functor from linearly ordered sets to sets. In fact, $\mathrm{Cut}(-)$ is representable by [1].

Definition 4.6. Let X and Y be simplicial sets. Define their join $X \star Y$ to be the simplicial set given by the following: For a finite linearly ordered set J, we set

$$(X \star Y)(J) = \coprod_{(J_1,J_2)\in\mathrm{Cut}(J)} X(J_1) \times Y(J_2)$$

where we declare that $X(\emptyset) = * = Y(\emptyset)$. Let $\alpha\colon J \to J'$ be a morphism of linearly ordered sets and consider a cut of J'. Consider the associated cut of J as in Lemma 4.4. Then there is a map

$$X(J_1') \times Y(J_2') \to X(J_1) \times Y(J_2).$$

This provides a unique map

$$(X \star Y)(J') \to (X \star Y)(J)$$

restricting to the above on each cut of J. This makes $X \star Y$ a simplicial set.

Example 4.7. Given two ordinary categories $\mathcal{C}$ and $\mathcal{D}$, we have that

$$\mathrm{N}(\mathcal{C} \star \mathcal{D}) \cong \mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D}).$$

In particular, we have that $\Delta^n \star \Delta^m = \Delta^{n+1+m}$.

Proof. We construct a map of simplicial sets

$$\mathrm{N}(\mathcal{C} \star \mathcal{D}) \to \mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D})$$

by observing that any n-simplex in $\mathrm{N}(\mathcal{C} \star \mathcal{D})$ determines a cut of $[n]$: at some point one jumps from morphisms in $\mathcal{C}$ to morphisms in $\mathcal{D}$. It is then easy to see that this map is an isomorphism. □

Lemma 4.8. *Given a simplicial set X, the join construction determines a functor $X \star -\colon \mathrm{sSet} \to \mathrm{sSet}_{X/}$, likewise it produces a functor $- \star X \colon \mathrm{sSet} \to \mathrm{sSet}_{X/}$.*

Proof. We need to show that for every $Y \in \mathrm{sSet}$, the simplicial set $X \star Y$ comes equipped with a map $X \to X \star Y$. This is obviously the case by the right half empty cut inclusion

$$X(J) \times Y(\emptyset) \subseteq \coprod_{(J_1,J_2)\in \mathrm{Cut}(J)} X(J_1) \times Y(J_2) = (X \star Y)(J).$$

Furthermore, given a morphism $Y \to Y'$ of simplicial sets, we claim that the canonical diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \star Y \arrow[d] \\ & X \star Y' \end{tikzcd}$$

commutes. This is immediate, and functoriality is also obvious. □

Lemma 4.9. *Let K be a simplicial set equipped with a map $p\colon K \to \Delta^1$. Then there is a functorial factorization into a composite*

$$K \to K_0 \star K_1 \xrightarrow{c} \Delta^1$$

where $K_i = p^{-1}(i)$ and the map c is the map $K_0 \star K_1 \to \Delta^0 \star \Delta^0 \cong \Delta^1$.

Proof. We need to construct the map $K \to K_0 \star K_1$. For any $n \geq 0$, we have that

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \Delta^1) = \mathrm{Hom}([n],[1]) = \mathrm{Cut}([n]).$$

Thus, for every n-simplex $x\colon \Delta^n \to K$, the composite $px\colon \Delta^n \to K \to \Delta^1$ determines a cut $([i],[j]) \in \mathrm{Cut}([n])$, so that the map $px\colon [n] \to [1]$ sends the first i points to 0 and the rest to 1. This, by definition, determines a point of $K_0([i]) \times K_1([j])$ which in turn determines an n-simplex of $K_0 \star K_1$.

It is then easy to see that this in fact determines a map of simplicial sets $K \to K_0 \star K_1$ and that this construction is functorial in $\mathrm{sSet}_{/\Delta^1}$. Concretely, for a commutative triangle

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \Delta^1 \\ K' \arrow[ur] & \end{tikzcd}$$

the induced diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K_0 \star K_1 \arrow[r] \arrow[d] & \Delta^1 \\ K' \arrow[r] & K'_0 \star K'_1 \arrow[ur] & \end{tikzcd}$$

commutes as well. $\square$

Corollary 4.10. *Given a map $\varphi\colon K \to X \star Y$ over Δ^1, i.e. a morphism in $\mathrm{sSet}_{/\Delta^1}$, there is a factorization into $K \to K_0 \star K_1 \xrightarrow{f \star g} X \star Y$.*

Proof. We observe that the factorization provided by Lemma 4.9 for the map $X \star Y \to \Delta^1$ is given by $X \star Y \to X \star Y \to \Delta^1$. Since this factorization is functorial we obtain a commutative diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d] & X \star Y \arrow[d, "\cong"] \\ K_0 \star K_1 \arrow[r, "f \star g"] & X \star Y \end{tikzcd}$$

so the claim follows. $\square$

Proposition 4.11. *X and Y are ∞-categories if and only if $X \star Y$ is an ∞-category.*

Proof. Consider a map

$$\Lambda^n_j \to X \star Y$$

for $n \geq 2$ and $0 < j < n$. We want to show that it extends over Δ^n. We can post compose this map with the canonical map $X \star Y \to \Delta^1$ and obtain a factorization

$$\Lambda^n_j \to (\Lambda^n_j)_0 \star (\Lambda^n_j)_1 \to X \star Y.$$

There are several possibilities for what this first map is: First recall that any map $\Lambda^n_j \to \Delta^1$ factors uniquely over Δ^n (since Δ^1 is the nerve of category and Λ^n_j is an inner horn). There are three cases we will consider now:

(1) The map $\Lambda^n_j \to \Delta^1$ is constant at 0,
(2) The map $\Lambda^n_j \to \Delta^1$ is constant at 1,
(3) The map $\Lambda^n_j \to \Delta^1$ is not constant.

In the first case, we find that the map $\Lambda^n_j \to X \star Y$ factors through $X \to X \star Y$, and thus can be extended over Δ^n if and only if X is an ∞-category: If an extension of the composite

$$\Lambda^n_j \to X \to X \star Y$$

to Δ^n exists, then the composite $\Delta^n \to X \star Y \to \Delta^1$ is constant at 0, so that the map $\Delta^n \to X \star Y$ in fact factors through the inclusion $X \to X \star Y$. Similarly, in the second case we find that the map $\Lambda^n_j \to X \star Y$ factors through $Y \to X \star Y$, and thus can be extended over Δ^n if and only if Y is an ∞-category.

Lastly, let us consider the case where the map $\Lambda^n_j \to \Delta^1$ is not constant. Observe that this map factors uniquely through a non-constant map $\Delta^n \to \Delta^1$. The non constant maps correspond precisely to the non half-empty cuts of $[n]$, so there is a $0 \leq k < n$ such that the map $\Delta^n \to \Delta^1$ is isomorphic to the canonical map $\Delta^k \star \Delta^\ell \to \Delta^1$. It follows that $(\Lambda^n_j)_0$ consists of all those m-simplices of Λ^n_j, which are represented by maps $[m] \to [n]$ whose image is contained in $\{0, \dots, k\}$ (so that it lies in the fibre over 0) and such that there exists a number different from j which is not in the image of $[m] \to [n]$ (so that it lies in the horn). Since $k < n$, we can be sure that n does not lie in the image of the map $[m] \to [n]$ representing an m-simplex of $(\Lambda^n_j)_0$. In other words, we find that $(\Lambda^n_j)_0 \cong \Delta^k$. Likewise, we find that $(\Lambda^n_j)_1 \cong \Delta^\ell$. The factorization of Corollary 4.10 hence reads as

$$\Lambda^n_j \to \Delta^k \star \Delta^\ell \to X \star Y$$

which is the desired extension. The Proposition follows. $\square$

Lecture 19 – 14.01.2019. We now want to come to the construction of slice categories of ∞-categories. For this we observe that if the functor $S \star -\colon \mathrm{sSet} \to \mathrm{sSet}_{S/}$ admits a right adjoint $\mathrm{sSet}_{S/} \to \mathrm{sSet}$

$$(p\colon S \to X) \mapsto X_{p/}$$

we obtain that a map from $Y \to X_{p/}$ is the same thing as a map $S \star Y \to X$ in $\mathrm{sSet}_{S/}$. Specializing to $Y = \Delta^n$ we obtain a simplicial set:

Definition 4.12. For $p\colon S \to X$, the association $n \mapsto \mathrm{Hom}_{\mathrm{sSet}_{S/}}(X \star \Delta^n, S)$ determines a simplicial set which we call $X_{p/}$.

Lemma 4.13. *If an ordinary category $\mathcal{C}$ is (co)complete and $x \in \mathcal{C}$ is an object, then $\mathcal{C}_{x/}$ and $\mathcal{C}_{/x}$ are (co)complete as well. The forgetful map $\mathcal{C}_{x/} \to \mathcal{C}$ preserves limits and connected colimits (i.e. colimits indexed over connected categories), and the forgetful map $\mathcal{C}_{/x} \to \mathcal{C}$ preserves colimits and connected limits.*

Proof. We first observe that $\mathcal{C}_{/x} \cong (\mathcal{C}^{\mathrm{op}}_{x/})^{\mathrm{op}}$, so that it suffices to treat the case of colimits. We now show that the forgetful map $\mathcal{C}_{/x} \to \mathcal{C}$ preserves colimits: Consider a diagram $F\colon I \to \mathcal{C}_{/x}$. The colimit of the underlying diagram $I \to \mathcal{C}_{/x} \to \mathcal{C}$ canonically comes with a map to x, and it is easy to see that this produces a colimit of F.

The case of colimits of a diagram $F\colon I \to \mathcal{C}_{x/}$ is slightly more complicated. We observe that any such diagram is equivalently given by a diagram $G\colon I^{\triangleleft} \to \mathcal{C}$ whose restriction to the cone point is given by the object x. We observe that there are canonical functors $\Delta^0 \to I^{\triangleleft} \leftarrow I$ which are inclusions. In particular we have a canonical map

$$x = \operatorname*{colim}_{\Delta^0} G_{|\Delta^0} \to \operatorname*{colim}_{I^{\triangleleft}} G.$$

We claim that this morphism is a colimit of F: Suppose given a functor $\bar{F}\colon I^{\triangleright} \to \mathcal{C}_{/x}$, i.e. a compatible family of maps $F(i) \to (x \to y)$ in $\mathcal{C}_{x/}$. We wish to show that there exists a unique map

$$\operatorname*{colim}_{I^{\triangleleft}} G \to y$$

compatible with both maps from x. This comes from the observation that $G_{|I} = F$, so that for $i \in I$, there is a canonical map $G(i) = F(i) \to y$, and for the cone point, we have $G(*) = x$, so that there is a canonical map to y as well. These are compatible since F takes values in the slice $\mathcal{C}_{x/}$. This shows that F admits a colimit, namely the map

$$x \to \operatorname*{colim}_{I^{\triangleleft}} G$$

described above.

To finish the proof of the lemma, we need to show that if I is connected, then the canonical map

$$\operatorname*{colim}_{I} F \to \operatorname*{colim}_{I^{\triangleleft}} G$$

is an isomorphism, so that the functor $\mathcal{C}_{x/} \to \mathcal{C}$ preserves connected colimits. We first construct a canonical map in the other direction: For this it suffices to construct a map $G(j) \to \operatorname*{colim}_I F$ for $j \in I^{\triangleleft}$, compatible in j. If $j \in I$, then $G(j) = F(j)$ so that there is a canonical map to $\operatorname*{colim}_I F$. So we need to construct a map $x = G(*) \to \operatorname*{colim}_I F$. For this we choose any object $i \in I$, and get a map

$$x = G(*) \to G(i) = F(i) \to \operatorname*{colim}_I F$$

as wanted. We need to show that these maps assemble into a map

$$\operatorname*{colim}_{I^{\triangleleft}} G \to \operatorname*{colim}_I F.$$

In other words, we need to show that for any morphism in $I^{\triangleleft}$, the corresponding triangle commutes. It suffices to treat morphisms of the form $* \to j$ for some $j \in I$ (for morphisms in I, it holds by construction). Concretely we need to show that for any two objects $i, j \in I$, the two maps

$$x = G(*) \to G(i) = F(i) \to \operatorname*{colim}_I F$$

and

$$x = G(*) \to G(j) = F(j) \to \operatorname*{colim}_I F$$

are the same maps. This is were the assumption that I is connected enters: We find a sequence of morphisms connecting i to j in I. By induction on the length, we may assume that the length is one, so that there is in fact a map $i \to j$ in I. In this case, it follows from the property of colimits that the triangle

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d] & \operatorname*{colim}_I F \\ F(j) \arrow[ur] & \end{tikzcd}$$

commutes. On the other hand, by assumption on F, the triangle

$$\begin{tikzcd} x \arrow[r] \arrow[dr] & F(i) \arrow[d] \\ & F(j) \end{tikzcd}$$

also commutes, so that the claim is shown.

By construction, the composite

$$\operatorname*{colim}_I F \to \operatorname*{colim}_{I^{\triangleleft}} G \to \operatorname*{colim}_I F$$

is the identity. The other composite gives a map

$$\operatorname*{colim}_{I^{\triangleleft}} G \to \operatorname*{colim}_{I} F \to \operatorname*{colim}_{I^{\triangleleft}} G$$

whose restriction to $G(j) \to \operatorname*{colim}_{I^{\triangleleft}} G$ for $j \in I$ is the canonical map, and whose restriction to the cone point is given by $x \to G(i) \to \operatorname*{colim}_{I^{\triangleleft}} G$. This map is the canonical map $G(*) \to \operatorname{colim} G$, so that the above composite is also the identity. □

Lemma 4.14. *The functors $S \star -$ and $- \star S \colon \mathrm{sSet} \to \mathrm{sSet}_{S/}$ preserve colimits.*

Proof. It suffices to check that it preserves coequalizers, which are calculated underlying by Lemma 4.13, and that it preserves coproducts. For the latter, we recall that the coproduct $(S \to A) \amalg (S \to B)$ in $\mathrm{sSet}_{S/}$ is given by the canonical map to the pushout $S \to A \amalg_S B$ in sSet, see again Lemma 4.13 for the description of colimits in such slices. Then observe that

$$(S * (\coprod_{i \in I} A_i))_n = S_n \amalg \coprod_{i \in I} (A_i)_n \amalg \coprod_{k+l=n-1} S_k \times \coprod_{i \in I} (A_i)_l$$

whereas

$$\coprod_{i \in I} (S * A_i)_n = \coprod_{i \in I} \Big(S_n \amalg (A_i)_n \amalg \coprod_{k+l=n-1} S_k \times (A_i)_l \Big) / \sim$$

where the relation identitfies $\coprod_I S_n$ to S_n. For coequalizers, the statement follows similarly from the explicit description of the simplices of the join. □

Corollary 4.15. *The functors $S \star -$ and $- \star S \colon \mathrm{sSet} \to \mathrm{sSet}$ preserve pushouts.*

Proof. The functors $S \star -$ and $- \star S \colon \mathrm{sSet} \to \mathrm{sSet}_{S/}$ preserve all colimits by Lemma 4.14 and the forgetful functor $\mathrm{sSet}_{S/} \to \mathrm{sSet}$ preserves connected colimits by Lemma 4.13. Pushouts are connected colimits, so the corollary follows. □

Corollary 4.16. *The functor $(p \colon S \to X) \mapsto X_{p/}$ is right adjoint to $S \star - \colon \mathrm{sSet} \to \mathrm{sSet}_{S/}$. Likewise, the functor $(p \colon S \to X) \mapsto X_{/p}$ is right adjoint to $- \star S \colon \mathrm{sSet} \to \mathrm{sSet}_{S/}$.*

Proof. By definition, the adjunction property holds for representables. By Lemma 4.14, the functors $S \star -$ and $- \star S$ preserve colimits, so that the adjunction bijection prolongs from representables to all simplicial sets. □

Example 4.17. Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object, which we view as a functor $x \colon \Delta^0 \to \mathcal{C}$. We obtain slices $\mathcal{C}_{x/}$ and $\mathcal{C}_{/x}$. For a general simplicial set K we will write $K^{\triangleleft} = \Delta^0 \star K$ and $K^{\triangleright} = K \star \Delta^0$ and call these constructions cone and cocone over K.

Observation 4.18. Let us spell out explicitly the unit and counit of the slice/join adjunction. For a fixed simplicial set S, the counit of the adjunction is given by a natural map as follows: Let $p \colon S \to X$ be an object of $\mathrm{sSet}_{S/}$, so we obtain the slice $X_{p/}$. The counit is then the map $S \star X_{p/} \to X$ in $\mathrm{sSet}_{S/}$ given by

$$S_n \amalg \mathrm{Hom}_{\mathrm{sSet}_{S/}}(S \star \Delta^n, X) \amalg \coprod_{k+l=n-1} S_k \times \mathrm{Hom}_{\mathrm{sSet}_{S/}}(S \star \Delta^l, X) \to X_n$$

which is given by p in the first component, induced by precomposition with $\Delta^n \to S \star \Delta^n$ on the second component, and induced by precomposition with $\Delta^k \star \Delta^l \to S \star \Delta^l$ for each k-simplex of S on the last component.

Likewise, the unit is the map $X \to (S \star X)_{\mathrm{can}/}$, where $\mathrm{can}\colon S \to S \star X$ is the canonical map. It is given by joining an n-simplex of X with S, which produces a map

$$X_n \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \to \mathrm{Hom}_{\mathrm{sSet}_{S/}}(S \star \Delta^n, S \star X).$$

Definition 4.19. Let $\mathcal{C}$ be an ordinary category. We define a new category $\mathrm{Tw}(\mathcal{C})$, the twisted arrow category of $\mathcal{C}$ as follows: Objects are the morphisms of $\mathcal{C}$. A morphism in $\mathrm{Tw}(\mathcal{C})$ from $f'\colon x' \to y'$ to $f\colon x \to y$ is given by a commutative diagram

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\alpha"] & y \\ x' \arrow[r, "f'"] & y' \arrow[u, "\beta"'] \end{tikzcd}$$

We also write that the pair (α, β) is a morphism from f' to $f = \beta f' \alpha$. Composition is obtained by glueing together such diagrams.

Lecture 20 – 17.01.2019.

Lemma 4.20. *The slice construction induces a functor* $\mathrm{Tw}(\mathrm{sSet}) \to \mathrm{sSet}$. *In particular, for*

$$A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$$

there is an induced map

$$X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}.$$

The same holds true for the other slice.

Proof. The objects of $\mathrm{Tw}(\mathrm{sSet})$ are given by maps $p\colon S \to X$ of simplicial sets, and such an object is sent to $X_{/p}$. We need to argue how this is functorial in morphisms of the twisted arrow category, i.e. we need to produce a canonical map

$$X_{\varphi/} \to Y_{f\varphi i/}$$

as the pair (i, f) is a morphism from φ to $f\varphi i$ in $\mathrm{Tw}(\mathrm{sSet})$.

First, we construct maps $X_{\varphi/} \to X_{\varphi i/}$ and $X_{\varphi/} \to Y_{f\varphi/}$ which correspond the the morphisms

$$\begin{tikzcd} A \arrow[r, "\varphi i"] \arrow[d, "i"] & X \arrow[d, equal] \\ B \arrow[r, "\varphi"] & X \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f\varphi"] \arrow[d, equal] & Y \\ B \arrow[r, "\varphi"] & X \arrow[u, "f"'] \end{tikzcd}$$

in $\mathrm{Tw}(\mathrm{sSet})$. Using those constructions, we similarly obtain maps

$$X_{\varphi i/} \to Y_{f\varphi i/} \leftarrow Y_{f\varphi/}$$

and we will then show that the diagram

$$\begin{tikzcd} X_{\varphi/} \arrow[r] \arrow[d] & Y_{f\varphi/} \arrow[d] \\ X_{\varphi i/} \arrow[r] & Y_{f\varphi i/} \end{tikzcd}$$

commutes. This is already part of functoriality in the twisted arrow category because the pair (i, f) satisfies

$$(i, \mathrm{id}) \circ (\mathrm{id}, f) = (i, f) = (\mathrm{id}, f) \circ (i, \mathrm{id})$$

as the following diagrams show:

$$\begin{tikzcd} A \arrow[r, "f\varphi i"] \arrow[d, equal] & Y \\ A \arrow[r, "\varphi i"] \arrow[d, "i"] & X \arrow[u, "f"'] \arrow[d, equal] \\ B \arrow[r, "\varphi"] & X \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f\varphi i"] \arrow[d, "i"] & Y \arrow[d, equal] \\ B \arrow[r, "f\varphi"] \arrow[d, equal] & Y \\ B \arrow[r, "\varphi"] & X \arrow[u, "f"'] \end{tikzcd}$$

The map $X_{\varphi/} \to X_{\varphi i/}$ is adjoint to a map $A \star X_{\varphi/} \to X$ under A which we define to be the canonical composite

$$A \star X_{\varphi/} \to B \star X_{\varphi/} \to X$$

consisting of the map induced by i and the counit of the adjunction. Likewise, the map $X_{\varphi/} \to X_{f\varphi/}$ is adjoint to a map $B \star X_{\varphi/} \to Y$ under B which we define to be the composite

$$B \star X_{\varphi/} \to X \to Y$$

consisting of the counit, followed by f.

To see that the diagram

$$\begin{tikzcd} X_{\varphi/} \arrow[r] \arrow[d] & Y_{f\varphi/} \arrow[d] \\ X_{\varphi i/} \arrow[r] & Y_{f\varphi i/} \end{tikzcd}$$

commutes, we observe that both composites are adjoint to the map

$$A \star X_{\varphi/} \to B \star X_{\varphi/} \to X \to Y.$$

It is then easy to see that this construction is functorial in Tw(sSet). For the other slice, the argument is similar. □

Lemma 4.21. *The slice/join adjunction induces a bijection of lifting problems between diagrams of the kind*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X_{\varphi/} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/} \end{tikzcd}$$

and diagrams of the kind

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend left, "\varphi"] & A \star T \amalg_{A \star S} B \star S \arrow[r] \arrow[d] & X \arrow[d] \\ & B \star T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Proof. Exercise 71. □

An analogue of Lemma 3.26 and Theorem 3.32 holds for joins and slices in place of product and mapping simplicial sets.

Lemma 4.22. *Let $i\colon A \to B$ and $g\colon S \to T$ be monomorphisms. Then the induced map*

$$i \hat{\star} g\colon A \star T \amalg_{A \star S} B \star S \to B \star T$$

is a monomorphism and in addition satisfies that it is

1. *inner anodyne if i is right anodyne or g is left anodyne,*
2. *is left anodyne if i is left anodyne,*
3. *is right anodyne if g is right anodyne.*

Proof. For (1) let us prove the case where i is right anodyne. We claim that the set which contains all monomorphisms $i\colon A \to B$ such that the map $i \hat{\star} g$ is inner anodyne (for any monomorphism $g\colon S \to T$) is a saturated class: This is because it is the set of all monomorphisms which has the LLP wrt morphisms of the form

$$X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$

for an inner fibration $f\colon X \to Y$ and an arbitrary map $\varphi\colon T \to X$. It hence suffices to show that the horn inclusions $\Lambda^n_j \to \Delta^n$ for $0 < j \leq n$ are in this set. Now we claim that the set of monomorphisms $g\colon S \to T$ such that the map $(\Lambda^n_j \to \Delta^n) \hat{\star} g$ is inner anodyne is also a saturated set. It hence suffices to finally prove that the claim holds for g the boundary inclusions $\partial\Delta^m \to \Delta^m$. In this case we have to see that

$$\Lambda^n_j \star \Delta^m \cup \Delta^n \star \partial\Delta^m \to \Delta^n \star \Delta^m$$

is inner anodyne. This follows from Exercise 72, which shows that the former is given by Λ^{n+1+m}_j which is now an inner horn because $j \leq n < n+1+m$.

The case where g is left anodyne follows from a similar calculation using that

$$\partial\Delta^m \star \Delta^n \cup \Delta^m \star \Lambda^n_j \to \Delta^{m+1+n}$$

is isomorphic to the inclusion $\Lambda^{m+1+n}_{m+1+j} \to \Delta^{m+1+n}$ and $0 \leq j < n$, so that this is again an inner horn.

Let us now prove (2). Using the same reduction arguments as before, it suffices to treat the case where $i\colon \Lambda^n_j \to \Delta^n$ with $0 \leq j < n$ and where $g\colon \partial\Delta^m \to \Delta^m$ is the boundary inclusion. Then we get, as before that the map $i \hat{\star} g$ is given by $\Lambda^{n+1+m}_j \to \Delta^{n+1+m}$ with $0 \leq j < n$ which is clearly a left anodyne map. The case (3) is analogues. □

Theorem 4.23. *Let $A \xrightarrow{i} B \xrightarrow{\varphi} X \xrightarrow{f} Y$ be composable maps and assume that i is a monomorphism and that f is an inner fibration.*

1. *The induced map*
$$X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$
is a left fibration.
2. *If the map $f\colon X \to Y$ is a left fibration, then also the induced map*
$$X_{/\varphi} \to X_{/\varphi i} \times_{Y_{/f\varphi i}} Y_{/f\varphi}$$
is a left fibration.
3. *If the map $i\colon A \to B$ is right anodyne, then the map*
$$X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$
is a trivial fibration.

(4) *If the map $f\colon X \to Y$ is a trivial fibration, then the map*
$$X_{\varphi/} \to X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/}$$
is a trivial fibration.

Proof. Again we consider a general lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X_{\varphi/} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/} \end{tikzcd}$$

This lifting problem is equivalent (Exercise) to the following lifting problem

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend left, "\varphi"] & A \star T \amalg_{A\star S} B \star S \arrow[r] \arrow[d] & X \arrow[d] \\ & B \star T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

To prove (1) we thus need to see the the left vertical map is inner anodyne provided $S \to T$ is left anodyne. This follows from Lemma 4.22 part (1). To prove (2) we one needs that the map
$$S \star B \amalg_{S\star A} T \star A \to T \star B$$
is left anodyne provided the map $S \to T$ is left anodyne. This is the content of Lemma 4.22 part (2). To prove (3) we need to observe that if $S \to T$ is a monomorphism and $A \to B$ is right anodyne, then the map $A \star T \amalg_{A\star S} B \star S \to B \star T$ is inner anodyne, again by Lemma 4.22 part (1). To prove (4) we only need to use that the map $A \star T \amalg_{A\star S} B \star S \to B \star T$ is always a monomorphism and that trivial fibrations satisfy the RLP wrt monomorphisms. $\square$

Let us spell out some explicit special cases:

Corollary 4.24. *Suppose given maps $A \to B \to X \to Y$ as before where $A \to B$ is a monomorphism and where $X \to Y$ is an inner fibration.*

(1) *If $Y = *$, so that X is an ∞-category we get that $X_{\varphi/} \to X_{\varphi i/}$ is a left fibration, and that $X_{/\varphi} \to X_{/\varphi i}$ is a right fibration. In particular if $A = \emptyset$, then the map $X_{\varphi/} \to X$ is a left fibration and $X_{/\varphi} \to X$ is a right fibration. In particular, $X_{\varphi/}$ and $X_{/\varphi}$ are ∞-categories if X is.*
(2) *If $Y = *$ so that X is an ∞-category, we see that $X_{\varphi/} \to X_{\varphi i/}$ is a trivial fibration if $A \to B$ is right anodyne and that $X_{/\varphi} \to X_{/\varphi i}$ is a trivial fibration if $A \to B$ is left anodyne.*
(3) *If $f\colon X \to Y$ is a trivial fibration, consider the case where $A = \emptyset$. Then the map $X_{\varphi/} \to X \times_Y Y_{f\varphi/}$ is a trivial fibration. Furthermore the map $X \times_Y Y_{f\varphi/} \to Y_{f\varphi/}$ is a pullback of $X \to Y$ and thus also a trivial fibration.*

Yet another special case of this is the following. Suppose $\mathcal{C}$ is an ∞-category and that $f\colon x \to y$ is a morphism in $\mathcal{C}$. We can consider the situation $\Delta^0 \to \Delta^1 \to \mathcal{C} \to *$ and obtain maps
$$\mathcal{C}_{/x} \leftarrow \mathcal{C}_{/f} \to \mathcal{C}_{/y}$$

corresponding to the two restrictions of f to Δ^0 (likewise for the other slice). Since the inclusion $\{0\} \to \Delta^1$ is left anodyne, it follows from Corollary 4.24 part (2) that the map $\mathcal{C}_{f/} \to \mathcal{C}_{x/}$ is a trivial fibration so that we can choose a section to obtain a composite

$$\mathcal{C}_{/x} \to \mathcal{C}_{/f} \to \mathcal{C}_{/y}$$

which we informally think of as the functor of post composition with f. The same works for the other slice to obtain a functor $\mathcal{C}_{y/} \to \mathcal{C}_{x/}$ which we think of as pre composition with f.

5. Joyal lifting and applications

Lecture 21 – 21.01.2019.

Definition 5.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is called conservative if it detects equivalences, i.e. if whenever $f\colon x \to y$ is a morphism in $\mathcal{C}$ such that $F(f)\colon fx \to fy$ is an equivalence in $\mathcal{D}$, then f itself is an equivalence.

Observation 5.2. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is conservative if and only if its opposite functor $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is conservative.

Proposition 5.3. *Left and right fibrations between ∞-categories are conservative.*

Proof. By passing to opposite categories it suffices to treat the case of left fibrations. Suppose given a morphism $f\colon \Delta^1 \to \mathcal{C}$ which becomes an equivalence in $\mathcal{D}$. Consider the diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ \Delta^2 \arrow[r, dashed] \arrow[ur, dotted] & \mathcal{D} \end{tikzcd}$$

where the map $\Lambda^2_0 \to \mathcal{C}$ is given by f on the edge $\Delta^{\{0,1\}}$ and by the identity on $\Delta^{\{0,2\}}$. Since the image in $\mathcal{D}$ is an equivalence, there exists a dashed arrow making the diagram commute. Since $\mathcal{C} \to \mathcal{D}$ is a left fibration there also exists the dotted arrow. This proves that f admits a left inverse in $\mathcal{C}$ which becomes a left inverse of $p(f)$ after applying p and thus an equivalence after applying p. Running the same argument for this morphism proves that it itself admits left inverse showing that the first constructed left inverse of f is an equivalence. Thus also f is an equivalence. $\square$

Definition 5.4. A inner fibration $\mathcal{C} \to \mathcal{D}$ of ∞-categories is called an isofibration if every lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^1 \arrow[r, "f"] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

in which f represents an equivalence of $\mathcal{D}$ has a solution which represents an equivalence of $\mathcal{C}$.

Lemma 5.5. *An inner fibration $\mathcal{C} \to \mathcal{D}$ between ∞-categories is an isofibration if and only if the induced functor $\mathrm{N}(h\mathcal{C}) \to \mathrm{N}(h\mathcal{D})$ is an isofibration.*

Proof. Exercise 80. $\square$

Corollary 5.6. *A functor $\mathcal{C} \to \mathcal{D}$ between ∞-categories is an isofibration if and only if $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an isofibration.*

Proof. Exercise 81. □

Proposition 5.7. *Left and right fibrations between ∞-categories are conservative isofibrations.*

Proof. Left and right fibrations are conservative by Proposition 5.3. Now let $p\colon \mathcal{C} \to \mathcal{D}$ be a left fibration and consider a lifting problem as in Definition 5.4. Since $\mathcal{C} \to \mathcal{D}$ is a left fibration and $\{0\} \to \Delta^1$ is left anodyne, a lift as needed exists. By conservativity, any such lift is an equivalence. For right fibrations, use Corollary 5.6. □

The $\mathcal{D} = \Delta^0$ case of the following theorem is already very interesting.

Theorem 5.8. *Let $\mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories and let $\phi\colon \Delta^1 \to \mathcal{C}$ be a morphism in $\mathcal{C}$. Then a lifting problem*

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

in which the top composite is ϕ can be solved if ϕ is an equivalence in $\mathcal{C}$.

Proof. To prove the lifting property provided ϕ is an equivalence, we consider a diagram

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

where the top horizontal composite is an equivalence in $\mathcal{C}$, say ϕ, and wish to show the existence of the dashed arrow. We observe that the map $\Lambda^n_0 \to \Delta^n$ is isomorphic to the join-pushout product

$$\{0\} \star \Delta^{-2+n} \amalg_{\{0\}\star\partial\Delta^{-2+n}} \Delta^1 \star \partial\Delta^{n-2} \to \Delta^1 \star \Delta^{-2+n}$$

where Δ^{-2+n} short for $\Delta^{\{2,\dots,n\}}$. This was done in an earlier exercise. The map $\Delta^{\{0,1\}} \to \Lambda^n_0$ identifies with the canonical composite

$$\Delta^1 \to \Delta^1 \star \partial\Delta^{-2+n} \to \{0\} \star \Delta^{-2+n} \amalg_{\{0\}\star\partial\Delta^{-2+n}} \Delta^1 \star \partial\Delta^{n-2}.$$

The above diagram is, by adjunction, equivalent to the diagram

$$\begin{tikzcd}
\{0\} \arrow[rr] \arrow[d] & & \mathcal{C}_{/\Delta^{-2+n}} \arrow[d] \\
\Delta^1 \arrow[rr, "\phi'"] \arrow[urr, dashed] \arrow[drr, "\phi"'] & & \mathcal{C}_{/\partial\Delta^{-2+n}} \times_{\mathcal{D}_{/\partial\Delta^{-2+n}}} \mathcal{D}_{/\Delta^{-2+n}} \arrow[d] \\
 & & \mathcal{C}
\end{tikzcd}$$

We claim that all of the three top horizontal maps in the diagram

$$\begin{tikzcd}
\mathcal{C}_{/\Delta^{-2+n}} \arrow[r] & \mathcal{C}_{/\partial\Delta^{-2+n}} \times_{\mathcal{D}_{/\partial\Delta^{-2+n}}} \mathcal{D}_{/\Delta^{-2+n}} \arrow[r] \arrow[d] & \mathcal{C}_{/\partial\Delta^{-2+n}} \arrow[r] \arrow[d] & \mathcal{C} \\
& \mathcal{D}_{/\Delta^{-2+n}} \arrow[r] & \mathcal{D}_{/\partial\Delta^{-2+n}} &
\end{tikzcd}$$

are right fibrations and thus conservative by Proposition 5.3. The first one is the dual version of Theorem 4.23 part (1), the last one is explicitly stated in Corollary 4.24 part (1) and the middle map is a pullback of $\mathcal{D}_{/\Delta^{-2+n}} \to \mathcal{D}_{/\partial\Delta^{-2+n}}$ which is a right fibration by the same reasoning as the first map, hence also the pullback is a right fibration.

It thus follows from the assumption that ϕ is an equivalence that ϕ' is also an equivalence. Hence the dashed arrow exists by the fact that right fibrations are isofibrations by Proposition 5.7. □

Remark. One can ask whether the statement of the theorem can be promoted to an "if and only if". In other words, if any such lifting problem can be solved, can one conclude that ϕ is an equivalence. The answer to this question is no, as the following example shows. Suppose that $\mathcal{C}$ is an ordinary category and ϕ is a morphism which admits a left inverse ψ, but is not an isomorphism. Such categories exist, we leave explicit examples as an exercise to the reader. Consider the functor $\mathrm{N}(\mathcal{C}) \to \Delta^0$. We claim that any lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathrm{N}(\mathcal{C}) \arrow[d] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

can be solved: If $n \geq 3$ we have seen that any map from a horn extends (uniquely) to Δ^n, **??**. If $n = 2$, this is possible since ϕ admits a left inverse: The map $\Lambda^2_0 \to \mathrm{N}(\mathcal{C})$ in the diagram is uniquely determined by its restriction to $\Delta^{\{0,2\}}$, let us denote by g the corresponding morphism of $\mathcal{C}$. Then the string $(\phi, g \circ \psi)$ determines a 2-simplex in $\mathrm{N}(\mathcal{C})$ which extends the given map from Λ^2_0. But ϕ is not an isomorphism. However, we do have the following statement.

Corollary 5.9. *An inner fibration $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is conservative if and only if for every $n \geq 2$ and every lifting problem*

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "\phi"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

in which $p(\phi)$ is an equivalence in $\mathcal{D}$, there exists a solution.

Proof. Suppose that p is conservative. Then the assumption that $p(\phi)$ is an equivalence implies that ϕ itself is an equivalence, and hence any such lifting problem can be solved by Theorem 5.8. Conversely, suppose that any such lifting problem has a solution. To show that p is conservative, let us consider a morphism $\phi\colon \Delta^1 \to \mathcal{C}$ such that $p(\phi)$ is an equivalence in $\mathcal{D}$. Consider the map $\Lambda^2_0 \to \mathcal{C}$ whose restriction to $\Delta^{\{0,1\}}$ is ϕ and whose restriction to

$\Delta^{\{0,2\}}$ is the identity. Since ϕ becomes an equivalence in $\mathcal{D}$, there exists the solid arrows in the lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "\phi"] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

which can be solved by assumption. This provides a left inverse ψ of ϕ. It follows as in the proof of Proposition 5.3 that $p(\psi)$ is an equivalence. Running the same argument for ψ in place of ϕ we again find that ψ itself admits a left inverse and hence is an equivalence. Thus also ϕ is an equivalence and consequently, p is conservative. $\square$

Remark. The opposite of the inclusion $\Delta^{\{0,1\}} \to \Lambda^n_0$ is given by the map $\Delta^{\{n-1,n\}} \to \Lambda^n_n$. Since inner fibrations and conservative functors are invariant under passing to opposites, we find that the analogues statements of Theorem 5.8 and Corollary 5.9 where we replace the inclusion $\Delta^{\{0,1\}} \to \Lambda^n_0$ by the map $\Delta^{\{n-1,n\}} \to \Lambda^n_n$ hold as well.

Notice that the important direction in Theorem 5.8 for us is that such lifting problems can be solved provided p is conservative, as the next corollary shows.

Corollary 5.10. *∞-groupoids are Kan complexes.*

Proof. By definition, ∞-groupoids are precisely the ∞-categories $\mathcal{C}$ where the canonical map $\mathcal{C} \to *$ is conservative. Thus the claim follows from Corollary 5.9 $\square$

Using this we are now in the position to define the ∞-category of ∞-categories:

Definition 5.11. The ∞-category Cat_∞ of ∞-categories is the coherent nerve of the simplicial category with objects ∞-categories and hom simplicial sets given by the maximal ∞-groupoid inside the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. (This uses that forming the maximal ∞-groupoid is a monoidal functor from ∞-categories to Kan complexes: it is right adjoint to the inclusion and thus preserves products).

Definition 5.12. A functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a Joyal (or categorical) equivalence, if the corresponding 1-simplex in Cat_∞ is an equivalence in the sense of Definition 2.22.

Remark. Concretely, this means that there is a 2-simplex $\sigma\colon \Delta^2 \to \mathrm{Cat}_\infty$ such that $\sigma_{|\Delta^{\{0,1\}}} = f$ and $\sigma_{|\Delta^{\{0,2\}}} = \mathrm{id}_\mathcal{C}$. From Observation 2.64 we find that for $g = \sigma_{|\Delta^{\{1,2\}}}$ we have specified a 1-simplex in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\simeq$ from gf to $\mathrm{id}_\mathcal{C}$. In other words, f is an equivalence if and only if there is a functor $g\colon \mathcal{D} \to \mathcal{C}$ and natural equivalences $gf \simeq \mathrm{id}_\mathcal{C}$ and $fg \simeq \mathrm{id}_\mathcal{D}$.

Definition 5.13. Two functors $f, f'\colon \mathcal{C} \to \mathcal{D}$ are called (naturally) equivalent if the corresponding morphisms in Cat_∞ are equivalent in the sense of Definition 2.22.

Remark. Unwinding the definitions, we find that f is equivalent to f' precisely if there exists a natural equivalence $\tau\colon f \to f'$, i.e. τ is an equivalence between f and f' in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

We continue with more applications of Joyal's extension theorem.

Corollary 5.14. *Equivalences in an ∞-category $\mathcal{C}$ are represented precisely by those maps $\Delta^1 \to \mathcal{C}$ which extend over the canonical map $\Delta^1 \to J$.*

Proof. The fact that any map $\Delta^1 \to \mathcal{C}$ which extends over J is an equivalence is an exercise, see Exercise 62. Conversely, an equivalence is represented by a map $\Delta^1 \to \mathcal{C}^{\simeq} \subseteq \mathcal{C}$. To show that this map extends over the extension $\Delta^1 \to J$, it suffices to observe that $\mathcal{C}^{\simeq}$ is a Kan complex by Corollary 5.10 and that the map $\Delta^1 \to J$ is anodyne (its geometric realization is a homotopy equivalence as both are contractible). See also the proof of Lemma 8.4 for a purely simplicial proof of the fact that this map is anodyne. $\square$

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Corollary 5.15. *The pullback of a conservative inner fibration $\mathcal{C} \to \mathcal{D}$ along any functor $\mathcal{D}' \to \mathcal{D}$ of ∞-categories is again a conservative inner fibration.*

Proof. We use the lifting criterion for conservative inner fibrations established in Corollary 5.9 and consider the diagram

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C}' \arrow[r] \arrow[d, "q"] & \mathcal{C} \arrow[d, "p"] \\
 & \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & \mathcal{D}' \arrow[r] & \mathcal{D}
\end{tikzcd}$$

in which the composite $\Delta^1 \to \mathcal{D}'$ represents an equivalence. and want to show that a dashed arrow exists. Since p is conservative, the dotted arrow exists. Hence the dashed arrow exists, because the right square is a pullback, thus q is conservative. $\square$

Proposition 5.16. *A inner fibration $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is an isofibration if and only if the induced functor $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a Kan fibration.*

Proof. First, let us suppose that p is an isofibration. First, we show that the induced functor $p^{\simeq}\colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is also an isofibration. It is again an inner fibration by a similar argument that shows that $\mathcal{C}^{\simeq}$ is itself an ∞-category: Consider a lifting problem

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & \mathcal{D}^{\simeq} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

A dotted arrow exists since p is an inner fibration. We claim that the dotted arrow must already land in $\mathcal{C}^{\simeq}$ giving rise to the dashed arrow. This simply follows from the fact that its restriction to the spine lands in $\mathcal{C}^{\simeq}$ which implies that the whole n-simplex lies in $\mathcal{C}^{\simeq}$. Clearly, the further lifting property of isofibrations is satisfied by $p^{\simeq}$: The definition says that any lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{D}^{\simeq} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

has a solution (we simply spell out that certain 1-simplices are required to be equivalences).

Next, we observe that $p^{\simeq}$ is clearly conservative as is any functor from an ∞-groupoid. We thus now know that $p^{\simeq}\colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a conservative inner fibration, and an isofibration. To see that it is a Kan fibration, we first show that it is a left fibration. The left horn $\Delta^0 \to \Delta^1$ can be extended because $p^{\simeq}$ is an isofibration. To deal with the higher dimensional left horns, we use the criterion for conservative inner fibrations given by Corollary 5.9: It tells us that lifts exists provided certain edges of the horn map to equivalences. This condition is tautologically fulfilled because $\mathcal{C}^{\simeq}$ is an ∞-groupoid. Running the same argument (using the version of Joyal lifting with the right outer horn) we also find that $p^{\simeq}$ is a right fibration, and hence a Kan fibration.

The converse if obvious: The map $\{0\} \to \Delta^1$ is a horn inclusion, and thus admits a lift for $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ because we assume it to be a Kan fibration. This shows that p is an isofibration. $\square$

Proposition 5.17. *An inner fibration $\mathcal{C} \to \mathcal{D}$ is an isofibration if and only if it has the RLP wrt $\Delta^0 \to J$. In particular, the isofibrations are precisely the Joyal fibrations between ∞-categories according to Definition 3.22.*

Proof. It is clear that having the lifting property wrt $\Delta^0 \to J$ implies that the map is an isofibration. To show the converse we observe that every diagram

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ J \arrow[r] & \mathcal{D} \end{tikzcd}$$

factors through the subcategories of equivalences of $\mathcal{C}$ and $\mathcal{D}$, respectively. It thus suffices to show that a map $f\colon \mathcal{C} \to \mathcal{D}$ which is an isofibration if and only if the induced map $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a Kan fibration which was done in Proposition 5.16. $\square$

Corollary 5.18. *Isofibrations are stable under pullback.*

6. Pointwise criterion for natural equivalences

In this section we aim to prove the following theorem.

Theorem 6.1. *Let $K \to L$ be a map between simplicial sets which induces a bijection $K_0 \to L_0$. Then, for every ∞-category $\mathcal{C}$, the induced functor*

$$\mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$$

is conservative.

Corollary 6.2. *The canonical functor*

$$\mathrm{Fun}(K, \mathcal{C}) \to \prod_{x \in K_0} \mathcal{C}$$

is conservative. In other words, Let $f\colon \Delta^1 \to \mathrm{Fun}(K, \mathcal{C})$ be a natural transformation between functors $F, G\colon K \to \mathcal{C}$. If for all $x \in K$, the induced morphism $\Delta^1 \to \mathcal{C}$ is an equivalence, then f is an equivalence.

Lecture 23 – 28.01.2019. In order to prove Theorem 6.1 we will prove the following preliminary lemma.

Lemma 6.3. *Let $S \to T$ be a monomorphism such that $S_0 \to T_0$ is a bijection and let $\mathcal{C}$ be an ∞-category. Consider a diagram*

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(T, \mathcal{C}) \arrow[r] \arrow[d] & \prod_{T_0} \mathcal{C} \arrow[d, "\cong"] \\ \Delta^1 \arrow[r, "\phi"] \arrow[ur, dashed] & \mathrm{Fun}(S, \mathcal{C}) \arrow[r] & \prod_{S_0} \mathcal{C} \end{tikzcd}$$

in which the lower composite represents an equivalence. Then there exists a dashed arrow making the square commute.

Proof. We first claim that the set of monomorphisms $S \to T$ which induce a bijection on 0-simplices is a saturated class and that this saturated class is generated by the boundary inclusion $\partial\Delta^n \to \Delta^n$ for $n \geq 1$. This follows again from a relative skeletal argument, using that we only have to attach higher dimensional simplices if the inclusion induces a bijection on 0-simplices. Then we claim that the set of monomorphisms $S \to T$ which satisfy the conclusion of the lemma is a saturated class. To see the pushout property, consider a pushout of monomorphisms

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

and then a lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C}^T \arrow[r] \arrow[d] & \mathcal{C}^{T'} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dotted] \arrow[urr, dashed] & \mathcal{C}^S \arrow[r] & \mathcal{C}^{S'} \end{tikzcd}$$

in which the map $\Delta^1 \to \mathcal{C}^S$ represents a transformation which is objectwise an equivalence. Then the same holds for the further composite to $\mathcal{C}^{S'}$. So the dashed arrow to $\mathcal{C}^{T'}$ exists, and since the right square is a pullback, the dotted arrow to $\mathcal{C}^T$ exists as well. To show that the composition $S \to T \to U$ satisfies the conclusion of the lemma if $S \to T$ and $T \to U$ does, consider a lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[dd] & \mathcal{C}^U \arrow[d] \\ & \mathcal{C}^T \arrow[d] \\ \Delta^1 \arrow[r] \arrow[uur, dotted] \arrow[ur, dashed] & \mathcal{C}^S \end{tikzcd}$$

such that the lower map represents an natural transformation which is objectwise an equivalence. By assumption, a dashed lift to $\mathcal{C}^T$ exists, but to find the further dotted lift, we need to know that the dashed lift to $\mathcal{C}^T$ represents again a natural transformation which is objectwise an equivalence. This follows from the (needed!) assumption that all monomorphisms in the set under consideration induce bijections on 0-simplices. The case of transfinite compositions follows from the one just considered: Just the case of limit ordinals needs to be done, and in this case, any map from $\{0\}$ to a colimit over an ordinal factors through some earlier stage, which is then again dealt with by the successor step just done. Finally we need to argue that retracts behave well. So let $S \to T$ be a retract of $U \to V$ and assume $U \to V$ satisfies the conlusion of the lemma. We consider a diagram

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathcal{C}^T \arrow[r] \arrow[d] & \mathcal{C}^V \arrow[r] \arrow[d] & \mathcal{C}^T \arrow[d] \\
\Delta^1 \arrow[r] \arrow[urr, dashed] & \mathcal{C}^S \arrow[r] & \mathcal{C}^U \arrow[r] & \mathcal{C}^S
\end{tikzcd}$$

in which the right horizontal composites are the identity and where the map $\Delta^1 \to \mathcal{C}^S$ is objectwise an equivalence. This property remains so after passing to $\mathcal{C}^U$. Thus a dashed arrow exists, post composing this with the map $\mathcal{C}^V \to \mathcal{C}^T$ proves the claim.

It hence suffices to prove the conclusion of the lemma for the boundary inclusions of dimension ≥ 1. There is an induced lifting problem

$$\begin{tikzcd}
\Delta^1 \times \{j\} \arrow[r] & \{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^1 \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

in which the upper composite represents an equivalence in $\mathcal{C}$, and where j is any 0-simplex of $\partial\Delta^n$. The proof of this fact is very similar to the proof we gave in Proposition 6.6: One considers the same filtration of $\Delta^1 \times \Delta^n$ starting with

$$\{0\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n$$

by adding the missing simplices in a clever order, and then observing that all these simplices are either attached along inner horns, or an outer horn, but where the outer edge is labeled with an equivalence. Then one uses Joyal lifting for this case to deduce the claim. $\square$

Remark. Let us spell out the situation for $n = 1$ and $n = 2$.

We recall that the following lemma was used in the proof of Lemma 3.30 and its proof was deferred to Exercise 63 part (1). Since we will use the explicit retraction, we now include a proof of this lemma:

Lemma 6.4. *The inclusion $\Lambda^n_0 \to \Delta^n$ is a retract of the map*

$$\{0\} \times \Delta^n \cup \Delta^1 \times \Lambda^n_0 \to \Delta^1 \times \Delta^n.$$

Proof. Consider the maps

$$[n] \xrightarrow{s} [1] \times [n] \xrightarrow{r} [n]$$

given by $s(x) = (1, x)$ and $r(x, y) = y$ if $(x, y) \neq (0, 1)$ and $r(0, 1) = 0$.

As in Lemma 3.28, we have to show that

(1) $rs = \mathrm{id}$,
(2) $s(\Lambda^n_0) \subseteq \{0\} \times \Delta^n \cup \Delta^1 \times \Lambda^n_0$, and
(3) $r(\{0\} \times \Delta^n \cup \Delta^1 \times \Lambda^n_0) \subseteq \Lambda^n_0$.

Part (1) is obvious. (2) is obvious as well: $s(\Lambda^n_0) = \{1\} \times \Lambda^n_0 \subseteq \Delta^1 \times \Lambda^n_0$. To see (3), we first observe that the composite

$$[n] \xrightarrow{\iota_0} [1] \times [n] \xrightarrow{r} [n]$$

sends k to 0 if $k = 0, 1$ and k to k if $k \geq 2$. In particular, 1 is not in the image, and thus $r(\{0\} \times \Delta^n) \subseteq \Lambda^n_0$. To see that also $r(\Delta^1 \times \Lambda^n_0) \subseteq \Lambda^n_0$, consider a composite

$$[m] \xrightarrow{(\beta,\alpha)} [1] \times [n] \xrightarrow{r} [n]$$

where α represents an m-simple of $\Delta^1 \times \Lambda^n_0$. Obviously, the image of $r \circ (\beta, \alpha)$ is contained in Image$(\alpha) \cup \{0\}$, thus this composite represents again a simplex of Λ^n_0. $\square$

Proof of Theorem 6.1. Since the functor we consider is an inner fibration (restriction along a monomorphism) we can apply Corollary 5.9 to prove conservativity. I.e. we consider a diagram

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathcal{C}) \arrow[d] \\
 & \Delta^n \arrow[r] & \mathrm{Fun}(L, \mathcal{C})
\end{tikzcd}$$

and assume the composite $\Delta^1 \to \mathrm{Fun}(L, \mathcal{C})$ to be an equivalence. By Lemma 6.4, the map $\Lambda^n_0 \to \Delta^n$ is a retract of the pushout product map

$$\{0\} \times \Delta^n \amalg_{\{0\} \times \Lambda^n_0} \Delta^1 \times \Lambda^n_0 \to \Delta^1 \times \Delta^n$$

so that we can consider the diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r] \arrow[d] & \{0\} \times \Delta^n \amalg_{\{0\} \times \Lambda^n_0} \Delta^1 \times \Lambda^n_0 \arrow[r] \arrow[d] & \Lambda^n_0 \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathcal{C}) \arrow[d] \\
\Delta^n \arrow[r, "s"] & \Delta^1 \times \Delta^n \arrow[r, "r"] \arrow[urr, dashed] & \Delta^n \arrow[r] & \mathrm{Fun}(L, \mathcal{C})
\end{tikzcd}$$

so that it suffices to find the dashed arrow in this diagram. By adjunction this relates to the lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathrm{Fun}(K \times \Delta^n, \mathcal{C}) \arrow[d] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(L \times \Delta^n \amalg_{L \times \Lambda^n_0} K \times \Lambda^n_0, \mathcal{C})
\end{tikzcd}$$

Now notice that the right vertical map is restriction along a monomorphism which induces a bijection on 0-simplices. To apply Lemma 6.3 we need to check that for every 0-simplex of Δ^n, the composite

$$\Delta^1 \to \Delta^1 \times \Delta^n \to \mathrm{Fun}(L, \mathcal{C})$$

represents an objectwise equivalence in $\mathcal{C}$. This of course relies on the explicit choice of the maps s and r which we have specified in Lemma 6.4.

The thing to observe is that for $0 \leq k \leq n$, the inclusion $\Delta^1 \to \Delta^1 \times \Delta^n$ given by sending i to (i, k), followed by the map $r\colon \Delta^1 \times \Delta^n$, is

(1) either the map which is constant at k (if $k \neq 1$) in which case clearly the map

$$\Delta^1 \to \Delta^1 \times \Delta^n \to \Delta^n \to \mathrm{Fun}(K_0, \mathcal{C})$$

is also constant, and thus objectwise an equivalence;

(2) or the map is given by the canonical inclusion $\Delta^{\{0,1\}} \to \Delta^n$ (if $k = 1$), in which case the resulting morphism of $\mathrm{Fun}(L, \mathcal{C})$ is an objectwise equivalence by assumption. In either case, a lift exists, so the theorem is proven.

□

Corollary 6.5. *Proposition 3.41 holds true. More precisely, given a monomorphism $K \to L$ of simplicial sets which is a bijection on 0-simplices, then the fibre of the induced map $\mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ over any point $\Delta^0 \to \mathrm{Fun}(K, \mathcal{C})$ is an ∞-groupoid. In particular, for an ∞-category $\mathcal{C}$, the ∞-category $\mathrm{map}_{\mathcal{C}}(x, y)$ is an ∞-groupoid.*

Proof. By Theorem 6.1 the functor $\mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ is an conservative inner fibration, and so the pullback along $\Delta^0 \to \mathrm{Fun}(K, \mathcal{C})$ is a conservative inner fibration as well (by Corollary 5.15). But $X \to *$ is a conservative inner fibration if and only if X is an ∞-groupoid. □

Proposition 6.6. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration and let $i\colon K \to L$ be a monomorphism of simplicial sets. Suppose that*

(1) *p is an isofibration, or*
(2) *i induces a bijection on 0-simplices.*

Then the induced functor

$$\mathcal{C}^L \to \mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L$$

is an isofibration.

Proof. By Theorem 3.32, we know that this map is an inner fibration. It thus suffices to show that any lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C}^L \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L \end{tikzcd}$$

in which the bottom horizontal map is an equivalence, has a solution which is again an equivalence. By Theorem 6.1 this is the case if for every object of L, the induced morphism in $\mathcal{C}$ is an equivalence. In particular, we see that if $K \to L$ is a bijection on 0-simplices, the above right vertical map is conservative, so that any lift of an equivalence is automatically an equivalence. By adjunction, this is lifting problem is thus equivalent to the lifting problem

$$\begin{tikzcd} \{0\} \times L \amalg_{\{0\} \times K} \Delta^1 \times K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^1 \times L \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

in which the composite of the dashed map with the map $\Delta^1 \times \Delta^0 \to \Delta^1 \times L$ given by an arbitrary object of L is an equivalence. We claim that the set of morphisms $K \to L$ for which the conclusion holds is a saturated class, a very similar argument was worked out in Lemma 6.3, we leave this version as Exercise 86. It hence suffices to show it for the boundary inclusions $\partial\Delta^n \to \Delta^n$ in case (1) and the boundary inclusions with $n \geq 1$ in case (2). Let us do case (1) first: If $n = 0$, then we have the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

where the lower horizontal map is an equivalence. A dashed arrow representing an equivalence in $\mathcal{C}$ exists because $\mathcal{C} \to \mathcal{D}$ is an isofibration by assumption. For the remaining cases $n \geq 1$ we we need to consider diagrams of the form

$$\begin{tikzcd} \{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

and we will not use that p is an isofibration so that the following argument settles both the remaining cases of (1) and (2). One constructs a filtration on $\Delta^1 \times \Delta^n$, starting with $\{0\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n$ by adding the missing simplices. As in the proof of Lemma 6.3, one finds that the relevant simplices are either attached along inner horn inclusions, or along outer horns where one outer edge is labelled with an equivalence. Using Joyal lifting, the proposition follows. □

Corollary 6.7. *Let $K \to L$ be a monomorphism of simplicial sets which induces a bijection on 0-simplices. Then the map*

$$\{0\} \times L \amalg_{\{0\}\times K} J \times K \to J \times L$$

has the LLP with respect to inner fibrations between ∞-categories.

Proof. Let us consider an inner fibration $p\colon \mathcal{C} \to \mathcal{D}$ and a lifting problem

$$\begin{tikzcd} \{0\} \times L \amalg_{\{0\}\times K} J \times K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ J \times L \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

This lifting problem is, by adjunction equivalent to the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \mathcal{C}^L \arrow[d] \\ J \arrow[r] \arrow[ur, dashed] & \mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L \end{tikzcd}$$

and by Proposition 6.6 the right vertical map is an isofibration. The lifting problem can thus be solved by Proposition 5.17. □

Corollary 6.8. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and $K \to L$ be a monomorphism of simplicial sets. Then*

$$(\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L)^{\simeq} = (\mathcal{C}^K)^{\simeq} \times_{(\mathcal{D}^K)^{\simeq}} (\mathcal{D}^L)^{\simeq}.$$

Proof. Both simplicial sets are subsets of the pullback $\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L$, thus we easily find the inlcusion "$\subseteq$". To show the converse, it suffices to prove that the right hand side is in fact an ∞-groupoid, as then it certainly contains the smallest ∞-groupoid contained in $\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L$, which is the left hand side. By Proposition 6.6 applied to the isofibration $\mathcal{D} \to \Delta^0$, we find that $\mathcal{D}^L \to \mathcal{D}^K$ is also an isofibration. By Proposition 5.16, the map $(\mathcal{D}^L)^{\simeq} \to (\mathcal{D}^K)^{\simeq}$ is hence a Kan fibration between Kan complexes, so that any pullback along a map from a Kan complex is again a Kan complex. $\square$

Proposition 6.9. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then f is a Joyal equivalence if and only if for every ∞-category $\mathcal{E}$, the induced map*

$$f^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is a Joyal equivalence.

Proof. To prove the "only if", choose an inverse $\mathcal{D} \to \mathcal{C}$ and choose natural transformations (η_0, η_1) whitnessing that g is inverse to f. Then the quadruple $(f^*, g^*, \eta_0^*, \eta_1^*)$ determines an equivalence between $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$.

We now show that f is in fact a Joyal equivalence if for all ∞-categories $\mathcal{E}$, the functor f^* is a Joyal equivalence. We obtain, by Exercise 88, a bijection

$$f^*\colon \pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \xrightarrow{\cong} \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq})$$

Now consider the case where $\mathcal{E} = \mathcal{C}$. Then this bijection shows the existence of a functor $g\colon \mathcal{D} \to \mathcal{C}$ such that $f^*(g) = gf$ is equivalent to $\mathrm{id}_{\mathcal{C}}$. Now taking $\mathcal{E} = \mathcal{D}$, we see that

$$f^*(fg) = fgf \simeq f$$

and thus that $fg \simeq \mathrm{id}_{\mathcal{D}}$ as needed. $\square$

Definition 6.10. A map $f\colon X \to Y$ between simplicial sets is called a Joyal (or categorical) equivalence if for all ∞-categories $\mathcal{C}$, the induced map

$$\mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$$

is a Joyal equivalence between ∞-categories.

Observation 6.11.
(1) This does not change the definition if X and Y are already ∞-categories by Proposition 6.9.
(2) A Joyal equivalence between Kan complexes is precisely a homotopy equivalence.

Proposition 6.12. *A trivial Kan fibration $f\colon X \to Y$ is a Joyal equivalence.*

Proof. Consider the following pullback squares of simplicial sets

$$\begin{tikzcd} \mathrm{Hom}_{/Y}(Y,X) \arrow[r] \arrow[d] & \mathrm{Hom}(Y,X) \arrow[d, "f_*"] \\ \Delta^0 \arrow[r, "\mathrm{id}_Y"] & \mathrm{Hom}(Y,Y) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_{/Y}(X,X) \arrow[r] \arrow[d] & \mathrm{Hom}(X,X) \arrow[d, "f_*"] \\ \Delta^0 \arrow[r, "f"] & \mathrm{Hom}(X,Y) \end{tikzcd}$$

The right vertical maps in each square are trivial fibrations, thus so are their pullbacks. But a trivial fibration over Δ^0 has a source a contractible Kan complex. Choose a 0-simplex $s \in \mathrm{Hom}_{/Y}(Y,X)$ so that $f_*(s) = fs = \mathrm{id}_Y$: Recall that f_*, as every trivial fibration, admits a section, and that any 0-simplex $s \in \mathrm{Hom}(Y,X)$ with $fs = \mathrm{id}_Y$ lies in $\mathrm{Hom}_{/Y}(Y,X)$. Observe that thus sf determines a 0-simplex in $\mathrm{Hom}_{/Y}(X, X)$ and thus that there must be a 1-simplex connecting it to the identity (again because these spaces of sections are contractible). Explicitly, we can find a map

$$\Delta^1 \to \mathrm{Hom}_{/Y}(X, X)$$

whose restriction to 0 is sf and whose restriction to 1 is id_X. For an arbitrary ∞-category we can compose this with the canonical map

$$\mathrm{Hom}_{/Y}(X, X) \to \mathrm{Hom}(X, X) \to \mathrm{Hom}(\mathcal{C}^X, \mathcal{C}^X)$$

and see that the resulting map

$$\Delta^1 \to \mathrm{Hom}_{/Y}(X, X) \to \mathrm{Hom}(\mathcal{C}^X, \mathcal{C}^X)^{\simeq}$$

determines a natural equivalence between $(sf)^*$ and id. Since $(fs)^* = \mathrm{id}$ we have shown that f^* and s^* determine inverse equivalences of $\mathcal{C}^X$ and $\mathcal{C}^Y$ for any ∞-category Y, so that f^* is a Joyal equivalence and thus f itself is a Joyal equivalence. □

Corollary 6.13. *An inner anodyne map is a Joyal equivalence.*

Proof. Let $i\colon A \to B$ be an inner anodyne map and $\mathcal{C}$ be an ∞-category. We need to show that $\mathcal{C}^B \to \mathcal{C}^A$ is a Joyal equivalence. By Theorem 3.32 part (2) this map is a trivial fibration, and thus by Proposition 6.12 the claim is shown. □

Corollary 6.14. *Every simplicial set is Joyal equivalent to an ∞-category.*

Proof. By the small object argument, Proposition 3.9, for any simplicial set X one can factor the map $X \to *$ into an inner anodyne map followed by an inner fibration. This produces a map $X \to \mathcal{C}$ where $\mathcal{C}$ is an ∞-category and $X \to \mathcal{C}$ is inner anodyne and thus a Joyal equivalence by Corollary 6.13. □

Lemma 6.15. *A Kan fibration $p\colon X \to Y$ between Kan complexes which induces a surjection on π_0 is in fact surjective on 0-simplices. Likewise, a Kan fibration between Kan complexes which induces an injection on π_0 and a surjection on π_1 has the property that any lifting problem*

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

can be solved.

Proof. A homotopy equivalence induces a bijection on simplicial path components. Thus for a 0-simplex $y\colon \Delta^0 \to Y$, we find a commutative diagram

$$\begin{tikzcd} \{0\} \arrow[r, "x'"] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r, "h"] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $h_{|\{1\}} = y$. Since p is a Kan fibration, a dashed arrow exists. Its restriction to $\{1\}$ provides a preimage of y in X.

To see the second claim, pick two objects x, x' of X which give rise to the top horizontal map in the commutative diagram

$$\begin{tikzcd} \partial\Delta^1 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r, "h"] \arrow[ur, dashed] & Y \end{tikzcd}$$

which we wish to show admits a dashed arrow. The assumption that p induces a bijection on π_1 implies that the same is true not for loops at a point of x but at homotopy classes of paths from x to x' (This is where we use that p induces an injection on path components: The assumptions imply that there exists a path from x to x' in X which we can use to compare the set of homotopy classes of paths from x to x' to the set of homotopy classes of loops at x). One can thus find a path $\alpha\colon \Delta^1 \to X$ which becomes equivalent to h after applying p. This means that we can find a 2-cell $\sigma\colon \Delta^2 \to Y$ such that $\sigma_{|\Delta^{\{0,1\}}} = p(\alpha)$, $\sigma_{|\Delta^{\{0,2\}}} = h$ and $\sigma_{|\Delta^{\{1,2\}}} = \mathrm{id}_{x'}$. Since we can lift both $p(\alpha)$ and the identity, we obtain a lifting problem

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed] & Y \end{tikzcd}$$

which can be solved as p is a Kan fibration. The resulting map solves the original lifting problem. ☐

Remark. In fact, a Kan fibration which is also a weak equivalence is a trivial fibration. This is a classical fact in simplicial homotopy theory. We will deduce it from the previous lemma together with the following observation:

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Lemma 6.16. *A functor $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a trivial fibration if and only if it is a Joyal equivalence and an isofibration.*

Proof. Trivial fibrations are Joyal equivalences by Proposition 6.12 and isofibrations because a trivial fibration is a left fibration which in turn is a conservative isofibration by Proposition 5.7.

To see the converse, we want to show that for any monomorphism $K \to L$ and any lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ L \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

has a solution. This is equivalent to the lifiting problem

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C}^L \arrow[d, "\overline{p}"] \\ \Delta^0 \arrow[r] \arrow[ur, dashed] & \mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L \end{tikzcd}$$

which amounts to showing the surjectivity of the right hand map on 0-simplices. To prove this, we may pass to the underlying groupoid cores, as this does not change the 0-simplices.

Since p is an isofibration, so is $\overline{p}$, by Proposition 6.6. In particular, the induced map

$$(\mathcal{C}^L)^{\simeq} \xrightarrow{\overline{p}} (\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{C}^L)^{\simeq}$$

is a Kan fibration. Furthermore, by Corollary 6.8 the latter is further equal to (along the canonical map) the pullback of the groupoid cores, so that we find that the map

$$(\mathcal{C}^L)^{\simeq} \to (\mathcal{C}^K)^{\simeq} \times_{(\mathcal{D}^K)^{\simeq}} (\mathcal{D}^L)^{\simeq}$$

is a Kan fibration. We wish to show that it is surjective on 0-simplices. By Lemma 6.15 it suffices to show that this map induces a surjection on π_0. For this we observe that since $\mathcal{C} \to \mathcal{D}$ is a Joyal equivalence, so are the maps

$$\mathcal{C}^K \to \mathcal{D}^K \text{ and } \mathcal{C}^L \to \mathcal{D}^L$$

by an exercise, Exercise 91. Hence, as they are isofibrations, passing to groupoid cores gives us Kan fibrations, which are in addition Joyal equivalences.

We then consider the diagram

$$\begin{tikzcd}
(\mathcal{C}^L)^{\simeq} \arrow[r] & (\mathcal{C}^K)^{\simeq} \times_{(\mathcal{D}^K)^{\simeq}} (\mathcal{D}^L)^{\simeq} \arrow[r] \arrow[d] & (\mathcal{D}^L)^{\simeq} \arrow[d] \\
 & (\mathcal{C}^K)^{\simeq} \arrow[r] & (\mathcal{D}^K)^{\simeq}
\end{tikzcd}$$

and note that all maps are Kan fibrations and that both the composite and the lower horizontal map are in Joyal equivalences, and thus homotopy equivalences. We wish to show that the second horizontal top map induces a bijection on π_0, so that the first map induces a surjection on π_0. For this we observe that the lower horizontal map has the RLP wrt $\partial\Delta^1 \to \Delta^1$ by Lemma 6.15. As a pullback, so does the second top horizontal map. This implies that this map is injective on π_0.

Using Lemma 6.15, we find that the map

$$(\mathcal{C}^L)^{\simeq} \to (\mathcal{C}^K)^{\simeq} \times_{(\mathcal{D}^K)^{\simeq}} (\mathcal{D}^L)^{\simeq}$$

induces a surjection on 0-simplices. This proves the lemma. □

Corollary 6.17. *A Kan fibration $p\colon X \to Y$ between Kan complexes is a trivial Kan fibration if and only if it is a homotopy equivalence. In particular, among Kan fibrations, the trivial Kan fibrations satisfy the 3-for-2 property.*

Proof. A Kan fibration which is a homotopy equivalence is an isofibration which is a Joyal equivalence. The in particular follows from the 3-for-2 property for homotopy equivalences. □

Remark. We hence find that given an isofibration which is in addition a Joyal equivalence, $p\colon \mathcal{C} \to \mathcal{D}$, the induced map

$$(\mathcal{C}^L)^{\simeq} \xrightarrow{\overline{p}} (\mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{C}^L)^{\simeq}$$

is in fact a trivial fibration: We claim that in the following composite

$$(\mathcal{C}^L)^{\simeq} \to (\mathcal{C}^K)^{\simeq} \times_{(\mathcal{D}^K)^{\simeq}} (\mathcal{D}^L)^{\simeq} \to (\mathcal{D}^L)^{\simeq}$$

both the composite, and the latter map are trivial fibrations: The composite is a Kan fibration which is a homotopy equivalence and thus a trivial fibration. The latter map is a pullback of the map $(\mathcal{C}^K)^\simeq \to (\mathcal{D}^K) \simeq$ which is a trivial Kan fibration by the same reasoning. By 3-for-2 for trivial fibrations (among Kan fibrations) it follows that the map

$$(\mathcal{C}^L)^\simeq \to (\mathcal{C}^K)^\simeq \times_{(\mathcal{D}^K)^\simeq} (\mathcal{D}^L)^\simeq$$

is also a trivial fibration.

7. Fully faithful and essentially surjective functors

The goal of this section is to prove that as in ordinary category theory, functor which are essentially surjective and fully faithful are in fact invertible. The proof we present follows an argument which we learned from Gijs Heuts. It uses some results from classical simplicial homotopy theory.

We wish to emphasize however, that the proof of this fact in ordinary category theory, which we presented in Exercise 37, goes through verbatim in ∞-categories once enough technology is established: The main thing to prove is that an essentially surjective and fully faithful functor admits a right adjoint. This will then be an inverse. Of course, for this argument to make sense we will have to speak of adjunctions and the way we want to do it will require the straightening–unstraightening equivalence. We will come back to this approach in a later chapter.

Definition 7.1. A functor $f\colon \mathcal{C} \to \mathcal{D}$ is called fully faithful if for all objects $x, y \in \mathcal{C}$, the induced map $\mathrm{map}_{\mathcal{C}}(x,y) \to \mathrm{map}_{\mathcal{D}}(fx, fy)$ is a Joyal equivalence, i.e. a homotopy equivalence.

Definition 7.2. A functor $f\colon \mathcal{C} \to \mathcal{D}$ is called essentially surjective if the induced functor $hf\colon h\mathcal{C} \to h\mathcal{D}$ is. In other words, if for every object $d \in \mathcal{D}$, there exists an object $x \in \mathcal{C}$ and an equivalence $fx \simeq d$ in $\mathcal{D}$.

Lemma 7.3. *Let $f, g\colon \mathcal{C} \to \mathcal{D}$ be functors and let $\tau\colon \Delta^1 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be a natural transformation from f to g. Then the diagram*

$$\begin{array}{ccc}
\mathrm{map}_{\mathcal{C}}(x,y) & \longrightarrow & \mathrm{map}_{\mathcal{D}}(fx,fy) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{D}}(gx,gy) & \longrightarrow & \mathrm{map}_{\mathcal{D}}(fx,gy)
\end{array}$$

commutes up to homotopy. If τ is a natural equivalence, then the lower horizontal and right vertical maps are equivalences. In particular, if $\mathcal{D} = \mathcal{C}$ and $g = \mathrm{id}$, we find that if f is equivalent to $\mathrm{id}_{\mathcal{C}}$, then the map

$$\mathrm{map}_{\mathcal{C}}(x,y) \to \mathrm{map}(fx,fy)$$

is a homotopy equivalence.

Proof. Recall that there is a functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{\Delta^1}, \mathcal{D}^{\Delta^1})$ induced by postcomposition. The given transformation τ thus induces a functor $\Delta^1 \to \mathrm{Fun}(\mathcal{C}^{\Delta^1}, \mathcal{D}^{\Delta^1})$ which is (by adjunction) a functor

$$\mathcal{C}^{\Delta^1} \to \mathcal{D}^{\Delta^1 \times \Delta^1}.$$

Unravelling this construction we find that this functor sends a morphism $x \to y$ to the diagram

$$\begin{tikzcd} fx \arrow[r, "\tau_x"] \arrow[d] & gx \arrow[d] \\ fy \arrow[r, "\tau_y"] & gy \end{tikzcd}$$

We hence find a diagram

$$\begin{tikzcd} \mathcal{C}^{\Delta^1} \arrow[r] & \mathcal{D}^{\Delta^1 \times \Delta^1} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{D}^{\Delta^1} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1} \arrow[d] \\ & \mathcal{D}^{\Delta^1} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1} \arrow[r] & \mathcal{D}^{\Delta^1} \end{tikzcd}$$

here the horizontal functor from $\mathcal{D}^{\Delta^1 \times \Delta^1}$ is given by restriction along the map $\Lambda^2_1 \to \Delta^1 \times \Delta^1$ which singles out one corner of the square and the vertical functor is restriction along the map $\Lambda^2_1 \to \Delta^1 \times \Delta^1$ which singles out the other corner. The diagonal map is given by restriction along the inclusion $\Delta^1 \to \Delta^1 \times \Delta^1$ sending 0 to $(0,0)$ and 1 to $(1,1)$. The remaining two functors are given composition. Both of the triangles commute up to a natural equivalence (one has to choose a section of the trivial fibration $\mathcal{D}^{\Delta^2} \to \mathcal{D}^{\Lambda^2_1}$).

Now we fix two objects x and y of $\mathcal{C}$ and consider the inclusion $\mathrm{map}_{\mathcal{C}}(x,y) \to \mathcal{C}^{\Delta^1}$. By inspection we see that the diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,y) \arrow[r, dashed] \arrow[d] & \{\tau_x\} \times \mathrm{map}_{\mathcal{D}}(gx,gy) \arrow[d] \\ \mathcal{C}^{\Delta^1} \arrow[r] & \mathcal{D}^{\Delta^1} \times_{\mathcal{D}} \mathcal{D}^{\Delta^1} \end{tikzcd}$$

commutes, i.e. that the lower composite factors as the dashed arrow indicates.

The same holds for the restriction of the vertical map to $\mathrm{map}_{\mathcal{C}}(x,y)$ so that in total we obtain a diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \{\tau_x\} \times \mathrm{map}_{\mathcal{D}}(gx,gy) \arrow[d] \\ \mathrm{map}_{\mathcal{D}}(fx,fy) \times \{\tau_y\} \arrow[r] & \mathrm{map}_{\mathcal{D}}(fx,gy) \end{tikzcd}$$

which commutes up to a natural equivalence as needed. □

Proposition 7.4. *A Joyal equivalence between ∞-categories is fully faithful and essentially surjective.*

Proof. By Exercise 90 we see that the functor $h\mathcal{C} \to h\mathcal{D}$ is an equivalence and thus essentially surjective, so that $\mathcal{C} \to \mathcal{D}$ is essentially surjective according to Definition 7.2. To show fully faithfulness, choose an inverse g of f. Then, for pair of objects $x, y \in \mathcal{C}$ we get

$$\mathrm{map}_{\mathcal{C}}(x,y) \to \mathrm{map}_{\mathcal{D}}(fx,fy) \to \mathrm{map}_{\mathcal{C}}(gfx,gfy) \to \mathrm{map}_{\mathcal{D}}(fgfx,fgfy)$$

Since the functor gf is naturally equivalent to $\mathrm{id}_{\mathcal{C}}$ and the functor fg is naturally equivalent to $\mathrm{id}_{\mathcal{D}}$ we see that the first two maps compose to an equivalence and the latter two compose to an equivalence. This implies that the middle map is itself an equivalence (it has both a left and a right inverse) and thus that the first map is also an equivalence (it is a right-inverse of an equivalence). $\square$

Remark. Alternatively, one can argue as follows: We have seen that every ∞-category gives rise to a category enriched in the homotopy category of Kan complexes because composition is well-defined up to homotopy see **??**. Having this, we find that a functor $f\colon \mathcal{C} \to \mathcal{D}$ is a Joyal equivalence if and only if the induced functor of $h(\mathrm{Kan})$ enriched categories is an equivalence. Likewise, it is fully faithful and essentially surjective if and only if the induced functor of $h(\mathrm{Kan})$ enriched categories is fully faithful. From this analysis we deduce that if f is naturally equivalent to g, then f is fully faithful if and only if g is. In the above argument, we can apply this to gf which is equivalent to id and thus must be fully faithful itself, so that the required map is in fact a homotopy equivalence.

Lemma 7.5. *The inclusion of a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is a fully faithful functor.*

Proof. We first observe that for every ∞-category $\mathcal{D}$, the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C}_0)$ is the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ on those functors which factor through $\mathcal{C}_0 \subseteq \mathcal{C}$, see Exercise 48. Having this we deduce that the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathcal{C}_0) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \mathcal{C}_0 \times \mathcal{C}_0 \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

is a pullback. This implies the lemma by passing to fibres over objects (x, y) of $\mathcal{C}_0 \times \mathcal{C}_0$. $\square$

Lemma 7.6. *Let $\mathcal{C}$ be an ∞-category and let x, y be objects of $\mathcal{C}$. Then the map*

$$\mathrm{map}_{\mathcal{C}^{\simeq}}(x, y) \to \mathrm{map}_{\mathcal{C}}(x, y)$$

is the inclusion of those path components whose points are equivalences of $\mathcal{C}$.

Proof. We will show that any map $\Delta^n \to \mathrm{map}_{\mathcal{C}}(x, y)$ lifts to $\mathrm{map}_{\mathcal{C}^{\simeq}}(x, y)$ if for every $i \in \Delta^n$, the corresponding morphism from x to y is an equivalence. For this we consider a map $\Delta^n \to \mathrm{map}_{\mathcal{C}}(x, y)$ such that for all $i \in \Delta^n$, the restriction of its adjoint map

$$\{i\} \times \Delta^1 \to \Delta^n \times \Delta^1 \to \mathcal{C}$$

represents an equivalence in $\mathcal{C}$. Then we observe that, for all $\epsilon = 0, 1$ we also have that

$$\Delta^n \times \{\epsilon\} \to \Delta^n \times \Delta^1 \to \mathcal{C}$$

is constant (at either x or y) and hence also represents an equivalence of $\mathcal{C}$. Since all morphisms in $\Delta^n \times \Delta^1$ are composites of ones of the previous form, we deduce that the map

$$\Delta^n \times \Delta^1 \to \mathcal{C}$$

factors through $\mathcal{C}^{\simeq}$. This shows the claim. $\square$

We now aim to prove the converse of this proposition. In order to do so we shall need the following preparatory statements.

Lemma 7.7. *If $f\colon \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective, then so is $f^{\simeq}\colon \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$.*

Proof. Essentially surjectivity is clear as $\mathcal{C}^{\simeq}$ and $\mathcal{C}$ have the same objects. Fully faithfulness is also fine: As $\mathrm{map}_{\mathcal{C}^{\simeq}}(x,y) \subseteq \mathrm{map}_{\mathcal{C}}(x,y)$ is a collection of path components, it suffices to know the conclusion of the lemma in the case where $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories (by means of the homotopy category). This is an explicit and easy check. $\square$

We will make use of the following fundamental property of Kan fibrations, see for instance [GJ09, Lemma 7.3].

Lemma 7.8. *Let $f\colon X \to Y$ be a Kan fibration between Kan complexes. Let x be a point in X and let F be the fibre of f over the point $y = f(x)$. Then there exists a long exact sequence*

$$\dots \longrightarrow \pi_{n+1}(Y,y) \xrightarrow{\partial} \pi_n(F,x) \longrightarrow \pi_n(X,x) \xrightarrow{f_*} \pi_n(Y,y) \xrightarrow{\partial} \dots \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y)$$

natural in morphisms of fibrations, which is an exact sequence of groups for $n \geq 1$ and is an exact sequence of pointed sets for $n = 0$.

We will make use of Whitehead's theorem:

Proposition 7.9. *Let $f\colon X \to Y$ be a map between Kan complexes which induces a bijection on path components. Then f is a homotopy equivalence if and only if for all points x in X and all $n \geq 1$, the induced map*

$$f_*\colon \pi_n(X,x) \to \pi_n(Y,y)$$

is a bijection.

Corollary 7.10. *A fully faithful and essentially surjective functor $f\colon X \to Y$ between Kan complexes is a homotopy equivalence.*

Proof. The functor induces an equivalence of homotopy categories so we find that the map f induces a bijection $\pi_0(X) \to \pi_0(Y)$. We wish to show that for all x in X and all $n \geq 1$, also the induced map

$$\pi_n(X,x) \to \pi_n(Y,y)$$

is a bijection, where $y = f(x)$. For this we consider the following diagram

$$\begin{tikzcd}
\mathrm{map}_X(x,x) \arrow[r] \arrow[d] & P_x(X) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, X) \arrow[d] \\
\Delta^0 \arrow[r, "(\mathrm{id},x)"] & \Delta^0 \times X \arrow[r, "(x,\mathrm{id}_X)"] & X \times X
\end{tikzcd}$$

Since right most vertical map is an isofibration between Kan complexes, it is a Kan fibration. Hence so is the middle vertical map. This construction is clearly natural in X. We thus obtain a diagram of Kan fibre sequences

$$\begin{tikzcd}
\mathrm{map}_X(x,x) \arrow[r] \arrow[d] & P_x(X) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{map}_Y(y,y) \arrow[r] & P_y(Y) \arrow[r] & Y
\end{tikzcd}$$

where the vertical maps are induced by the map $f\colon X \to Y$. We will now show that $P_x(X)$ is contractible: This is because it sits also in the pullback diagram

$$\begin{tikzcd} P_x(X) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, X) \arrow[d, "s"] \\ \Delta^0 \arrow[r, "x"] & X \end{tikzcd}$$

where t stands for the target map, i.e. the map obtained from restriction along $\{0\} \to \Delta^1$. Since this map is anodyne and X is a Kan complex, the resulting map $\mathrm{Fun}(\Delta^1, X) \to X$ is a trivial fibration, and hence so is the map $P_x(X) \to \Delta^0$. This implies that $P_x(X)$ is contractible.

We hence obtain that for all $n \geq 1$ there is a commutative diagram

$$\begin{tikzcd} \pi_n(X,x) \arrow[r] \arrow[d] & \pi_{n-1}(\mathrm{map}_X(x,x), \mathrm{id}_x) \arrow[d] \\ \pi_n(Y,y) \arrow[r] & \pi_{n-1}(\mathrm{map}_Y(y,y), \mathrm{id}_y) \end{tikzcd}$$

in which the horizontal maps are isomorphisms by the long exact sequence of Lemma 7.8 and the contractability of $P_x(X)$ and $P_y(Y)$. Since f is fully faithful, the map $\mathrm{map}_X(x,x) \to \mathrm{map}_Y(y,y)$ is a homotopy equivalence, and hence induces bijections on all homotopy groups. It follows that the left vertical map is also an isomorphism. □

Corollary 7.11. *A fully faithful and essentially surjective functor between ∞-categories induces a homotopy equivalence on groupoid cores.*

Proof. By Lemma 7.7 the induced functor on groupoid cores is still essentially surjective and fully faithful so that Corollary 7.10 applies. □

For Lemma 7.13 we will need Reedy's lemma, which is the following statement. It holds in fact in any model category. For a proof for general model categories we refer to [Hir03, Prop. 13.1.2].

Lemma 7.12. *Consider a pullback diagram*

$$\begin{tikzcd} C \times_A B \arrow[r] \arrow[d] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "\simeq", "f"'] & A \end{tikzcd}$$

in which p is a fibration and where f is a weak equivalence between fibrant objects. Then also the map $C \times_A B \to B$ is a weak equivalence.

Remark. For simplicial sets, let us assume that we have long exact sequences in homotopy groups for Kan fibrations at our disposal: In this case we easily find from a diagram chase using the long exact sequence for the vertical fibrations that the induced map

$$\pi_i(C \times_A B, x) \to \pi_i(B, x')$$

is a bijection for every basepoint x of $C \times_A B$ and for every $i \geq 1$. To prove Reedy's lemma in the case of interest it hence remains to see that the map $C \times_A B \to B$ induces a bijection on path components. The arguments here are very similar to the ones used in the proof of Lemma 6.15.

Let's prove surjectivity of this map first: pick a point b in B representing a class $[b] \in \pi_0(B)$ and consider the point $p(b)$ in A. Since $C \to A$ induces a bijection on path components, we can find a point c in C and a path $\Delta^1 \to A$ connecting $f(c)$ to $p(b)$. This gives, as in Lemma 6.15, a lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & B \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

which can be solved as p is a Kan fibration. We thus find that there exists a b' in B such that $p(b') = f(c)$ and such that $[b'] = [b]$ in $\pi_0(B)$. The pair (c, b') thus determines an element of $\pi_0(C \times_A B)$ which is sent to $[b]$ in $\pi_0(B)$. This shows that the map $\pi_0(C \times_A B) \to \pi_0(B)$ is surjective.

To show injectivity, consider two points (c, b) and (c', b') of $C \times_A B$ and assume that there is a path $\alpha \colon \Delta^1 \to B$ connecting b and b' in B. Then $p(\alpha) \colon \Delta^1 \to A$ connects $p(b) = f(c)$ to $p(b') = f(c')$. Since the map f is a homotopy equivalence, there is a path $\beta \colon \Delta^1 \to C$ such that $f_*(\beta)$ is equivalent to $p_*(\alpha)$. Precisely, we find a 2-cell $\sigma \colon \Delta^2 \to A$ such that

(1) $\sigma_{|\Delta^{\{0,1\}}} = f_*(\beta)$,
(2) $\sigma_{|\Delta^{\{1,2\}}} = \mathrm{id}_{f(c')}$, and
(3) $\sigma_{|\Delta^{\{0,2\}}} = p_*(\alpha)$.

Since we can lift both $p_*(\alpha)$ and $\mathrm{id}_{f(c')}$ along p, similarly as in Lemma 6.15, we find a diagram

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d] & B \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed] & A \end{tikzcd}$$

which admits a dashed arrow as indicated since p is a Kan fibration. It follows that there exists a path $\gamma \colon \Delta^1 \to B$ connecting b and b' such that $p_*(\gamma) = f_*(\beta)$. Hence γ and β combine to a map $\Delta^1 \to C \times_A B$ connecting (c, b) and (c', b'). This shows that the map in question is also injective.

Lecture 25 – 04.02.2019.

Lemma 7.13. *Consider a diagram of Kan complexes*

$$\begin{tikzcd} C \arrow[r, two heads] \arrow[d, "\simeq"] & A \arrow[d, "\simeq"] & B \arrow[l] \arrow[d, "\simeq"] \\ C' \arrow[r, two heads] & A' & B' \arrow[l] \end{tikzcd}$$

in which the vertical maps are weak equivalences and the left horizontal maps are Kan fibrations. Then the induced map on pullbacks

$$C \times_A B \to C' \times_{A'} B'$$

is again a weak equivalence.

Proof. We first reduce the case in question to the situation where also the maps $B \to A$ and $B' \to A'$ are fibrations: By the small object argument we find a commutative diagram

$$\begin{tikzcd} A \arrow[d] & D \arrow[l, two heads] \arrow[d] & B \arrow[l, "\simeq"'] \arrow[d] \\ A' & D' \arrow[l, two heads] & B' \arrow[l, "\simeq"'] \end{tikzcd}$$

by functorially factoring the map $B \to A$ as a weak equivalence followed by a fibration (in our case an anodyne map followed by a Kan fibration). We observe that it follows that both D and D' are fibrant. We obtain a commutative diagram

$$\begin{tikzcd} C \times_A B \arrow[r] \arrow[d, "\simeq"] & C' \times_{A'} B' \arrow[d, "\simeq"] \\ C \times_A D \arrow[r] & C' \times_{A'} D' \end{tikzcd}$$

and wish to show that the top horizontal map is an equivalence. We claim that both vertical maps are equivalences: For instance the left vertical map sits inside a pullback diagram

$$\begin{tikzcd} C \times_A B \arrow[r] \arrow[d] & C \times_A D \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r, "\simeq"] & D \arrow[r] & A \end{tikzcd}$$

so that Reedy's lemma will imply that the top horizontal map is an equivalence because the map $C \times_A D \to D$ is a fibration as it is pulled back from $C \to A$ which is a fibration by assumption. The argument for the right vertical map above is analogous.

We may thus assume that in the statement of the lemma, all horizontal maps are in fact fibrations.We observe that the map in question factors as the composite

$$C \times_A B \to (C' \times_{A'} A) \times_A B = C' \times_{A'} B \to C' \times_{A'} B'.$$

Now we use Reedy's lemma three times:

(1) The map $C \to C' \times_{A'} A$ is an equivalence: By Reedy's lemma, the map $C' \times_{A'} A \to C'$ is a weak equivalence as it sits in the following pullback

$$\begin{tikzcd} C' \times_{A'} A \arrow[r] \arrow[d] & C' \arrow[d, two heads] \\ A \arrow[r, "\simeq"] & A' \end{tikzcd}$$

Hence in the composite

$$C \to C' \times_{A'} A \to C'$$

both the second map and the composite are equivalences. By 3-for-2 for equivalences the claim follows.

(2) The map $C \times_A B \to (C' \times_{A'} A) \times_A B$ is an equivalence: It sits in the pullback square

$$\begin{tikzcd} C \times_A B \arrow[r] \arrow[d] & (C' \times_{A'} A) \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & C' \times_{A'} A \arrow[r] & A \end{tikzcd}$$

in which the middle vertical map is a fibration, as it is pulled back from the map $B \to A$ which is now a fibration by assumption. The lower horizontal map is an equivalence by the previous step, so we conclude again using Reedy's lemma.

(3) The map $C' \times_{A'} B \to C' \times_{A'} B'$ is an equivalence: It sits in a pullback square

$$\begin{tikzcd} C' \times_{A'} B \arrow[r] \arrow[d] & C' \times_{A'} B' \arrow[r] \arrow[d] & C' \arrow[d] \\ B \arrow[r] & B' \arrow[r] & A' \end{tikzcd}$$

in which the right vertical map is a fibration, as it is pulled back from $C' \to A'$ which is also a fibration by assumption. Now the map $B \to B'$ is an equivalence, so we conclude again using Reedy's lemma.

□

We will need a similar invariance statement for inverse limits, a more general version of this result is [Hir03, Theorem 19.9.1]:

Lemma 7.14. *Consider a natural transformation between functors* $\mathbb{N}^{\mathrm{op}} \to \mathrm{sSet}$

$$\begin{tikzcd} \dots \arrow[r, two heads] \arrow[d, "\simeq"] & X_3 \arrow[r, two heads] \arrow[d, "\simeq"] & X_2 \arrow[r, two heads] \arrow[d, "\simeq"] & X_1 \arrow[r, two heads] \arrow[d, "\simeq"] & X_0 \arrow[d, "\simeq"] \\ \dots \arrow[r, two heads] & Y_3 \arrow[r, two heads] & Y_2 \arrow[r, two heads] & Y_1 \arrow[r, two heads] & Y_0 \end{tikzcd}$$

and assume that all horizontal maps are fibrations, that all vertical maps are weak equivalences and that all objects are Kan. Then the induced map

$$\lim_i X_i \to \lim_i Y_i$$

is an equivalence.

Remark. Again, one can give a proof of this using long exact sequences in homotopy groups: It turns out that there is an exact sequence

$$0 \longrightarrow \lim_i^1 \pi_{k+1}(X_i) \longrightarrow \pi_k(\lim_i X_i) \longrightarrow \lim_i \pi_k(X_i) \longrightarrow 0$$

so applying (carefully!) a diagram chase argument one can show that in our situation, the induced map on inverse limits induces a bijection on all homotopy groups. For this argument see for instance [Hir15].

Lemma 7.15. *Consider a commutative diagram of Kan complexes*

$$\begin{tikzcd} Y' \arrow[r, "f"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ X' \arrow[r, "f'"] & X \end{tikzcd}$$

in which the map p is a fibration. Suppose that f' is a homotopy equivalence, and that for each 0-simplex x' of X' the induced map $p'^{-1}(x') \to p^{-1}(x)$ between vertical fibres is an equivalence as well. Then the map f is a homotopy equivalence.

Proof. By Reedy's lemma we may assume without loss of generality that f' is the identity: We can simply replace the map p be the canonical map $X' \times_X Y \to X'$ and leave the fibres unchanged while knowing that the map $X' \times_X Y \to Y$ is an equivalence.

If we have long exact sequences in homotopy groups available, it is an easy diagram chase to see that for every point y' in Y' and every $i \geq 1$, the map

$$\pi_i(Y', y') \to \pi_i(Y, y)$$

is a bijection, where $y = f(y')$. It hence remains to show that the map f induces a bijection on path components.

To show injectivity assume given two points x, y in Y' whose images under f in Y are connected by a path $\alpha\colon \Delta^1 \to Y$. In other words, we have $\alpha(0) = f(x)$ and $\alpha(1) = f(y)$. It follows that $p\alpha\colon \Delta^1 \to X$ is a path between $p(f(x)) = p'(x)$ and $p(f(y)) = p'(y)$. Consider the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & Y' \arrow[d, "p"] \\ \Delta^1 \arrow[r, "p\alpha"] \arrow[ur, dashed, "\beta"] & X \end{tikzcd}$$

which Kan be solved since p' is a fibration. We hence have $p'\beta = p\alpha$, and since $p' = pf$ we obtain $pf\beta = p\alpha$. Furthermore we have $f\beta(0) = f(x) = \alpha(0)$. We hence obtain a lifting problem

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(f\beta,\alpha)"] \arrow[d] & Y \arrow[d, "p"] \\ \Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed, "\tau"] & X \end{tikzcd}$$

where σ is a degeneration of the path $p\alpha$. This lifting problem can be solved since p is a Kan fibration. Restricting the dashed arrow τ to $\Delta^{\{1,2\}}$ we obtain a path from $f(y)$ to $f(\beta(1))$ which is sent by p to the constant path at $p'(y)$. In other words, $\tau_{|\Delta^{\{1,2\}}}$ is a path in $p^{-1}(p'(y))$. Since f restricts to a homotopy equivalence on this fibre, we obtain that there is also a path between y and $\beta(1)$ inside $p'^{-1}(p'(y))$. Since $\beta(1)$ is connected (via β) with x, we deduce that x and y are connected by a path in Y'.

To show surjectivity, consider a point y of Y and let $x = p(y)$. By definition, y lies in the fibre F_x of p over x. This shows that F_x is not empty. By assumption, the map f' restricts to a homotopy equivalence $F_x \simeq F'_x$ where the latter denotes the fibre of p' over x. Pick a point y in F'_x which corresponds to $[y]$ under the bijection $\pi_0(F'_x) \to \pi_0(F_x)$. Then the map $\pi_0(Y') \to \pi_0(Y)$ sends $[y']$ to $[y]$ so the map in question is also surjective. □

Lecture 26 – 07.02.2019. We are now in the position to prove the characterization of Joyal equivalences as the essentially surjective and fully faithful functors.

Theorem 7.16. *A fully faithful and essentially surjective functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a Joyal equivalence.*

Proof. We will prove the theorem by showing that for any simplicial set X, the canonical functor

$$f_*\colon (\mathcal{C}^X)^{\simeq} \to (\mathcal{D}^X)^{\simeq}$$

is a homotopy equivalence. Once this is shown, one can consider $X = \mathcal{D}$ and, by inverting the homotopy equivalence, we obtain a diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "g"] \arrow[d] & (\mathcal{C}^{\mathcal{D}})^{\simeq} \arrow[d, "f_*"] \\ \Delta^1 \arrow[r, "h"] & (\mathcal{D}^{\mathcal{D}})^{\simeq} \end{tikzcd}$$

where h is a path from $\mathrm{id}_{\mathcal{D}}$ to fg, i.e. h provides a natural equivalence between fg and $\mathrm{id}_{\mathcal{D}}$. To see that also gf is naturally equivalent to $\mathrm{id}_{\mathcal{C}}$ we consider the homotopy equivalence

$$f_*\colon (\mathcal{C}^{\mathcal{C}})^{\simeq} \to (\mathcal{D}^{\mathcal{C}})^{\simeq}$$

and observe that both $\mathrm{id}_{\mathcal{C}}$ is sent to f and that gf is sent to fgf. But since fg is connected to $\mathrm{id}_{\mathcal{D}}$, we find that fgf is also connected to f through a natural equivalence. Since the above map is a homotopy equivalence, this implies that there also must be a path between $\mathrm{id}_{\mathcal{C}}$ and fg, so that any such path also provides a natural equivalence $fg \simeq \mathrm{id}_{\mathcal{C}}$ and thus that f and g are mutually inverse functors. Hence, f is a Joyal equivalence.

We will now prove the remaining claim. We first consider the case where $X = \Delta^0$. This is equivalent to the statement that f induces a homotopy equivalence of groupoid cores which we settled in Corollary 7.10. Next, we treat the case $X = \Delta^1$. We recall that the source–target map $\mathcal{C}^{\Delta^1} \to \mathcal{C} \times \mathcal{C}$ is an isofibration and hence the resulting map

$$(\mathcal{C}^{\Delta^1})^{\simeq} \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$$

is a Kan fibration which fits into the commutative square

$$\begin{tikzcd} (\mathcal{C}^{\Delta^1})^{\simeq} \arrow[r] \arrow[d] & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \arrow[d, "\simeq"] \\ (\mathcal{D}^{\Delta^1})^{\simeq} \arrow[r] & \mathcal{D}^{\simeq} \times \mathcal{D}^{\simeq} \end{tikzcd}$$

in which the right vertical map is a homotopy equivalence by the previous step (and the observation that products of Joyal equivalences are again Joyal equivalences). According to Lemma 7.15, to show that the left vertical map is a homotopy equivalence it will hence suffice to show that the induced map on fibres over a point $(x, y) \in \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ is a homotopy equivalence as well. We will argue momentarily that the fibre of the respective horizontal maps over the point (x, y), respectively over the point (px, py), is given by the corresponding mapping space, so that the induced map on fibres is given by

$$\mathrm{map}_{\mathcal{C}}(x, y) \to \mathrm{map}_{\mathcal{D}}(px, py).$$

This map is a homotopy equivalence for all pairs (x, y) by the assumption that p is fully faithful. We thus conclude the case $X = \Delta^1$, once the claim about the fibres is justified. For this we observe that the diagram

$$\begin{tikzcd} (\mathcal{C}^{\Delta^1})^{\simeq} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d] \\ \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \arrow[r] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

is a pullback, as the right vertical functor is conservative by Theorem 6.1 so that we can allude to Exercise 79.

Next, we deal with the case $X = I^n$, the n-dimensional spine. We will prove that the map

$$(\mathcal{C}^{I^n})^{\simeq} \to (\mathcal{D}^{I^n})^{\simeq}$$

is a homotopy equivalence by induction on n. The case $n = 1$ was done in the previous step. Then we claim that there is a pullback diagram as follows.

$$\begin{tikzcd}
(\mathcal{C}^{I^n})^{\simeq} \arrow[r] \arrow[d] & (\mathcal{C}^{I^{n-1}})^{\simeq} \arrow[d] \\
(\mathcal{C}^{\Delta^1})^{\simeq} \arrow[r] & \mathcal{C}^{\simeq}
\end{tikzcd}$$

As $I^n = I^{n-1} \amalg \Delta^1$, we find that the diagram is a pullback before applying groupoid cores, and the two maps with target $\mathcal{C}$ are isofibrations (as $\mathcal{C}$ is an ∞-category and the map $\mathcal{C} \to \Delta^0$ is an isofibration). As in the proof of Corollary 6.8, it hence suffices to observe that the pullback of groupoid cores is itself an ∞-groupoid. This is the case because the right vertical map is a Kan fibration (as it is an isofibration before applying the groupoid core). The map $\mathcal{C} \to \mathcal{D}$ induces a map from this square to the corresponding square where $\mathcal{C}$ is replaced by $\mathcal{D}$ throughout. On all spots except the top left spot, this map is a homotopy equivalence by the inductive assumption. We thus conclude by Lemma 7.13.

Next, we deal with the case $X = \Delta^n$. For this we consider the diagram

$$\begin{tikzcd}
(\mathcal{C}^{\Delta^n})^{\simeq} \arrow[r] \arrow[d] & (\mathcal{D}^{\Delta^n})^{\simeq} \arrow[d] \\
(\mathcal{C}^{I^n})^{\simeq} \arrow[r] & (\mathcal{D}^{I^n})^{\simeq}
\end{tikzcd}$$

induced by the functor $\mathcal{C} \to \mathcal{D}$ and the inclusion $I^n \to \Delta^n$. By the previous step, the lower horizontal map is a homotopy equivalence, and by Theorem 3.32 the vertical maps are trivial fibrations before applying the groupoid core, and thus remain so after applying the groupoid core (the square obtained by restricting to groupoid cores is a pullback since trivial fibrations are conservative). Since trivial fibrations are homotopy equivalences we conclude by 3-for-2 for homotopy equivalences.

Next we deal with an arbitrary but finite dimensional simplicial set X. We prove the statement by induction over the dimension. For 0-dimensional X this follows again since products of Joyal equivalences are Joyal equivalences. Let us prove the inductive step and assume that X is an n-dimensional simplicial set. Consider its skelatal pushout

$$\begin{tikzcd}
\coprod \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X) \arrow[d] \\
\coprod \Delta^n \arrow[r] & X
\end{tikzcd}$$

which induces a pullback square

$$\begin{tikzcd}
(\mathcal{C}^X)^{\simeq} \arrow[r] \arrow[d] & (\mathcal{C}^{\mathrm{sk}_{n-1}(X)})^{\simeq} \arrow[d] \\
\prod(\mathcal{C}^{\Delta^n})^{\simeq} \arrow[r] & \prod(\mathcal{C}^{\partial\Delta^n})^{\simeq}
\end{tikzcd}$$

in which the lower horizontal map is a product of Kan fibrations, and hence itself a Kan fibration. We conclude this case by Lemma 7.13.

To prove the general case, we now write an arbitrary simplicial set X as the $\mathbb{N}$-indexed colimit over its skeleta. We then obtain an isomorphism

$$(\mathcal{C}^X)^{\simeq} \cong \lim_n (\mathcal{C}^{\mathrm{sk}_n(X)})^{\simeq}$$

and all transition maps in this diagram are Kan fibrations (as they are restrictions along monomorphisms). We now conclude using Lemma 7.14. □

8. Localizations

Lecture 1 – 24.04.2019. Next we want to study a further construction of ∞-categories which will play a role later as well: We wish to "universally invert" a chosen set of morphisms in a given ∞-category. Such a construction will be called a Dwyer–Kan localization.

Definition 8.1. Let $\mathcal{C}$ be an ∞-category and let $S \subseteq \mathcal{C}_1$ be a subset of the morphisms of $\mathcal{C}$. For an auxiliary ∞-category $\mathcal{D}$, we let $\mathrm{Fun}^S(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory consisting of those functors $f\colon \mathcal{C} \to \mathcal{D}$ such that $f(S) \subseteq \mathcal{D}^{\simeq}$, i.e. where f maps the morphisms of S to equivalences in $\mathcal{D}$. If S consists of all morphisms, we will write $\mathrm{Fun}^{\simeq}(\mathcal{C}, \mathcal{D})$ for $\mathrm{Fun}^{\mathcal{C}_1}(\mathcal{C}, \mathcal{D})$.

Definition 8.2. Let $\mathcal{C}$ be an ∞-category and let $S \subseteq \mathcal{C}_1$ be a subset of the morphisms of $\mathcal{C}$. A functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ is called a Dwyer–Kan localization of $\mathcal{C}$ along S, if for every auxiliary ∞-category $\mathcal{D}$, the restriction functor

$$\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and its essential image consists of those functors that send S to equivalences.

Remark. By Theorem 7.16 this is equivalent to saying that the restriction functor factors through a Joyal equivalence $\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \to \mathrm{Fun}^S(\mathcal{C}, \mathcal{D})$.

Lemma 8.3. *If a localization exists, it is uniquely determined up to Joyal equivalence.*

Proof. Let $i\colon \mathcal{C} \to X$ and $j\colon \mathcal{C} \to Y$ be localizations of $\mathcal{C}$ along S. By the universal property we obtain a diagram

$$\begin{tikzcd}
& \mathcal{C} \arrow[dl, "i"'] \arrow[dr, "j"] & \\
X \arrow[rr, shift left, "F"] & & Y \arrow[ll, shift left, "G"]
\end{tikzcd}$$

where F is a functor such that $Fi \simeq j$ and G is such that $Gj \simeq i$. We now want to show that $FG \simeq \mathrm{id}_Y$ and that $GF \simeq \mathrm{id}_X$. By the universal property, it again suffices to show that these equations hold after precomposing with j and i respectively. There we find that

$$FGj \simeq Fi \simeq j$$

and likewise that

$$GFi \simeq Gj \simeq i.$$

$\square$

In order to prove that localizations exist, we will first need the following lemma.

Lemma 8.4. *The map $\Delta^1 \to J$ is a localization at the unique morphism from 0 to 1.*

Proof. Let $\mathcal{D}$ be an ∞-category. We already know that the restriction map factors as follows:

$$\mathrm{Fun}(J, \mathcal{D}) \to \mathrm{Fun}^{\simeq}(\Delta^1, \mathcal{D}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{D})$$

and need to show that the first map is a Joyal equivalence. We will show that it is in fact a trivial fibration and conclude the lemma using Proposition 6.12. For this we will consider the following filtration $F_k(J)$ of J with $F_1(J) = \Delta^1$. We consider the non-degenerate k-simplex $\nu_k \colon \Delta^k \to J$ given by the string of composable morphisms

$$0 \to 1 \to 0 \to \dots$$

and we let $F_k(J)$ be the smallest sub simplicial set of J which contains this k-simplex. We observe that $\nu_1(\Delta^1) \subseteq J$ is the canonical inclusion. In addition, for each $k \geq 2$, we have that $\nu_k(\Delta^{\{0,1\}}) = \nu_1(\Delta^1) \subseteq J$. We claim that there is a pushout diagram as follows.

$$\begin{tikzcd}
\Lambda^k_0 \arrow[r] \arrow[d] & F_{k-1}(J) \arrow[d] \\
\Delta^k \arrow[r] & F_k(J)
\end{tikzcd}$$

To see this, we observe that clearly the image of ν_k union $F_{k-1}(J)$ equals $F_k(J)$. It then suffices to see that their intersection is given by Λ^k_0. For this we consider the composite $\Delta^{[k]\setminus\{i\}} \to \Delta^k \to F_k(J)$. For $i = 0$, this is given by the sequence of $k-1$ composable maps

$$1 \to 0 \to 1 \to \dots$$

which is not contained in $F_{k-1}(J)$. However, if $i \neq 0$, then it is given by a sequence starting with 0 of length $k-1$ and is hence contained in $F_{k-1}(J)$ by definition.

Lecture 2 – 29.04.2019. We want to show that the map

$$\mathrm{Fun}(J, \mathcal{D}) \to \mathrm{Fun}^{\simeq}(\Delta^1, \mathcal{D})$$

is a Joyal equivalence. This map factors as follows

$$\mathrm{Fun}(J, \mathcal{D}) \to \mathrm{Fun}^{\Delta^1}(F_k(J), \mathcal{D}) \to \mathrm{Fun}^{\Delta^1}(F_{k-1}(J), \mathcal{D}) \to \mathrm{Fun}^{\simeq}(\Delta^1, \mathcal{D}).$$

We will show that the map in the middle is a trivial fibration for all $k \geq 2$. It follows that also the map

$$\mathrm{Fun}(J, \mathcal{D}) \cong \lim_k \mathrm{Fun}^{\Delta^1}(F_k(J), \mathcal{D}) \to \mathrm{Fun}^{\simeq}(\Delta^1, \mathcal{D})$$

is a trivial fibration and hence a Joyal equivalence. We thus need to show that for every commutative diagram

$$\begin{tikzcd} & \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Fun}^{\Delta^1}(F_k(J),\mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\Delta^1}(\Delta^k,\mathcal{D}) \arrow[d] \\ \Delta^0 \arrow[r, "x"] & \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted] & \mathrm{Fun}^{\Delta^1}(F_{k-1}(J),\mathcal{D}) \arrow[r] & \mathrm{Fun}^{\Delta^1}(\Lambda^k_0,\mathcal{D}) \end{tikzcd}$$

there exists a dashed arrow making everything commute. We claim that it suffices to find a dotted arrow: It is clear that the right square is a pullback if we drop the superscript Δ^1, so that the pullback consists of all functors $F_k(J) \to \mathcal{D}$, whose restriction to Δ^k send Δ^1 to an equivalence. This shows that the right square is a pullback.

By adjunction, this lifting problem corresponds to the lifting problem

$$\begin{tikzcd} \Delta^1 \arrow[r] & \Lambda^k_0 \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n,\mathcal{D}) \arrow[r, "\mathrm{ev}_x"] \arrow[d] & \mathcal{D} \\ & \Delta^k \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\partial\Delta^n,\mathcal{D}) & \end{tikzcd}$$

and we observe that the top horizontal composite is an equivalence for every object x of Δ^n. Once we know this, the dashed arrow exists by Joyal's extension theorem, because the right vertical map is an inner fibration and the functor $\mathrm{Fun}(\Delta^n,\mathcal{D}) \to \prod_x \mathcal{D}$ is conservative by Theorem 6.1. $\square$

Lemma 8.5. *For every ∞-category $\mathcal{C}$, there exists a localization along all morphisms of $\mathcal{C}$.*

Proof. We first construct an anodyne map $f\colon \mathcal{C} \to X$ to a Kan complex (an ∞-groupoid) X as follows. We let Y be the pushout

$$\begin{tikzcd} \coprod_{\alpha\in\mathcal{C}_1} \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \coprod_{\alpha\in\mathcal{C}_1} J \arrow[r] & Y \end{tikzcd}$$

Since $\Delta^1 \to J$ is anodyne, as follows from the proof of Lemma 8.4, it follows that also f is anodyne. Then we take an inner anodyne map $g\colon Y \to X$ with X an ∞-category: Via the small object argument we can factor the map $Y \to *$ through an inner anodyne map followed by an inner fibration. Since g is inner anodyne, we see that the composite gf is anodyne and we claim that X is in fact an ∞-groupoid. For this we need to show that its homotopy category is a groupoid.

Since the map $Y \to X$ is inner anodyne, it is a Joyal equivalence, and thus induces an equivalence on homotopy categories. Furthermore, taking homotopy categories is left adjoint to the nerve functor, and hence preserves pushouts, so that there is a pushout of categories

$$\begin{tikzcd} \amalg_{\alpha\in\mathcal{C}_1}[1] \arrow[r] \arrow[d] & h\mathcal{C} \arrow[d] \\ \amalg_{\alpha\in\mathcal{C}_1} J \arrow[r] & hY \end{tikzcd}$$

Thus hY is obtained from $h\mathcal{C}$ by inverting all morphisms in $\mathcal{C}$ Exercise 97. In particular, it is a groupoid, and thus so is hX.

Finally, we claim that for every ∞-category $\mathcal{D}$, the restriction functor $\mathrm{Fun}(X, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ factors through a trivial fibration

$$\mathrm{Fun}(X, \mathcal{D}) \to \mathrm{Fun}^{\simeq}(\mathcal{C}, \mathcal{D}).$$

Since trivial fibrations are Joyal equivalences by Proposition 6.12, the map $\mathcal{C} \to X$ is a localization.

It is clear that $\mathrm{Fun}(X, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ factors through $\mathrm{Fun}^{\simeq}(\mathcal{C}, \mathcal{D})$ because every morphism in $\mathcal{C}$ maps to an equivalence in X, since X is an ∞-groupoid. The map of interest now factors as

$$\mathrm{Fun}(X, \mathcal{D}) \to \mathrm{Fun}(Y, \mathcal{D}) \to \mathrm{Fun}^{\simeq}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

The first map is a trivial fibration by Theorem 3.32 part (2) since $Y \to X$ is an inner fibration. The second map sits inside a diagram

$$\begin{tikzcd}
\mathrm{Fun}(Y, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\simeq}(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d] \\
\prod\limits_{\alpha \in \mathcal{C}_1} \mathrm{Fun}(J, \mathcal{C}) \arrow[r, two heads, "\simeq"] & \prod\limits_{\alpha \in \mathcal{C}_1} \mathrm{Fun}^{\simeq}(\Delta^1, \mathcal{C}) \arrow[r] & \prod\limits_{\alpha \in \mathcal{C}_1} \mathrm{Fun}(\Delta^1, \mathcal{D})
\end{tikzcd}$$

in which the big square is a pullback by definition of Y, and the right square is a pullback by inspection. It follows that the left square is a pullback as well. By Lemma 8.4 the left lower horizontal map is a trivial fibration, thus so is the upper horizontal map. This finishes the proof of the lemma. $\square$

Remark. We will see later that the association of sending $\mathcal{C}$ to the localization along all morphisms is a left adjoint to the inclusion of ∞-groupoids into ∞-categories (as ∞-functors between ∞-categories).

Proposition 8.6. *For every $S \subseteq \mathcal{C}_1$, there exists a localization of $\mathcal{C}$ along S.*

Proof. We observe that for every subset $S \subseteq \mathcal{C}_1$ there is a smallest subcategory $\mathcal{C}_S$ of $\mathcal{C}$ which contains S: this is clear for categories, and the statement here follows by pulling back the corresponding subcategory of the homotopy category of $\mathcal{C}$. It follows easily that a localization of $\mathcal{C}$ along $\mathcal{C}_S$ is a localization of $\mathcal{C}$ at S, see also Exercise 94.

We thus take a localization of $\mathcal{C}_S$ along all morphisms, more precisely an inner anodyne map $\mathcal{C}_S \to X$ to an ∞-groupoid X as in Lemma 8.5. Then we consider the pushout

$$\begin{tikzcd}
\mathcal{C}_S \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
X \arrow[r] & W
\end{tikzcd}$$

and then consider an inner anodyne map $g\colon W \to \mathcal{D}$ with $\mathcal{D}$ an ∞-category. Then, for an auxiliary ∞-category $\mathcal{E}$, consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \arrow[r, "g^*"] & \mathrm{Fun}(W, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^S(\mathcal{C}, \mathcal{E}) \arrow[d] \\
& \mathrm{Fun}(X, \mathcal{E}) \arrow[r] & \mathrm{Fun}^S(\mathcal{C}_S, \mathcal{E}).
\end{tikzcd}$$

We claim that the right hand square is pullback diagram: To see this, it suffices to observe that every functor $W \to \mathcal{E}$ sends the morphisms in the image of S to equivalences, which follows from the fact that they are sent to equivalences in X. Moreover, the lower horizontal map is a trivial fibration by the previous step, thus the upper horizontal map is also a trivial fibration. The map g^* is a trivial fibration, since $W \to \mathcal{D}$ is inner anodyne. Thus the upper composite is a trivial fibration, and thus a Joyal equivalence. This shows that the map $\mathcal{C} \to \mathcal{D}$ is a localization along S. □

Apart from the fact that the procedure of "universally inverting" morphisms produces many interesting examples of ∞-categories (even if the category we start out with is an ordinary category), we will use it to prove a certain factorization property of functors between ∞-categories later.

Example 8.7. Consider the 1-category Cat^1_∞ of ∞-categories, i.e. the full subcategory of sSet whose objects are the ∞-categories. Recall that the ∞-category Cat_∞ of ∞-categories is given by the homotopy coherent nerve $\mathrm{N}(\mathrm{Cat}^1_\infty)$ of this category with its canonical Kan enrichment given by $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$. The identity of Cat^1_∞ canonically refines to a functor between simplicial categories, with constant simplicial enrichment on the domain and the canonical simplicial enrichment on the target. In other words, we obtain a canonical functor of ∞-categories

$$\mathrm{Cat}^1_\infty \longrightarrow \mathrm{Cat}_\infty.$$

We observe that his functor sends Joyal equivalences to equivalences: By definition, a Joyal equivalence is a map of ∞-categories which becomes an equivalence in the ∞-category Cat_∞, see Definition 5.12. It follows that this functor induces a functor

$$\mathrm{Cat}^1_\infty[\mathrm{Joy}^{-1}] \longrightarrow \mathrm{Cat}_\infty$$

where Joy denotes the set of Joyal equivalences.

Example 8.8. Likewise, there is a canonical functor $\mathrm{Kan} \longrightarrow \widehat{\mathrm{An}}$ which sends homotopy equivalences to equivalences in $\widehat{\mathrm{An}}$. Hence there is an induced functor

$$\mathrm{Kan}[\mathrm{he}^{-1}] \longrightarrow \widehat{\mathrm{An}}.$$

Lemma 8.9. *The inclusions* $\mathrm{Kan} \to \mathrm{sSet}$ *and* $\mathrm{Cat}^1_\infty \to \mathrm{sSet}$ *induce equivalences*

$$\mathrm{Kan}[\mathrm{he}^{-1}] \simeq \mathrm{sSet}[\mathrm{we}^{-1}] \textit{ and } \mathrm{Cat}^1_\infty[\mathrm{Joy}^{-1}] \simeq \mathrm{sSet}[\mathrm{Joy}^{-1}].$$

Proof. The small object argument provides functors $F\colon \mathrm{sSet} \to \mathrm{Cat}^1_\infty$ and $G\colon \mathrm{sSet} \to \mathrm{Kan}$ by functorially factoring the map $X \to *$ into an inner anodyne map followed by an inner fibration, respectively by an anodyne map followed by a Kan fibration. We claim that these functors send Joyal equivalences to Joyal equivalences, respectively weak homotopy equivalences to weak homotopy equivalences: Suppose $X \to Y$ is such an equivalence, then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ Y \arrow[r] & \mathcal{D} \end{tikzcd}$$

where the horizontal maps are inner anodyne and hence Joyal equivalences. We find that $\mathcal{C} \to \mathcal{D}$ is a Joyal equivalence if and only $X \to Y$ is, by 3-for-2 for Joyal equivalences. The argument for weak homotopy equivalences is the same. Hence this functor induces a functor

$$\mathrm{sSet}[\mathrm{Joy}^{-1}] \longrightarrow \mathrm{Cat}^1_\infty[\mathrm{Joy}^{-1}]$$

which we claim to be an inverse to the canonical functor

$$\mathrm{Cat}^1_\infty[\mathrm{Joy}^{-1}] \longrightarrow \mathrm{sSet}[\mathrm{Joy}^{-1}]$$

induced by the inclusion.

We observe that the map $X \to \mathcal{C} = F(X)$ determines a natural transformation from the identity of sSet to the composite $iF\colon \mathrm{sSet} \to \mathrm{sSet}$. More precisely it determines a map

$$\Delta^1 \longrightarrow \mathrm{Fun}(\mathrm{sSet}, \mathrm{sSet}) \longrightarrow \mathrm{Fun}(\mathrm{sSet}, \mathrm{sSet}[\mathrm{Joy}^{-1}])$$

which we claim to land inside the full subcategory consisting of those functors that send Joyal equivalences to equivalences: This is because we have just argued that F (and also clearly i) send Joyal equivalences to Joyal equivalences. Hence, by the universal property of localizations, we obtain a map

$$\Delta^1 \longrightarrow \mathrm{Fun}(\mathrm{sSet}[\mathrm{Joy}^{-1}], \mathrm{sSet}[\mathrm{Joy}^{-1}])$$

whose restriction to 0 and 1 are given by the identity and a functor, whose restriction to sSet is induced by the composite Fi. We observe that for fixed object $X \in \mathrm{sSet}$, the resulting morphism is given by $X \to F(X)$ which we argued to be a Joyal equivalence. In particular the 1-simplex given above is a natural equivalence between the identity of $\mathrm{sSet}[\mathrm{Joy}^{-1}]$ and the functor induced by Fi.

It remains to show that also the functor induced by the composite iF is an equivalence. For this we argue analogously and find that the map $X \to FX$ induces a transformation $\mathrm{id} \to iF$ which is pointwise a Joyal equivalence. This shows the corollary for Joyal equivalences, and the argument for weak homotopy equivalences is the same. □

The following theorem is very important, but at the moment beyond the scope of this lecture. It holds in greater generality for simplicial model categories, see [Lur17, Theorem 1.3.4.20]. We will sketch a proof of this result later, see Corollary 11.26 and Corollary 11.27.

Theorem 8.10. *The canonical functors* $\mathrm{sSet}[\mathrm{we}^{-1}] \to \mathrm{An}$ *and* $\mathrm{sSet}[\mathrm{Joy}^{-1}] \to \mathrm{Cat}_\infty$ *are equivalences of* ∞*-categories.*

We finish this section with a useful factorization construction.

Definition 8.11. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We define the path-fibration $P(f)$ of f by the pullback

$$\begin{tikzcd} P(f) \arrow[r] \arrow[d] & \mathrm{Fun}(J, \mathcal{D}) \arrow[d, "s"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

of simplicial sets, where s is the source map.

Consider the map $\mathcal{C} \to \mathrm{Fun}(J, \mathcal{D})$ which is adjoint to the map $\mathcal{C} \times J \to \mathcal{C} \to \mathcal{D}$. It sends an object x to the identity morphism of $f(x)$. Clearly the composition of this functor with s is given by f so we obtain an induced map $c\colon \mathcal{C} \to P(f)$ which is a section of the canonical map $P(f) \to \mathcal{C}$. In particular, c is a monomorphism. Furthermore, the composite of $P(f) \to \mathrm{Fun}(J, \mathcal{D})$ with the target map $\mathrm{Fun}(J, \mathcal{D}) \to \mathcal{D}$ produces a composite

$$\mathcal{C} \xrightarrow{c} P(f) \xrightarrow{t} \mathcal{D}.$$

This composite is given by f, so we have produced a factorization of f through $P(f)$.

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Lemma 8.12. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and let $\mathcal{C} \xrightarrow{c} P(f) \xrightarrow{t} \mathcal{D}$ be the factorization just constructed. Then c is Joyal equivalence and t is an isofibration. In particular, any map $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories can be factored as a Joyal equivalence which is a monomorphism, followed by an isofibration.*

Proof. We claim that the source map $s\colon \mathrm{Fun}(J, \mathcal{D}) \to \mathcal{D}$ is a trivial fibration. By Lemma 6.16 it suffices to show that it is a Joyal equivalence and an isofibration. We have already seen that $\Delta^0 \to J$ is a Joyal equivalence, so it follows from Proposition 6.9 that also the induced map $\mathrm{Fun}(J, \mathcal{D}) \to \mathrm{Fun}(\Delta^0, \mathcal{D}) \cong \mathcal{D}$ is a Joyal equivalence. By Proposition 6.6 the restriction along a monomorphism mapping into an ∞-category is an isofibration, so we conclude that $\mathrm{Fun}(J, \mathcal{D}) \to \mathcal{D}$ is a trivial fibration as claimed. Thus, as a pullback of this map, also the functor $P(f) \to \mathcal{C}$ is a trivial fibration. Since the composite

$$\mathcal{C} \to P(f) \to \mathcal{C}$$

is the identity (by construction), it follows from 3-for-2 for Joyal equivalences that the map $\mathcal{C} \to P(f)$ is a Joyal equivalence as claimed.

To see that $P(f) \to \mathcal{D}$ is an isofibration we observe that the square

$$\begin{tikzcd} P(f) \arrow[r] \arrow[d] & \mathrm{Fun}(J, \mathcal{D}) \arrow[d] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "f \times \mathrm{id}"] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

is also a pullback and that the right vertical map is an isofibration again by Proposition 6.6. It follows that $P(f) \to \mathcal{C} \times \mathcal{D}$ is also an isofibration, Corollary 5.18. As the projection $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ is also an isofibration, the lemma is proven. $\square$

9. Fat joins, fat slices and mapping spaces

In this section we will construct an alternative join, and show that it is Joyal equivalent to the construction of Definition 4.6. This is used to compare different models of mapping spaces in an ∞-category.

Definition 9.1. Let X and Y be simplicial sets. We define a new simplicial set $X \diamond Y$ to be the pushout

$$\begin{tikzcd} X \times Y \times \partial\Delta^1 \arrow[r] \arrow[d] & X \amalg Y \arrow[d] \\ X \times Y \times \Delta^1 \arrow[r] & X \diamond Y \end{tikzcd}$$

Here, the top horizontal arrow sends the triple $(x, y, 0)$ to x and the triple $(x, y, 1)$ to y.

Lemma 9.2. *For fixed Y, the association $X \mapsto X \diamond Y$ extends to a functor* $\mathrm{sSet} \to \mathrm{sSet}_{Y/}$*. As such, it preserves colimits. The same holds for the association $X \mapsto Y \diamond X$.*

Proof. The first statement is an easy calculation: we need to see that for any morphism $X \to X'$ the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X \diamond Y \arrow[d] \\ & X' \diamond Y \end{tikzcd}$$

commutes. This follows simply from the fact that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X \amalg Y \arrow[d] \\ & X' \amalg Y \end{tikzcd}$$

commutes. To see that this functor commutes with colimits, it again suffices to show that it commutes with coproducts and with coequalizers. So let X, X' and Y be simplicial sets. We need to show that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \diamond Y \arrow[d] \\ X' \diamond Y \arrow[r] & (X \amalg X') \diamond Y \end{tikzcd}$$

is a pushout in simplicial sets, which follows immediately from the definition. We recall that coequalizers in $\mathrm{sSet}_{Y/}$ are calculated underlying, see **??**. So let

$$X \rightrightarrows X' \longrightarrow C$$

be a coequalizer diagram. We need to show that applying $- \diamond Y$ gives again a coequalizer diagram. Then functors $- \times Y \times \partial\Delta^1$ and $- \times Y \times \Delta^1$ are left adjoint and hence preserve this coequalizer diagram. We claim that the diagram

$$X \amalg Y \rightrightarrows X' \amalg Y \longrightarrow C \amalg Y$$

is also a coequalizer: Suppose given a map $X' \amalg Y \to T$ whose precomposition with the two given maps give the same map $X \amalg Y \to T$, we find that there is a unique map from $C \to T$ and from Y to T as needed. Now since two colimits always commute, we may take the pushout over the coequalizers and obtain the coequalizer of the pushout. This proves the lemma. □

Observation 9.3. There exists a canonical map of simplicial sets $X \diamond Y \to \Delta^1$ induced by the commutative diagram

$$\begin{tikzcd} X \times Y \times \partial\Delta^1 \arrow[r] \arrow[d] & X \amalg Y \arrow[d] \\ X \times Y \times \Delta^1 \arrow[r, "\mathrm{pr}"] & \Delta^1 \end{tikzcd}$$

where the right vertical map is given by $X \to \Delta^{\{0\}}$ a combined with $Y \to \Delta^{\{1\}}$.

Lemma 9.4. *Let X and Y be simplicial sets. Then there exists a canonical map $X \diamond Y \to X \star Y$ which commutes with the projections to Δ^1 and the inclusions of X and Y. This map is functorial in X and Y.*

Proof. Recall from Lemma 4.9 that for every map $p\colon K \to \Delta^1$ of simplicial sets, there exists a canonical factorization into $K \to K_0 \star K_1 \to \Delta^1$, where $K_i = p^{-1}(\{i\})$. We apply this to the map $X \diamond Y \to \Delta^1$ we have just constructed. We find that $(X \diamond Y)_0$ is given by the pushout

$$\begin{tikzcd}
X \times Y \times \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\
X \times Y \times \{0\} \arrow[r] & X
\end{tikzcd}$$

and likewise that $(X \diamond Y)_1$ is given by Y. $\square$

We will now need Lemma 7.12 (Reedy's lemma) in the following context:

Proposition 9.5. *Consider a pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "p'"] & \mathcal{D} \arrow[d, "p"] \\
\mathcal{C}' \arrow[r, "f'"] & \mathcal{D}'
\end{tikzcd}$$

in which the map p is an isofibration and the map f' is a Joyal equivalence. Then the map f is also a Joyal equivalence.

Proof. We will show that for every ∞-category $\mathcal{E}$, the induced map

$$\pi_0(\mathrm{Fun}(\mathcal{E}, \mathcal{C})^{\simeq}) \xrightarrow{f_*} \pi_0(\mathrm{Fun}(\mathcal{E}, \mathcal{D})^{\simeq})$$

is a bijection. Once this is shown, we can choose $\mathcal{E} = \mathcal{D}$ and find a functor $g\colon \mathcal{D} \to \mathcal{C}$ such that $f_*(g) = fg \simeq \mathrm{id}_{\mathcal{D}}$. It is then easy to see that g is an inverse of f.

We then observe that the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{E}, \mathcal{C})^{\simeq} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{E}, \mathcal{D})^{\simeq} \arrow[d] \\
\mathrm{Fun}(\mathcal{E}, \mathcal{C}')^{\simeq} \arrow[r] & \mathrm{Fun}(\mathcal{E}, \mathcal{D}')^{\simeq}
\end{tikzcd}$$

is a pullback diagram of Kan complexes, in which the right vertical map is a Kan fibration and the lower horizontal map is a homotopy equivalence. In Lemma 7.12 we have shown that this implies that the top horizontal arrow is also a homotopy equivalence.

In fact, in the proof we have explicitly shown that the induced map on π_0 is a bijection. For sake of completeness, we reproduce the argument here. So let us assume given a pullback diagram of Kan complexes

$$\begin{tikzcd}
C \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & A
\end{tikzcd}$$

such that the map $B \to A$ is a Kan fibration and the map $C \to A$ is a homotopy equivalence. We aim to show that the map $\pi_0(C \times_A B) \to \pi_0(B)$ is a bijection.

Let's prove surjectivity of this map first: pick a point b in B representing a class $[b] \in \pi_0(B)$ and consider the point $p(b)$ in A. Since $C \to A$ induces a bijection on path components, we can find a point c in C and a path $\Delta^1 \to A$ connecting $f(c)$ to $p(b)$. This gives, as in Lemma 6.15, a lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & B \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

which can be solved as p is a Kan fibration. We thus find that there exists a b' in B such that $p(b') = f(c)$ and such that $[b'] = [b]$ in $\pi_0(B)$. The pair (c, b') thus determines an element of $\pi_0(C \times_A B)$ which is sent to $[b]$ in $\pi_0(B)$. This shows that the map $\pi_0(C \times_A B) \to \pi_0(B)$ is surjective.

To show injectivity, consider two points (c, b) and (c', b') of $C \times_A B$ and assume that there is a path $\alpha\colon \Delta^1 \to B$ connecting b and b' in B. Then $p(\alpha)\colon \Delta^1 \to A$ connects $p(b) = f(c)$ to $p(b') = f(c')$. Since the map f is a homotopy equivalence, there is a path $\beta\colon \Delta^1 \to C$ such that $f_*(\beta)$ is equivalent to $p_*(\alpha)$. Precisely, we find a 2-cell $\sigma\colon \Delta^2 \to A$ such that

(1) $\sigma_{|\Delta^{\{0,1\}}} = f_*(\beta)$,
(2) $\sigma_{|\Delta^{\{1,2\}}} = \mathrm{id}_{f(c')}$, and
(3) $\sigma_{|\Delta^{\{0,2\}}} = p_*(\alpha)$.

Since we can lift both $p_*(\alpha)$ and $\mathrm{id}_{f(c')}$ along p, similarly as in Lemma 6.15, we find a diagram

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d] & B \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed] & A \end{tikzcd}$$

which admits a dashed arrow as indicated since p is a Kan fibration. It follows that there exists a path $\gamma\colon \Delta^1 \to B$ connecting b and b' such that $p_*(\gamma) = f_*(\beta)$. Hence γ and β combine to a map $\Delta^1 \to C \times_A B$ connecting (c, b) and (c', b'). This shows that the map in question is also injective. $\square$

Lemma 9.6. *Consider a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, two heads] \arrow[d, "\simeq"] & \mathcal{D} \arrow[d, "\simeq"] & \mathcal{E} \arrow[l] \arrow[d, "\simeq"] \\ \mathcal{C}' \arrow[r, two heads] & \mathcal{D}' & \mathcal{E}' \arrow[l] \end{tikzcd}$$

in which the vertical maps are Joyal equivalences and the left horizontal maps are isofibrations. Then the induced map on pullbacks

$$\mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \mathcal{C}' \times_{\mathcal{D}'} \mathcal{E}'$$

is again a Joyal equivalence.

Proof. Copying the proof of Lemma 7.13 we again first explain how to reduce to the case where the two maps $\mathcal{E} \to \mathcal{D}$ and $\mathcal{E}' \to \mathcal{D}'$ are also isofibrations. Here we simply functorially factor the maps as in Lemma 8.12 as a monomorphism which is a Joyal equivalence followed by an isofibration to obtain a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, two heads] \arrow[d, "\simeq"] & \mathcal{D} \arrow[d, "\simeq"] & \mathcal{P} \arrow[l, two heads] \arrow[d, "\simeq"] & \mathcal{E} \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\ \mathcal{C}' \arrow[r, two heads] & \mathcal{D}' & \mathcal{P}' \arrow[l, two heads] & \mathcal{E}' \arrow[l, "\simeq"'] \end{tikzcd}$$

in which all vertical maps are Joyal equivalences. We notice that $\mathcal{P}$ and $\mathcal{P}'$ are also ∞-categories. We then obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{C}' \times_{\mathcal{D}'} \mathcal{E}' \arrow[d] \\ \mathcal{C} \times_{\mathcal{D}} \mathcal{P} \arrow[r] & \mathcal{C}' \times_{\mathcal{D}'} \mathcal{P}' \end{tikzcd}$$

in which we claim the vertical maps to be Joyal equivalences: For instance, the left vertical map sits inside a diagram

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{C} \times_{\mathcal{D}} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, two heads] \\ \mathcal{E} \arrow[r, "\simeq"] & \mathcal{P} \arrow[r] & \mathcal{D} \end{tikzcd}$$

in which both squares are pullbacks. We may thus apply Proposition 9.5 to the left square. The argument for the right vertical map above is the same. We have thus reduced to the case where all horizontal maps in the diagram of the statement of the lemma are isofibrations.

Now it is a 3-fold application of Proposition 9.5 to obtain the lemma as in Lemma 7.13, by considering the following factorization of the map in question:

$$\mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to (\mathcal{C}' \times_{\mathcal{D}'} \mathcal{D}) \times_{\mathcal{D}} \mathcal{E} \cong \mathcal{C}' \times_{\mathcal{D}'} \mathcal{E} \to \mathcal{C}' \times_{\mathcal{D}'} \mathcal{E}'.$$

□

Corollary 9.7. *Suppose given a diagram of simplicial sets*

$$\begin{tikzcd} X \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\ X' & Y' \arrow[l] \arrow[r] & Z' \end{tikzcd}$$

in which all vertical maps are Joyal equivalences and the left horizontal maps are monomorphisms. Then the induced map

$$X \amalg_Y Z \to X' \amalg_{Y'} Z'$$

is again a Joyal equivalence.

Proof. Let $\mathcal{C}$ be an ∞-category. By definition, we need to show that the map

$$\mathrm{Fun}(X' \amalg_{Y'} Z', \mathcal{C}) \to \mathrm{Fun}(X \amalg_Y Z, \mathcal{C})$$

is a Joyal equivalence. This map is isomorphic to the map

$$\mathrm{Fun}(X', \mathcal{C}) \times_{\mathrm{Fun}(Y',\mathcal{C})} \mathrm{Fun}(Z', \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}) \times_{\mathrm{Fun}(Y,\mathcal{C})} \mathrm{Fun}(Z, \mathcal{C})$$

which is a Joyal equivalence by Lemma 9.6 and the assumptions. □

Corollary 9.8. *For all monomorphisms $A \to B$, the pushout product map*

$$J \times A \amalg_{\{0\}\times A} \{0\} \times B \longrightarrow J \times B$$

is a Joyal equivalence.

Proof. We consider the following diagram

$$\begin{tikzcd}
\{0\} \times A \arrow[r] \arrow[d] & \{0\} \times B \arrow[d] \arrow[ddr, bend left] & \\
J \times A \arrow[r] \arrow[drr, bend right] & J \times A \amalg_{\{0\}\times A} \{0\} \times B \arrow[dr] & \\
& & J \times B
\end{tikzcd}$$

The horizontal arrows are monomorphisms and the left vertical arrow and the right bended arrow are Joyal equivalences: This is because $\{0\} \to J$ is a Joyal equivalence and Joyal equivalences are closed under finite products as shown in Exercise 102. By Corollary 9.7 the right vertical map is also a Joyal equivalence, so the lemma follows from the 3-for-2 property for Joyal equivalences. □

Definition 9.9. A map of simplicial sets is called J-anodyne if it belongs the saturated set generated by inner anodyne maps and maps of the form $J \times A \amalg_{\{0\}\times A} \{0\} \times B \longrightarrow J \times B$ for a monomorphism $A \to B$.

Corollary 9.10. *Every J-anodyne map is a Joyal equivalence.*

Proof. We claim that monomorphisms which are Joyal equivalences themselves form a saturated set, which we leave as Exercise 103. It then suffices to know that inner anodyne maps are Joyal equivalences by **??** and that the maps of the form $J \times A \amalg_{\{0\}\times A} \{0\} \times B \longrightarrow J \times B$ are also Joyal equivalences, which is Corollary 9.8. □

Proof. It is closed under pushouts by because mapping a monomorphism an ∞-category gives an isofibration. If the mono is in addition a Joyal equivalence, the isofibration will also be a Joyal equivalence and thus a trivial fibration. Such are closed under pullbacks. Since both monomorphisms and Joyal equivalences are closed under retracts, it remains to show that it is closed under countable composition: Mapping a monomorphism which is a Joyal equivalence to an ∞-category produces an isofibration which is also a Joyal equivalence and thus a trivial fibration Lemma 6.16. Since trivial fibrations are closed under countable composition (the map from the inverse limit to any stage is a trivial fibration) the claim follows. □

Lecture 4 – 08.05.2019. We can use Corollary 9.10 to give a smaller generating set for J-anodyne maps, all of whose domains are finite simplicial sets. This will help to apply the small object argument in the way presented here, which needs this technical assumption.

Proposition 9.11. *The set of J-anodyne maps is the smallest saturated set containing inner anodyne maps and the map $\{0\} \to J$.*

Proof. For notational convenience, let us call the smallest saturated set containing inner anodyne maps and the map $\{0\} \to J$ the set of *super-J-anodyne* maps. It is clear that super-J-anodyne maps are J-anodyne, so it suffices to show that for all monomorphisms $A \to B$, the pushout product map

$$\{0\} \times B \amalg_{\{0\}\times A} J \times A \to J \times B$$

super-J-anodyne. The set of monomorphisms $A \to B$ for which this is the case is itself saturated, so it suffices to show the claim for $A \to B$ being the boundary inclusions $\partial\Delta^n \to \Delta^n$ for $n \geq 0$. We use the small object argument for the set of super-J-anodyne maps and obtain a factorization of the map we are interested in

$$\{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} J \times \partial\Delta^n \xrightarrow{i} \mathcal{C} \xrightarrow{p} J \times \Delta^n$$

in which i is super-J-anodyne and the second map satisfies the RLP with respect to inner horn inclusions and $\{0\} \to J$. Since $J \times \Delta^n$ is an ∞-category, we deduce that the map $\mathcal{C} \to J \times \Delta^n$ is an isofibration between ∞-categories. Since J-anodyne maps are Joyal equivalences by Corollary 9.10, and super-J-anodyne maps are J-anodyne, we find that the map $\mathcal{C} \to J \times \Delta^n$ is in fact a trivial fibration and thus admits a solution $s\colon J \times \Delta^n \to \mathcal{C}$ to the lifting problem

$$\begin{tikzcd}
\{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} J \times \partial\Delta^n \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\
J \times \Delta^n \arrow[r, equal] \arrow[ur, dashed, "s"] & J \times \Delta^n
\end{tikzcd}$$

Considering the diagram

$$\begin{tikzcd}
\{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} J \times \partial\Delta^n \arrow[r] \arrow[d, equal] & J \times \Delta^n \arrow[d, "s"] \\
\{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} J \times \partial\Delta^n \arrow[r, "i"] \arrow[d, equal] & \mathcal{C} \arrow[d, "p"] \\
\{0\} \times \Delta^n \amalg_{\{0\}\times\partial\Delta^n} J \times \partial\Delta^n \arrow[r] & J \times \Delta^n
\end{tikzcd}$$

we find that the map we are interested in is a retract of the super-J-anodyne map i and hence is itself super-J-anodyne. □

If the target of a monomorphism is an ∞-category, we can prove the following strengthening of Corollary 9.10.

Proposition 9.12. *Let $i\colon A \to \mathcal{B}$ be a monomorphism with $\mathcal{B}$ an ∞-category. Then i is J-anodyne if and only if i is a Joyal equivalence.*

Proof. By Corollary 9.10 it suffices to show the "if" part. We apply the small object argument and factor the map $A \to \mathcal{B}$ as a composition

$$A \to \mathcal{B}' \to \mathcal{B}$$

where the map $A \to \mathcal{B}'$ is J-anodyne and the map $\mathcal{B}' \to \mathcal{B}$ satisfies the RLP with respect to J-anodyne maps. Since $\mathcal{B}$ is an ∞-category so is $\mathcal{B}'$ and the map $\mathcal{B}' \to \mathcal{B}$ is an isofibration. By Corollary 9.10 and 3-for-2 for Joyal fibrations, the map $\mathcal{B}' \to \mathcal{B}$ is an isofibration and a Joyal equivalence and thus a trivial fibration by Lemma 6.16. Choosing a solution $s\colon \mathcal{B} \to \mathcal{B}'$

of the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{B}' \arrow[d] \\ \mathcal{B} \arrow[r, equal] \arrow[ur, dashed, "s"] & \mathcal{B} \end{tikzcd}$$

we find as in the proof of Proposition 9.11 that the map $A \to \mathcal{B}$ is a retract of the J-anodyne map $A \to \mathcal{B}'$ and thus is itself J-anodyne. □

We will later need the following theorem. Thanks to Hoang Kim Nguyen for the explanation of the needed reduction steps.

Theorem 9.13. *Let $p\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between ∞-categories and let $A \to B$ be a monomorphism which is in addition a Joyal equivalence. Then any lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

admits a solution as indicated by the dashed arrow.

Proof. Using the small object argument, we can factor the map $B \to \mathcal{D}$ through an inner andoyne map $B \to \mathcal{B}$ followed by a map $\mathcal{B} \to \mathcal{D}$ satisfying the RLP wrt J-anodyne maps. It follows that $\mathcal{B}$ is an ∞-category. Since inner anodyne maps are Joyal equivalences we obtain that the composite $A \to \mathcal{B}$ is a monomorphism and a Joyal equivalence. By Proposition 9.12 this map is J-anodyne. Since isofibrations between ∞-categories have the RLP with respect to J-anodyne maps, we can find a dashed arrow in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C} \arrow[dd] \\ B \arrow[d] & \\ \mathcal{B} \arrow[r] \arrow[uur, dashed] & \mathcal{D} \end{tikzcd}$$

which also solves the original lifting problem. □

We finish this intermezzo on J-anodyne maps with a nice fact about inner anodyne maps. The following lemma is taken from Stevenson [Ste18, Lemma 2.19]

Lemma 9.14. *Let $A \to \mathcal{C}$ be a monomorphism which is a bijection on 0-simplices, a Joyal equivalence and where $\mathcal{C}$ is an ∞-category. Then i is inner anodyne.*

Proof. By the small object argument we may factor this map as a composite $A \xrightarrow{j} \mathcal{B} \xrightarrow{p} \mathcal{C}$ with j an inner anodyne map and p an inner fibration. Since $\mathcal{C}$ is an ∞-category, p satisfies the assumptions of **??** and hence admits a section. As in the proof of Proposition 9.12 this section shows that i is a retract of j and is thus inner anodyne as well. □

Corollary 9.15. *Let $K \to L$ be a monomorphism which is a bijection on 0-simplices. Then the map*

$$\{0\} \times L \amalg_{\{0\}\times K} J \times K \to J \times L$$

is inner anodyne.

Proof. It suffices to prove the claim for the maps $\partial\Delta^n \to \Delta^n$ with $n \geq 1$. By construction, this map is then a bijection on 0-simplices, a Joyal equivalence by Corollary 9.8, and has target an ∞-category because Δ^n and J are ∞-categories. Applying Lemma 9.14 we conclude the corollary. □

We now come back to properties of the fat join which will be needed later.

Corollary 9.16. *Let $X \to X'$ be a Joyal equivalence between simplicial sets and let Y be a simplicial set. Then the map $X \diamond Y \to X' \diamond Y$ is a Joyal equivalence. Likewise, the map $Y \diamond X \to Y \diamond X'$ is a Joyal equivalence.*

Proof. Since $X \diamond Y$ is the pushout

$$\begin{tikzcd} X \times Y \times \partial\Delta^1 \arrow[r] \arrow[d] & X \amalg Y \arrow[d] \\ X \times Y \times \Delta^1 \arrow[r] & X \diamond Y \end{tikzcd}$$

in which the left vertical map is a monomorphism, it suffices by Corollary 9.7 to show that the maps induced by $X \to X'$ on the other three corners are Joyal equivalences. This follows from Exercise 102. □

Definition 9.17. A map of simplicial sets $f\colon X \to Y$ is said to admit a pre-inverse if there exists maps $g\colon Y \to X$ and $\tau\colon \Delta^1 \to \mathrm{Hom}(X, X)$ and $\tau'\colon \Delta^1 \to \mathrm{Hom}(Y, Y)$ such that

(1) $\tau_\varepsilon = \mathrm{id}_X$ and $\tau_{1+\varepsilon} = gf$, where $\varepsilon \in \{0, 1\} \cong \mathbb{Z}/2$,
(2) $\tau'_\varepsilon = \mathrm{id}_Y$ and $\tau'_{1+\varepsilon} = fg$, where again $\varepsilon \in \{0, 1\} \cong \mathbb{Z}/2$,
(3) for all objects x of X, the morphism $\tau(x)\colon \Delta^1 \to X$ represents a degenerate edge of X, and for all objects y of Y, $\tau'(y)\colon \Delta^1 \to Y$ represents a degenerate edge of Y.

Proposition 9.18. *Let X and Y be simplicial sets. Then the canonical map $X \diamond Y \to X \star Y$ of Lemma 9.4 is a Joyal equivalence.*

Proof. As noted in Lemma 9.2 and Lemma 4.14 both functors $- \diamond Y$ and $- \star Y$ commute with filtered colimits. We may therefore reduce the general situation to the case where X has only finitely many non-degenerate simplices. In this case we can write X as a pushout $X' \amalg_{\partial\Delta^n} \Delta^n$. Since pushouts are connected colimits, we also obtain isomorphisms

$$X \diamond Y \cong X' \diamond Y \amalg_{\partial\Delta^n \diamond Y} \Delta^n \diamond Y$$

and likewise for $\star$ in place of $\diamond$. It hence suffices to show that claim for X', $\partial\Delta^n$ and Δ^n. By the same reasoning, the statement for Δ^n (for all n) implies the one for $\partial\Delta^n$ and X' by induction. Hence it remains to show that

$$\Delta^n \diamond Y \to \Delta^n \star Y$$

is a Joyal equivalence. Now the inclusion $I^n \to \Delta^n$ is inner anodyne by Proposition 3.17, and thus in the diagram

$$\begin{tikzcd} I^n \diamond Y \arrow[r] \arrow[d] & I^n \star Y \arrow[d] \\ \Delta^n \diamond Y \arrow[r] & \Delta^n \star Y \end{tikzcd}$$

both vertical maps are Joyal equivalences: For the left vertical map this is Corollary 9.16 and for the right vertical map it is the fact that $- \star Y$ preserves inner anodyne maps, Lemma 4.22

part (1) (in fact it even sends right anodyne maps to inner anodyne maps). It hence suffices to prove the statement for I^n. Since $I^n \cong I^{n-1} \amalg_{\Delta^0} \Delta^1$ it finally suffices to treat the case where X is either Δ^0 or Δ^1. Now we observe that Δ^0 is a retract of Δ^1, and so the map

$$\Delta^0 \diamond Y \to \Delta^0 \star Y$$

is a retract of the map

$$\Delta^1 \diamond Y \to \Delta^1 \star Y.$$

Since retracts of Joyal equivalences are Joyal equivalences, see Exercise 103, it suffices to show that the latter is a Joyal equivalence for all Y. Performing the same reductions to Y, it suffices to finally show that the map

$$\Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1 \cong \Delta^3$$

is a Joyal equivalence. For this we will (almost) construct a *pre-inverse* for this map. In fact, we will construct a zig-zag of such pre-inverses connecting the identity to a now to be constructed inverse. To construct this map we first observe from the definitions that there is a canonical quotient map $(\Delta^1)^{\times 3} \to \Delta^1 \diamond \Delta^1$. We claim that the composite

$$\mathrm{can}\colon (\Delta^1)^{\times 3} \to \Delta^1 \diamond \Delta^1 \to \Delta^3$$

is given by the formula

$$(a,b,c) = \begin{cases} a & \text{if } c = 0, \\ b+2 & \text{if } c = 1. \end{cases}$$

This is just an explicit check of the definitions. We then consider the 3-simplex σ of $(\Delta^1)^{\times 3}$ represented by

$$(000) \to (100) \to (101) \to (111)$$

and its image in $\Delta^1 \diamond \Delta^1$. We first observe that the composite

$$\Delta^3 \to \Delta^1 \diamond \Delta^1 \to \Delta^3$$

is the identity.

Lecture 5 – 20.05.2019. We then must construct a map $\Delta^1 \to \mathrm{Hom}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$ which exhibits the map σ as pre-inverse of the canonical map $\Delta^1 \diamond \Delta^1 \to \Delta^3$. We will construct two maps $\Delta^1 \to \mathrm{Hom}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})$ connecting the identity to an auxiliary map Φ and Φ to $\sigma \circ \mathrm{can}$. Then show that they descend to maps between $\Delta^1 \diamond \Delta^1$ and finally that the resulting maps both have the property that for fixed object, the induced edge of $\Delta^1 \diamond \Delta^1$ is a degenerate edge. In the following picture, the left cube represents the identity of $(\Delta^1)^3$, the middle cube represents the map Φ and the right cube represents the composite $\sigma \circ \mathrm{can}$. There are evident maps from the middle cube to both outer cubes.

$$\begin{tikzcd}[column sep=small]
& 011 \arrow[rr] & & 111 \\
010 \arrow[ur] \arrow[rr] & & 110 \arrow[ur] & \\
& 001 \arrow[uu] \arrow[rr] & & 101 \arrow[uu] \\
000 \arrow[uu] \arrow[ur] \arrow[rr] & & 100 \arrow[uu] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& 011 \arrow[rr] & & 111 \\
000 \arrow[ur] \arrow[rr] & & 100 \arrow[ur] & \\
& 001 \arrow[uu] \arrow[rr] & & 001 \arrow[uu] \\
000 \arrow[uu] \arrow[ur] \arrow[rr] & & 100 \arrow[uu] \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& 111 \arrow[rr] & & 111 \\
000 \arrow[ur] \arrow[rr] & & 100 \arrow[ur] & \\
& 101 \arrow[uu] \arrow[rr] & & 101 \arrow[uu] \\
000 \arrow[uu] \arrow[ur] \arrow[rr] & & 100 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Now one needs to check that these maps descend, after post composition with the projection to the quotient $\Delta^1 \diamond \Delta^1$. To do so, we first observe that all cubes restrict to endomorphisms of $\Delta^1 \times \Delta^1 \times \partial\Delta^1$. Concretely, this means that if we only look at the front and back layer (that is we neglect the diagonal maps) and project to the third coordinate, only identity morphisms remain. Next we claim that there exist compatible endomorphisms of $\Delta^1 \amalg \Delta^1$: Both for Φ and $\sigma \circ \mathrm{can}$ the identity of $\Delta^1 \amalg \Delta^1$ is compatible. This shows that Φ and $\sigma \circ \mathrm{can}$ descend to the pushout $\Delta^1 \diamond \Delta^1$.

Alternatively, one sees that $\Delta^1 \diamond \Delta^1$ is the quotient of $(\Delta^1)^{\times 3}$ where the subsimplicial set $\Delta^1 \times \Delta^1 \times \{0\}$ is collapsed (via the first projection) to Δ^1 and the subsimplicial set $\Delta^1 \times \Delta^1 \times \{1\}$ is collapsed (via the second projection) to Δ^1. We thus have to see that Φ and $\sigma \circ \mathrm{can}$ followed by this projection are suitably invariant, i.e. that they satisfy $F(x, y, 0)$ is independent of y and $F(x, y, 1)$ is independent of x. This is an explicit check.

We observe that any morphism in $\Delta^1 \times \Delta^1 \times \partial\Delta^1$ which is mapped to an identity (a degenerate edge) of $\Delta^1 \amalg \Delta^1$ is also mapped to a degenerate edge in $\Delta^1 \diamond \Delta^1$. This shows that all maps from the middle to the left and right cube are degenerate edges in $\Delta^1 \diamond \Delta^1$. This implies that the horizontal maps between the cubes also descend 1-simplices of $\mathrm{Hom}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$ and furthermore that for any object x of $\Delta^1 \diamond \Delta^1$, these maps are degenerate which finally implies that the map σ is a pre-inverse to can. □

Corollary 9.19. *Let $X \to X'$ be a Joyal equivalence and Y be a simplicial set. Then the map $X \star Y \to X' \star Y$ is again a Joyal equivalence.*

Proof. In the commutative diagram

$$\begin{tikzcd} X \diamond Y \arrow[r] \arrow[d] & X' \diamond Y \arrow[d] \\ X \star Y \arrow[r] & X' \star Y \end{tikzcd}$$

the top horizontal map is a Joyal equivalence by Corollary 9.16 and the vertical maps are Joyal equivalences by Proposition 9.18. □

Definition 9.20. Let $p\colon Y \to W$ be an object of $\mathrm{sSet}_{Y/}$. We define the fat slice of p to be the simplicial set $W^{p/}$ defined by

$$(W^{p/})_n = \mathrm{Hom}_{\mathrm{sSet}_{Y/}}(Y \diamond \Delta^n, W)$$

and the simplicial set $W^{/p}$ to be given by

$$(W^{/p})_n = \mathrm{Hom}_{\mathrm{sSet}_{Y/}}(\Delta^n \diamond Y, W).$$

Lemma 9.21. *The functor $\mathrm{sSet}_{Y/} \to \mathrm{sSet}$ given by sending $p\colon Y \to W$ to $W^{/p}$ is right adjoint to the functor $- \diamond Y$. Likewise, the functor $p \mapsto W^{p/}$ is right adjoin to the functor $Y \diamond -$.*

Proof. By definition the adjunction bijection holds for representables, and hence for all simplicial sets since the functors $- \diamond Y$ and $Y \diamond -$ preserve colimits by Lemma 9.2. □

Lemma 9.22. *Let Y be a simplicial set and $p\colon Y \to W$ a map of simplicial sets. Then there are canonical maps $W_{/p} \to W^{/p}$ and $W_{p/} \to W^{p/}$.*

Proof. On n-simplices, we have to give a map

$$\mathrm{Hom}_{\mathrm{sSet}_{Y/}}(Y \star \Delta^n, W) \to \mathrm{Hom}_{\mathrm{sSet}_{/Y}}(Y \diamond \Delta^n, W).$$

For this, it suffices to recall that there is a map $Y \diamond \Delta^n \to Y \star \Delta^n$ in $\mathrm{sSet}_{Y/}$ which is natural with respect to maps in the simplex category Δ. Likewise for the other slice. □

As a consequence of Theorem 9.13 we can establish an analog of Theorem 4.23 for the fat slices.

Lemma 9.23. *Let*

$$S \xrightarrow{i} T \xrightarrow{f} \mathcal{C} \xrightarrow{p} \mathcal{D}$$

be maps of simplicial sets such that i is a monomorphism and p is an isofibration between ∞-categories. Then the functor

$$\mathcal{C}^{f/} \longrightarrow \mathcal{D}^{pf/} \times_{\mathcal{D}^{pfi/}} \mathcal{C}^{fi/}$$

is a left fibration. Likewise, the functor

$$\mathcal{C}^{/f} \longrightarrow \mathcal{D}^{/pf} \times_{\mathcal{D}^{/pfi}} \mathcal{C}^{/fi}$$

is a right fibration.

Proof. Let $A \to B$ be a left anodyne map and consider a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C}^{f/} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D}^{pf/} \times_{\mathcal{D}^{pfi/}} \mathcal{C}^{fi/} \end{tikzcd}$$

By adjunction, this is equivalent to the lifting problem

$$\begin{tikzcd} T \diamond A \amalg_{S \diamond A} S \diamond B \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ T \diamond B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

in which the left vertical map is a monomorphism. We wish to show that it is a Joyal equivalence, so that we can allude to Theorem 9.13 to conclude the lemma. We claim that in the diagram

$$\begin{tikzcd} T \diamond A \amalg_{S \diamond A} S \diamond B \arrow[r] \arrow[d] & T \star A \amalg_{S \star A} S \star B \arrow[d] \\ T \diamond B \arrow[r] & T \star B \end{tikzcd}$$

both horizontal maps are Joyal equivalences: For the lower horizontal map this is precisely Proposition 9.18 and for the upper horizontal map we use that these are pushouts along monomorphisms, so that a 3-fold application of Proposition 9.18 together with Corollary 9.7 gives the claim. Then we recall from Lemma 4.22 that the right vertical map is inner anodyne and hence a Joyal equivalence. Again, the argument for the other slice is the same. □

Corollary 9.24. *Let $p\colon Y \to \mathcal{C}$ be a diagram. Then the functor $\mathcal{C}^{p/} \to \mathcal{C}$ is a left fibration and the functor $\mathcal{C}^{/p} \to \mathcal{C}$ is a right fibration. In particular, both $\mathcal{C}^{p/}$ and $\mathcal{C}^{/p}$ are ∞-categories.*

Proof. This is Lemma 9.23 in the special case $\emptyset \to Y \to \mathcal{C} \to \Delta^0$. □

Proposition 9.25. *Let $p\colon Y \to \mathcal{C}$ be a diagram. Then the canonical functor*

$$\mathcal{C}_{p/} \longrightarrow \mathcal{C}^{p/}$$

is a Joyal equivalence. The same is true for the other slice.

Proof. By **??** it suffices to show that for every simplicial set K the canonical functor

$$\mathrm{Fun}(K, \mathcal{C}_{p/}) \longrightarrow \mathrm{Fun}(K, \mathcal{C}^{p/})$$

is a Joyal equivalence. By adjunction, this functor is isomorphic to

$$\mathrm{Fun}_{p/}(Y \star K, \mathcal{C}) \longrightarrow \mathrm{Fun}_{p/}(Y \diamond K, \mathcal{C}).$$

By definition, the subscript p on the left hand side denotes the pullback of simplicial sets

$$\begin{tikzcd} \mathrm{Fun}_{p/}(Y \star K, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(Y \star K, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "p"] & \mathrm{Fun}(Y, \mathcal{C}) \end{tikzcd}$$

and likewise for $\mathrm{Fun}_{p/}(Y \diamond K, \mathcal{C})$. We observe that the right vertical map is an isofibration because the map $Y \to Y \star K$ is a monomorphism and $\mathcal{C}$ is an ∞-category. The map $Y \diamond K \to Y \star K$ induces a map of the pullback diagrams which is an isomorphism on the lower corners, so by Lemma 9.6 it suffices to show that the functor

$$\mathrm{Fun}(Y \star K, \mathcal{C}) \longrightarrow \mathrm{Fun}(Y \diamond K, \mathcal{C})$$

is a Joyal equivalence which follows from Proposition 9.18. Notice that we also use here that the map $Y \to Y \diamond K$ is a monomorphism which follows immediately from the definitions. $\square$

Using ordinary slices one can also define right- and left mapping spaces in an ∞-category:

Definition 9.26. Let $\mathcal{C}$ be an ∞-category and let x and y be objects of $\mathcal{C}$. We define the right mapping space by the pullback

$$\begin{tikzcd} \mathrm{map}^R_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_{/y} \arrow[d] \\ \Delta^0 \arrow[r, "x"] & \mathcal{C} \end{tikzcd}$$

and the left mapping space by

$$\begin{tikzcd} \mathrm{map}^L_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"] & \mathcal{C} \end{tikzcd}$$

Remark. We notice that the map $\mathrm{map}^R_{\mathcal{C}}(x,y) \to \Delta^0$ is a right fibration and that the map $\mathrm{map}^L_{\mathcal{C}}(x,y) \to \Delta^0$ is a left fibration. By **??**, both are in fact Kan fibrations so that the left and right mapping spaces are Kan complexes.

We wish to compare these to the mapping space we have already defined in Definition 3.40. The following lemma will take care of this.

Lemma 9.27. *The following two diagrams are pullback diagrams.*

$$\begin{tikzcd} \mathcal{C}^{/y} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "(\mathrm{id}, y)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^{x/} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "(x, \mathrm{id})"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Proof. We show that the left square is a pullback diagram. The argument for the right square is analogues. We have to show that the diagram is a pullback on all n-simplices, so we consider the following diagram

$$\begin{tikzcd} \mathrm{Hom}_{/y}(\Delta^n \diamond \Delta^0, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^n \diamond \Delta^0, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^n \times \Delta^0 \times \Delta^1, \mathcal{C}) \arrow[d] \\ \mathrm{Hom}(\Delta^n, \mathcal{C}) \arrow[r, "(\mathrm{id}, y)"] \arrow[d] & \mathrm{Hom}(\Delta^n, \mathcal{C}) \times \mathrm{Hom}(\Delta^0, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^n \times \Delta^0 \times \partial\Delta^1, \mathcal{C}) \\ \Delta^0 \arrow[r, "y"] & \mathrm{Hom}(\Delta^0, \mathcal{C}) & \end{tikzcd}$$

We wish to show that the big top square is a pullback. The right most square is a pullback by the very definition of the fat join $\Delta^n \diamond \Delta^0$. The lower left square is a pullback by inspection. Combining the left two squares is a pullback by definition. It follows that the top left square is a pullback. Hence, combining the two top squares, we also obtain a pullback as needed. $\square$

Passing to fibres we obtain the following corollary.

Corollary 9.28. *The following diagrams are pullbacks*

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathcal{C}^{/y} \arrow[d] \\ \Delta^0 \arrow[r, "x"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] & \mathcal{C}^{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"] & \mathcal{C} \end{tikzcd}$$

Lecture 6 – 22.05.2019. We can then use the following lemma to compare the various definitions of mapping spaces.

Lemma 9.29. *Consider a diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{E}' \arrow[dl, "p'"] \\ & \mathcal{C} & \end{tikzcd}$$

where p and p' are isofibrations and where f is a Joyal equivalence. Then for all objects x in $\mathcal{C}$, the induced map on fibres $\mathcal{E}_x \to \mathcal{E}'_x$ is also a Joyal equivalence.

Proof. This follows immediately from Lemma 9.6. $\square$

We will see a partial converse of this lemma later.

Corollary 9.30. *Let $\mathcal{C}$ be an ∞-category and let x and y be objects of $\mathcal{C}$. Then the maps*

$$\mathrm{map}^R_{\mathcal{C}}(x, y) \to \mathrm{map}_{\mathcal{C}}(x, y) \leftarrow \mathrm{map}^L_{\mathcal{C}}(x, y)$$

are homotopy equivalences.

Proof. Apply Lemma 9.29 to the diagrams

$$\begin{tikzcd} \mathcal{C}_{/y} \arrow[rr] \arrow[dr] & & \mathcal{C}^{/y} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{x/} \arrow[rr] \arrow[dr] & & \mathcal{C}^{x/} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

using Corollary 9.24 and Proposition 9.25. □

Finally, we wish to compare the mapping spaces of the coherent nerve of a Kan enriched category with the mapping Kan complexes that are present in the simplicial category. The proofs are beyond the scope of these lectures, but see for instance [Lur09, Section 2.2.2].

Theorem 9.31. *Let $\mathcal{C}$ be a Kan enriched category and let x and y be objects of $\mathcal{C}$. Then there is a canonical map*

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{map}_{\mathrm{N}(\mathcal{C})}(x, y)$$

which is a homotopy equivalence. The homotopy class of this map is natural in x and y.

Corollary 9.32. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a weak equivalence of Kan enriched categories. Then then induced functor $\mathrm{N}(F)\colon \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{C}')$ is a Joyal equivalence of ∞-categories.*

Proof. The functor $\mathrm{N}(F)$ is essentially surjective if and only if F is weakly essentially surjective in the sense of **??**. Furthermore, Theorem 9.31 shows that F is weakly fully faithful if and only if $\mathrm{N}(F)$ is fully faithful. Hence we conclude by Theorem 7.16. □

Recall that we have defined the simplicial category CW whose objects are CW-complexes and whose simplicial set of maps is given by $\mathcal{S}(\mathrm{map}(X, Y))$. Its coherent nerve was denoted by An. We have furthermore defined the simplicial category Kan whose objects are Kan complexes and whose simplicial set of maps is given by the internal hom $\underline{\mathrm{Hom}}(A, B)$. We claim that there is a functor CW $\to$ Kan constructed as follows: On objects, it sends X to $\mathcal{S}(X)$. On morphisms we have to give a simplicial map

$$\mathcal{S}(\mathrm{map}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{S}(X), \mathcal{S}(Y))$$

compatible with composition. By adjunction, this is equivalent to giving a map

$$\mathcal{S}(\mathrm{map}(X, Y)) \times \mathcal{S}(X) \longrightarrow \mathcal{S}(Y).$$

We then recall that $\mathcal{S}$ is a right adjoint and hence preserves products. It hence suffices to give a simplicial map

$$\mathcal{S}(\mathrm{map}(X, Y) \times X) \longrightarrow \mathcal{S}(Y)$$

where we can use the ordinary composition map $\mathrm{map}(X, Y) \times X \to Y$ and apply the functor $\mathcal{S}$.

Corollary 9.33. *The previously described functor* $\mathrm{An} = \mathrm{N}(\mathrm{CW}) \to \mathrm{N}(\mathrm{Kan}) = \widehat{\mathrm{An}}$ *is a Joyal equivalence.*

Proof. We show that the functor CW $\to$ Kan is a weak equivalence of simplicial categories. It is weakly essentially surjective because every Kan complex X is homotopy equivalent to $\mathcal{S}(|X|)$ and thus up to equivalence in the image of the functor CW $\to$ Kan. To show that the functor is fully faithful, we have to show that the map

$$\mathcal{S}(\mathrm{map}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{S}(X), \mathcal{S}(Y))$$

is a homotopy equivalence. This is the case if and only if the composite

$$\mathrm{map}(X,Y) \longrightarrow |\mathcal{S}(\mathrm{map}(X,Y))| \longrightarrow |\underline{\mathrm{Hom}}(\mathcal{S}(X),\mathcal{S}(Y))|$$

is a homotopy equivalence. We will now need to use the fact that for any two Kan complexes A and B, the canonical map

$$|\underline{\mathrm{Hom}}(A,B)| \longrightarrow \mathrm{map}(|A|,|B|)$$

is a homotopy equivalence. □

Corollary 9.34. *The canonical functor* $\mathrm{An} \to \mathrm{Cat}_\infty$ *is fully faithful.*

Proof. The functor is given by applying the coherent nerve the the functor of simplicial categories $\mathrm{Kan} \to \mathrm{Cat}^1_\infty$. We thus only need to show that for any two Kan complexes X and Y, the canonical map

$$\underline{\mathrm{Hom}}(X,Y) \to \mathrm{Fun}(X,Y)^{\simeq}$$

is a homotopy equivalence. In fact, it is an isomorphism of simplicial sets, so the proposition follows. □

Remark. Later, we will be able to give a proof of this fact which is not based on comparing the mapping anima in An and Cat_∞ on the level of the simplicial categories which define these ∞-categories.

10. (Co)Cartesian fibrations

Let $p\colon X \to Y$ be an inner fibration between simplicial sets.

Definition 10.1. A morphism $f\colon \Delta^1 \to X$ is called p-cartesian if for $n \geq 2$ any lifting problem

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d] & X \arrow[d] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution. Dually, it is called p-cocartesian if any lifting problem

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution. One calls such an f a p-(co)cartesian lift of $p(f)$.

The definition of (co)cartesian morphisms can be rephrased in terms of slices as follows.

Lemma 10.2. *Let $p\colon X \to Y$ be an inner fibration and let $f\colon x \to y$ be a morphism in Y. Then f is p-cartesian if and only if the functor*

$$X_{/f} \longrightarrow X_{/y} \times_{Y_{/p(y)}} Y_{/pf}$$

is a trivial fibration. Dually, f is p-cocartesian if and only if the functor

$$X_{f/} \longrightarrow X_{x/} \times_{Y_{p(x)/}} Y_{pf/}$$

is a trivial fibration.

Proof. Let $\partial\Delta^n \to \Delta^n$ for $n \geq 0$ and consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & X_{/y} \times_{Y_{/p(y)}} Y_{/pf} \end{tikzcd}$$

which we wish to solve. By adjunction this corresponds to the lifting problem

$$\begin{tikzcd} \partial\Delta^n \star \Delta^1 \amalg_{\partial\Delta^n \star \Delta^0} \Delta^n \star \Delta^0 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \star \Delta^1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where the restriction to Δ^1 is given by f. We recall that the left vertical map is isomorphic to $\Lambda^{n+2}_{n+2} \to \Delta^{n+2}$ and that the inclusion of Δ^1 into Λ^{n+2}_{n+2} is the edge $\Delta^{\{n+1,n+2\}}$. The diagram can thus be solved for all $n \geq 0$. The cocartesian case is similiar. $\square$

Remark. Suppose that $p\colon X \to Y$ is an inner fibration between ∞-categories. Then the map $X_{/p} \to X^{/p}$ is a Joyal equivalence for any diagram $p\colon W \to X$ by Proposition 9.25. Furthermore, the map

$$X^{/f} \longrightarrow X^{/y} \times_{Y^{/p(y)}} Y^{/pf}$$

is also a right fibration by Lemma 9.23. Hence, f is cartesian if and only if this map is a trivial fibration. Likewise, f is cocartesian if and only if the map

$$X^{f/} \longrightarrow X^{x/} \times_{Y^{p(x)/}} Y^{pf/}$$

is a trivial fibration.

Lemma 10.3. *Let $p\colon \mathcal{E} \to \mathcal{D}$ and $q\colon \mathcal{D} \to \mathcal{C}$ be inner fibrations between ∞-categories and let $f\colon \Delta^1 \to \mathcal{E}$ be a morphism in $\mathcal{E}$ such that $p(f)$ is q-(co)cartesian. Then f is p-(co)cartesian if and only if f is qp-(co)cartesian.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{E}_{/f} \arrow[r] \arrow[drr] & \mathcal{E}_{/y} \times_{\mathcal{D}_{/p(y)}} \mathcal{D}_{/pf} \arrow[r, "\simeq"] & \mathcal{E}_{/y} \times_{\mathcal{D}_{/p(y)}} \mathcal{D}_{/p(y)} \times_{\mathcal{C}_{/qp(y)}} \mathcal{C}_{/qpf} \arrow[d, "\cong"] \\ & & \mathcal{E}_{/y} \times_{\mathcal{C}_{/qp(y)}} \mathcal{C}_{/qpf} \end{tikzcd}$$

in which the second horizontal map is a trivial fibration by the assumption that pf is q-cartesian. The first horizontal map and the diagonal map are both right fibrations and thus trivial fibrations if and only if they are Joyal equivalences. The lemma follows. $\square$

Remark. The previous lemma holds more generally without the assumption that the simplicial sets involved are ∞-categories, see [Lur09, Prop. 2.4.1.3]: Since the composition of trivial fibrations is again a trivial fibration we immediately see that if f is p-cartesian then it is also qp-cartesian. To see the converse, we want to show that the map $\mathcal{E}_{/f} \to \mathcal{E}_{/y} \times_{\mathcal{D}_{/p(y)}} \mathcal{D}_{/pf}$ (which is a Joyal equivalence by 3-for-2) in fact has contractible fibres and then allude to the general fact that a right fibration which has contractible fibres is a trivial fibration (or a Joyal equivalence and thus a trivial fibration), see [Lur09, Lemma 2.1.3.4] (or Theorem 10.21). The fibre we are interested in is the fibre of the map between fibres of the other two maps. These are contractible by assumption so we are done.

Lemma 10.4. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be an inner fibration between ∞-categories. Let $f\colon \Delta^1 \to \mathcal{E}$ be a morphism. Then f is an equivalence if and only if it is p-(co)cartesian and $p(f)$ is an equivalence.*

Proof. Suppose that f is an equivalence. Then so is $p(f)$ and the Joyal lifting theorem **??** implies that f is both cartesian and cocartesian. Assume now conversely that $p(f)$ is an equivalence so that it is q-cartesian where $q\colon \mathcal{C} \to \Delta^0$ is the projection, by Exercise 116. We find that f is p-cartesian if and only if f is qp-cartesian by Lemma 10.3, which in turn is the case if and only if f is an equivalence by another application of Exercise 116. $\square$

Lemma 10.5. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be an inner fibration and let $\sigma\colon \Delta^2 \to \mathcal{E}$ which we will depict as the diagram*

$$\begin{tikzcd} & y \arrow[dr, "\varphi"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] & & z \end{tikzcd}$$

Suppose that φ is p-cartesian. Then g is p-cartesian if and only if g is p-cartesian.

Proof. We observe that the inclusions $\Delta^{\{0,1\}} \to \Delta^2$ and $\Delta^{\{0,2\}} \to \Delta^2$ are left anodyne, so it follows that in the two commutative squares

$$\begin{tikzcd} \mathcal{E}_{/\sigma} \arrow[r] \arrow[d] & \mathcal{E}_{/z} \times_{\mathcal{C}_{/p(z)}} \mathcal{C}_{/p\sigma} \arrow[d] \\ \mathcal{E}_{/g} \arrow[r] & \mathcal{E}_{/z} \times_{\mathcal{C}_{/p(z)}} \mathcal{C}_{/pg} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_{/\sigma} \arrow[r] \arrow[d] & \mathcal{E}_{/y} \times_{\mathcal{C}_{/p(y)}} \mathcal{C}_{/p\sigma} \arrow[d] \\ \mathcal{E}_{/f} \arrow[r] & \mathcal{E}_{/y} \times_{\mathcal{C}_{/p(y)}} \mathcal{C}_{/pf} \end{tikzcd}$$

all vertical maps are trivial fibrations. We wish to show the the left lower horizontal map is a trivial fibration if and only if the right lower horizontal map is. Since each are isofibrations in any case, it suffices to show that one is a Joyal equivalence if and only if the other is. By means of the commutative squares, it hence suffices to show that the left top horizontal map is a Joyal equivalence if and only the right top horizontal map is. For this we consider the diagram

$$\begin{tikzcd} & \mathcal{E}_{/\sigma} \arrow[dl, bend right] \arrow[d] \arrow[dr, bend left] & \\ \mathcal{E}_{/y} \times_{\mathcal{C}_{/p(y)}} \mathcal{C}_{/p\sigma} & \mathcal{E}_{/\varphi} \times_{\mathcal{C}_{/p\varphi}} \mathcal{C}_{/p\sigma} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathcal{E}_{/z} \times_{\mathcal{C}_{/p(z)}} \mathcal{C}_{/p\sigma} \end{tikzcd}$$

Here, the right horizontal map is an equivalence because φ is p-cartesian, so that the map $\mathcal{E}_{/\varphi} \to \mathcal{E}_{/z} \times_{\mathcal{C}_{/p(z)}} \mathcal{C}_{/p\varphi}$ is an equivalence. The left horizontal map is an equivalence because it is induced by the restriction $\{1\} \to \Delta^{\{1,2\}}$ which is left anodyne. We thus find that the left bended map is an equivalence if and only if the right bended map is an equivalence, which was left to show. $\square$

The following notion will be convenient to use.

Definition 10.6. A commutative diagram of ∞-categories

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d, "p"] \\ \mathcal{C} \arrow[r] & \mathcal{C}' \end{tikzcd}$$

will be called homotopy cartesian if p is an isofibration and the induced map $\mathcal{E} \to \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}'$ is a Joyal equivalence. If $\mathcal{C} = \Delta^0$ we will also use the phrase homotopy fibre sequence for the composition $\mathcal{E} \to \mathcal{E}' \to \mathcal{C}'$.

Remark. More generally, one can neglect the condition that p be an isofibration as follows: One can factor p as a Joyal equivalence followed by an isofibration $\mathcal{E}' \xrightarrow{\simeq} \mathcal{E}'' \xrightarrow{q} \mathcal{C}'$ and then ask that the induced map $\mathcal{E} \to \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}''$ be a Joyal equivalence.

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Lemma 10.7. *Let $p \colon \mathcal{C} \to \mathcal{D}$ be an isofibration between ∞-categories and let x and y be objects of $\mathcal{C}$. Then the induced map*

$$\mathrm{map}_{\mathcal{C}}(x, y) \to \mathrm{map}_{\mathcal{D}}(px, py)$$

is a Kan fibration. The same holds true for map^L and map^R, even if p is only an inner fibration.

Proof. We consider the diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{D}}(p(x), p(y)) \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\ \mathcal{C}^{/y} \arrow[r] & \mathcal{D}^{/p(y)} \times_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ & \mathcal{D}^{/p(y)} \arrow[r] & \mathcal{D} \end{tikzcd}$$

respectively the ones with the ordinary slice and its variant using $\mathcal{C}_{x/}$ instead of $\mathcal{C}_{/y}$. All squares are pullbacks in this diagram: The lower one is by definition, as is the right large one, so the small square in the top right corner is a pullback. The combined large horizontal square is again a pullback by definition, so we deduce the the left small square is a pullback. The lower horizontal map in this diagram is a right fibration (in the case of $\mathcal{C}_{x/}$ is is a left fibration) hence so is the upper horizontal map. Since $\mathrm{map}_{\mathcal{D}}(px, py)$ is a Kan complex, it is in fact a Kan fibration as claimed. $\square$

Lemma 10.8. *Let $\mathcal{C} \to \mathcal{D}$ be an isofibration between ∞-categories and let z be an object of $\mathcal{D}$, and x and y objects of $\mathcal{C}$ with $px = py = z$. Then the diagram*

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}_z}(x,y) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(x,y) \arrow[d] \\ \Delta^0 \arrow[r, "\mathrm{id}_z"] & \mathrm{map}_{\mathcal{D}}(z,z) \end{tikzcd}$$

is a pullback.

Proof. To see the remaining claim we consider the diagram

$$\begin{tikzcd}
& F_{x,y} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{map}_{\mathcal{E}}(x,y) \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{map}_{\mathcal{C}}(z,z) \arrow[dl] \arrow[dd] \\
\mathcal{E}_z^{\Delta^1} \arrow[rr] \arrow[dd] & & \mathcal{E}^{\Delta^1} \arrow[rr] \arrow[dd] & & \mathcal{C}^{\Delta^1} \arrow[dd] & \\
& \Delta^0 \arrow[rr] \arrow[dl] & & \Delta^0 \arrow[rr] \arrow[dl] & & \Delta^0 \arrow[dl] \\
\mathcal{E}_z \times \mathcal{E}_z \arrow[rr] & & \mathcal{E} \times \mathcal{E} \arrow[rr] & & \mathcal{C} \times \mathcal{C} &
\end{tikzcd}$$

all horizontal composites are given by the fibre inclusion over the point corresponding to id_z. Since the middle and right vertical square are pullbacks, so is the left most vertical square. $\square$

Corollary 10.9. *Let $\mathcal{C} \to \mathcal{D}$ be an isofibration between ∞-categories and let z be an object of $\mathcal{D}$ and x and y be objects of $\mathcal{C}_z$. Then the diagram*

$$\begin{tikzcd}
\mathrm{map}^?_{\mathcal{C}_z}(x,y) \arrow[r] \arrow[d] & \mathrm{map}^?_{\mathcal{E}}(x,y) \arrow[d] \\
\Delta^0 \arrow[r, "\mathrm{id}"] & \mathrm{map}^?_{\mathcal{C}}(p(x),p(y))
\end{tikzcd}$$

is homotopy cartesian for $? = R, L$ or void.

Proof. For the ordinary mapping anima, this follows immediately since it is a pullback and the right vertical map is a Kan fibration. To see the claim for the right and left mapping anima, we use Lemma 10.7 and Corollary 9.30. $\square$

Lemma 10.10. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be an inner fibration between ∞-categories and let $f\colon \Delta^1 \to \mathcal{E}$ be a p-cartesian morphism from x to y. Then, for all objects z of $\mathcal{E}$, the induced map*

$$\mathcal{E}^{/f} \times_{\mathcal{E}} \{z\} \longrightarrow \left(\mathcal{E}^{/y} \times_{\mathcal{C}^{/p(y)}} \mathcal{C}^{/pf}\right) \times_{\mathcal{E}} \{z\}$$

is a trivial fibration as well.

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{/f} \times_{\mathcal{E}} \{z\} \arrow[r] \arrow[d] & \mathcal{E}^{/f} \arrow[d] \\
\left(\mathcal{E}^{/y} \times_{\mathcal{C}^{/p(y)}} \mathcal{C}^{/pf}\right) \times_{\mathcal{E}} \{z\} \arrow[r] \arrow[d] & \mathcal{E}^{/y} \times_{\mathcal{C}^{/p(y)}} \mathcal{C}^{/pf} \arrow[d] \\
\Delta^0 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

in which both the big square and the lower square are pullbacks, and hence so is the upper square. Since in this square the right vertical map is a trivial fibration by the assumption that f is p-cartesian, the claim follows. $\square$

Remark. Replacing the fat slice with the ordinary slice, the same statement holds true for inner fibrations between arbitrary simplicial sets.

Corollary 10.11. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be an inner fibration between ∞-categories and let $f\colon \Delta^1 \to \mathcal{E}$ be a p-cartesian morphism of $\mathcal{E}$ from x to y and let z be an object of $\mathcal{E}$. Then the diagram*

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{E}}(z,x) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}}(z,y) \arrow[d] \\
\mathrm{map}_{\mathcal{C}}(p(z),p(x)) \arrow[r] & \mathrm{map}_{\mathcal{C}}(p(z),p(y))
\end{tikzcd}$$

is a homotopy cartesian diagram of ∞-groupoids. Here the horizontal maps are induced by post composition with f and pf respectively.

Proof. We consider the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{E}}(z,x) \arrow[d] & \mathcal{E}^{/f} \times_{\mathcal{E}} \{z\} \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}}(z,y) \arrow[d] \\
\mathrm{map}_{\mathcal{C}}(p(z),p(x)) & \mathcal{C}^{/pf} \times_{\mathcal{C}} \{p(z)\} \arrow[l, "\simeq"'] \arrow[r] & \mathrm{map}_{\mathcal{C}}(p(z),p(y))
\end{tikzcd}$$

and observe that the left horizontal maps are trivial fibrations as the inclusion $\{0\} \to \Delta^1$ is left anodyne. It thus suffices to show that the right square is homotopy cartesian, and since the right vertical map is a Kan fibration by Lemma 10.7 it suffices to recall that the map

$$\mathcal{E}^{/f} \to \mathcal{E}^{/y} \times_{\mathcal{C}^{/p(y)}} \mathcal{C}^{/pf}$$

is a trivial fibration, so that the same remains true after applying $- \times_{\mathcal{E}} \{z\}$. $\square$

Remark. The statement of the corollary is not quite correct: The square we construct does not a priori commute (only up to homotopy). Since the right vertical map is a Kan fibration, it can always be replaced by a commutative diagram without changing the homotopy types of the participants and a concrete way of doing this is to consider the right square in the diagram appearing in the proof.

Remark. Given an inner fibration $p\colon \mathcal{E} \to \mathcal{C}$ and a morphism $f\colon \Delta^1 \to \mathcal{E}$ such that for *every* object z of $\mathcal{E}$ the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{E}}(z,x) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}}(z,y) \arrow[d] \\
\mathrm{map}_{\mathcal{C}}(p(z),p(x)) \arrow[r] & \mathrm{map}_{\mathcal{C}}(p(z),p(y))
\end{tikzcd}$$

is homotopy cartesian, then f is p-cartesian. The needed input is that the map

$$\mathcal{E}^{/f} \to \mathcal{E}^{/y} \times_{\mathcal{C}^{/(y)}} \mathcal{C}^{/pf}$$

is a trivial fibration if and only if it induces an equivalence on each fibre over points in $\mathcal{E}$. As before, this holds in general for right fibrations via a combinatorial argument (without $\mathcal{E}$ and $\mathcal{C}$ having to be ∞-categories). In the case where $\mathcal{E}$ and $\mathcal{C}$ are ∞-categories, we will prove this later for so called cartesian fibrations and deduce the version for right fibrations from this.

Corollary 10.12. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration and let x and y be objects of $\mathcal{E}$. Let $f\colon x' \to y$ be a p-cartesian morphism with $p(x') = p(x)$. Then the diagram*

$$\begin{tikzcd} \mathrm{map}_{\mathcal{E}_{p(x)}}(x,x') \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}}(x,y) \arrow[d] \\ \Delta^0 \arrow[r, "pf"] & \mathrm{map}_{\mathcal{C}}(p(x),p(y)) \end{tikzcd}$$

is homotopy cartesian.

Proof. By Corollary 10.11 we have a homotopy cartesian diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{E}}(x,x') \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}}(x,y) \arrow[d] \\ \mathrm{map}_{\mathcal{C}}(p(x),p(x')) \arrow[r] & \mathrm{map}_{\mathcal{C}}(p(x),p(y)) \end{tikzcd}$$

so that the induced map of vertical fibres is an equivalence over the point $\Delta^0 \to \mathrm{map}_{\mathcal{C}}(p(x),p(x'))$ corresponding to $\mathrm{id}_{p(x)}$. The left vertical fibre is given by $\mathrm{map}_{\mathcal{E}_{p(x)}}(x,x')$ by Corollary 10.9 so the claim is shown. □

Inner fibrations $p\colon \mathcal{E} \to \mathcal{C}$ which have a sufficient supply of cartesian morphisms thus are such that the mapping anima in $\mathcal{E}$ are controlled by those of $\mathcal{C}$ and all fibres.

Definition 10.13. An inner fibration $p\colon X \to Y$ is called a cartesian fibration if every lifting problem

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has a solution which is a p-cartesian morphism in X. Dually, p is called a cocartesian fibration if every lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a solution which is a p-cocartesian morphism in X.

Informally, an inner fibration is a (co)cartesian fibration if it admits a p-(co)cartesian lift of any morphism in $\mathcal{C}$ (which specified source or target).

Example 10.14. Right fibrations are cartesian fibrations and left fibrations are cocartesian fibrations.

Lemma 10.15. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a (co)cartesian fibration between ∞-categories. Then p is an isofibration.*

Proof. This follows from Lemma 10.4 which says that a p-cartesian lift of an equivalence is an equivalence. □

Lemma 10.16. *A cartesian fibration $\mathcal{E} \to \mathcal{C}$ is a right fibration if and only if every morphism in $\mathcal{E}$ is p-cartesian. Dually, a cocartesian fibration is a left fibration if and only if every morphism in $\mathcal{E}$ is p-cocartesian.*

Proof. By definition, in a cartesian fibration one can lift the right outer 1-horn. If furthermore every morphism in $\mathcal{E}$ is p-cartesian, this simply says that one can also lift all right outer horns of dimension greater or equals to two. Conversely, a right fibration admits some lift of the diagram in the definition, and by definition of a right fibration, every morphism in p-cartesian. The argument for cocartesian fibrations is the same. □

Proposition 10.17. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration between ∞-categories. Then p is a right fibration if and only if for all objects x of $\mathcal{C}$, the fibres $\mathcal{E}_x = \mathcal{E} \times_{\mathcal{C}} \{x\}$ are ∞-groupoids.*

Proof. Right fibrations are cartesian fibrations whose fibres are ∞-groupoids. Conversely, assume that $p\colon \mathcal{E} \to \mathcal{C}$ is a cartesian fibration whose fibres are ∞-groupoids. We will show that every morphism is p-cartesian and allude to Lemma 10.16. So let $f\colon \Delta^1 \to \mathcal{E}$ be a morphism from x to y and choose a p-cartesian lift φ of $p(f)$ with target y. We consider the diagram

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "{(f,\varphi)}"] \arrow[d] & \mathcal{E} \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

where the map σ is given by the diagram

$$\begin{tikzcd} & p(y) \arrow[dr, "p(f)"] & \\ p(x) \arrow[ur, "\mathrm{id}"] \arrow[rr, "p(f)"] & & p(y) \end{tikzcd}$$

Since φ is p-cartesian, there exists a dashed arrow in this diagram. The resulting 2-simplex τ is given by the diagram

$$\begin{tikzcd} & y \arrow[dr, "\varphi"] & \\ x \arrow[ur, "\psi"] \arrow[rr, "f"] & & y \end{tikzcd}$$

where now ψ is a morphism in the fibre $\mathcal{E}_x$ over x and is hence invertible by the assumption that all fibres are ∞-groupoids. We may thus apply Lemma 10.5 and see that f is cartesian because φ is. □

Corollary 10.18. *A cartesian fibration is conservative if and only if it is a right fibration.*

Proof. Right fibrations are conservative by Proposition 5.7. Conversely, given a conservative cartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$, the fibre $\mathcal{E}_x$ over each object x of $\mathcal{C}$ is an ∞-groupoid: Each morphism in the fibre is sent to the identity of x by p. We may thus apply Proposition 10.17. □

Definition 10.19. Let $p\colon X \to Y$ and $p'\colon X' \to Y$ be (co)cartesian fibrations. We say that a map $f\colon X \to X'$ is a morphism of (co)cartesian fibrations if $p'f = p$ and f sends p-cartesian morphisms to p'-cartesian morphisms.

Example 10.20. Suppose that $p\colon X \to Y$ is a cartesian fibration and that $p'\colon X' \to Y$ is a right fibration. Then any map $f\colon X \to X'$ with $p'f = p$ is a morphism of cartesian fibrations, because all morphisms in X' are p'-cartesian.

Lecture 8 – 29.05.2019.

Theorem 10.21. *Let $f\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of (co)cartesian fibrations $p\colon \mathcal{E} \to \mathcal{C}$ and $p'\colon \mathcal{E}' \to \mathcal{C}$ between ∞-categories. Then f is a Joyal equivalence if and only if for all objects z of $\mathcal{C}$, the induced map on fibres $\mathcal{E}_z \to \mathcal{E}'_z$ is a Joyal equivalence.*

Proof. The "only if" direction holds more generally for maps between isofibrations, see **??**. Let us hence assume that all induced maps $\mathcal{E}_x \to \mathcal{E}'_x$ are Joyal equivalences. We wish to show that f is a Joyal equivalence. We will show that f is fully faithful and essentially surjective and conclude the theorem from Theorem 7.16. To see that f is essentially surjective, we consider an object y' of $\mathcal{E}'$ and let $x = p'(y')$. Since the map $\mathcal{E}_x \to \mathcal{E}'_x$ is a Joyal equivalence, it is in particular essentially surjective. Hence there exists an object y in $\mathcal{E}_x$ and an equivalence between $f(y)$ and y' in $\mathcal{E}'_x$. It follows that f is essentially surjective.

To see that f is fully faithful we consider two objects x and y in $\mathcal{E}$ and need to show that the map

$$\mathrm{map}_{\mathcal{E}}(x,y) \longrightarrow \mathrm{map}_{\mathcal{E}'}(f(x),f(y))$$

is a homotopy equivalence.

To see this, we choose a p-cartesian lift $\hat{\alpha}\colon x' \to y$ of α (which implies that $p(x') = p(x)$) and note that $f(\hat{\alpha})$ is a p'-cartesian lift of $f(\alpha)$ by assumption. Furthermore, by **??** we have a diagram of homotopy fibre sequences

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{E}_{p(x)}}(x,x') \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}'_{p(x)}}(f(x),f(x')) \arrow[d] \\
\mathrm{map}_{\mathcal{E}}(x,y) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}'}(f(x),f(y)) \arrow[d] \\
\mathrm{map}_{\mathcal{C}}(p(x),p(y)) \arrow[r, equal] & \mathrm{map}_{\mathcal{C}}(p(x),p(y))
\end{tikzcd}$$

and the horizontal map on the base and the fibre is an equivalence by the assumption that f restricts to a fully faithful functor on the firbes. Thus the middle horizontal map is also an equivalence by Lemma 7.15. $\square$

11. Marked simplicial sets and marked anodyne maps

We have seen that a left/right fibration is a special kind of (co)cartesian fibration. Since left fibrations are determined by a right lifting property (with respect to left anodyne maps) one can ask whether (co)cartesian fibrations are also characterized by a lifting property. This is not true on the nose, but in it is true in the context of marked simplicial sets.

Definition 11.1. A marked simplicial set is a pair (X, S) where S is a subset of the 1-simplices of X which contains all degenerate 1-simplices. The elements of S will be called marked edges in X. There is a corresponding category sSet_+ of marked simplicial sets, where morphisms are required to send marked edges to marked edges.

Example 11.2. Let X be a simplicial set. Then we denote by $X^\flat$ the marked simplicial set where an edge is marked if and only if it is degenerate. We denote by $X^\sharp$ the marked simplicial set in which all morphisms are marked. This produces functors $(-)^\flat, (-)^\sharp\colon \mathrm{sSet} \to \mathrm{sSet}_+$. There are also two functors $\mathrm{sSet}_+ \to \mathrm{sSet}$: The one forgets the marking, and the other takes the smallest sub simplicial set spanned by the marked 1-simplices.

Example 11.3. Let $p\colon X \to Y$ be a map of simplicial sets. We denote by $X^\natural$ the marked simplicial set where an edge is marked if and only if it is p-cocartesian. Thus, if $p\colon X \to Y$ is a cocartesian fibration, then the map $X^\natural \to Y^\sharp$ is a map of marked simplicial sets.

Definition 11.4. We denote the smallest saturated set containing the following maps of marked simplicial sets *marked left anodyne.*

(1) For all $0 < i < n$, the maps $(\Lambda^n_i)^\flat \to (\Delta^n)^\flat$,
(2) for every $n > 0$, the map $(\Lambda^n_0)^{s\flat} \to (\Delta^n)^{s\flat}$ where the superscript $s\flat$ denotes all degenerate edges and the special edge $\Delta^{\{0,1\}}$ to be marked,
(3) the map $(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$, and
(4) for every ∞-groupoid X, the map $X^\flat \to X^\sharp$.

Remark. A different (but equivalent) generating set of the marked left anodyne maps is given by the following maps:

(1') For all $0 < i < n$, the maps $(\Lambda^n_i)^\flat \to (\Delta^n)^\flat$,
(2') the maps $(\Delta^1)^\sharp \times (\Delta^1)^\flat \cup \{0\} \times (\Delta^1)^\sharp \to (\Delta^1)^\sharp \times (\Delta^1)^\sharp$,
(3') the maps $(\Delta^1)^\sharp \times (\partial\Delta^n)^\flat \cup \{0\} \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$,
(4') the map $J^\flat \to J^\sharp$.

Remark. We could also equally well take the set generated by the maps (1), (2), (3′), and (4), we refer to [Lur09, Proposition 3.1.1.5] for details. This will be used in Lemma 11.8.

Proposition 11.5. *A map of marked simplicial sets $p\colon X \to Y$ has the right lifting property with respect to all marked left anodyne maps if and only if the following hold*

(1) *p is an inner fibration,*
(2) *An edge of X is marked if and only if it is p-cocartesian and its image is marked in Y,*
(3) *any lifting problem of marked simplicial sets*

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ (\Delta^1)^\sharp \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

can be solved.

Proof. Since $(-)^\flat$ is a left adjoint to the forgetful functor $\mathrm{sSet}_+ \to \mathrm{sSet}$, we find that p satisfies the RLP wrt the maps $(\Lambda^n_i)^\flat \to (\Delta^n)^\flat$ for $0 < i < n$ if and only if the underlying map of p is an inner fibration. We now assume that p satisfies the RLP wrt marked left anodyne maps and show that (2) and (3) hold: Since $(\Delta^1)^{s\flat} = (\Delta^1)^\sharp$ we find that (3) holds. To see that (2) holds, we first show any marked edge $f\colon \Delta^1 \to X$ is p-cocartesian. For this, consider a lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] & \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\ & \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

which we want to show to admit a solution. Since the above composite is marked, this gives rise to a diagram of marked simplicial sets

$$\begin{tikzcd} (\Lambda^n_0)^{s\flat} \arrow[r] \arrow[d] & X \arrow[d] \\ (\Delta^n)^{s\flat} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

which can be solved since the left vertical map is marked anodyne. To show the converse, consider a p-cocartesian morphism f from x to y in X such that $pf\colon \Delta^1 \to Y$ is marked. Consider the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ (\Delta^1)^\sharp \arrow[r, "pf"] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

which can be solved by a previous argument. The resulting morphism g is marked and thus p-cocartesian. Consider then the diagram

$$\begin{tikzcd} (\Lambda^2_0)^{s\flat} \arrow[r, "{(g,f)}"] \arrow[d] & X \arrow[d] \\ (\Delta^2)^{s\flat} \arrow[r] \arrow[ur, dashed, "\tau"] & Y \end{tikzcd}$$

which can again be solved by assumption. Here, the lower horizontal map is a degenerate 2-simplex on the morphism $pf = pg$. We denote $\tau_{|\Delta^{\{1,2\}}}$ by h. It follows from **??** that h is cocartesian and since $ph = \mathrm{id}$ and that h is an equivalence. We consider the diagram

$$\begin{tikzcd} J^\flat \arrow[r, "h"] \arrow[d] & X \arrow[d] \\ J^\sharp \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & Y \end{tikzcd}$$

which can again be solved as J is an ∞-groupoid. It follows that h is marked. We then observe that the RLP with respect to the map $(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ implies that a composition of marked morphisms is marked. Since f is a composition of g and h, we find that f is marked as needed.

We now prove that any map $p\colon X \to Y$ of marked simplicial sets with the properties (1)–(3) of the statement satisfies the RLP with respect to marked left anodyne maps. The lifting property with respect to $(-)^\flat$ applied to inner horn inclusions is clear. Consider a lifting problem

$$\begin{tikzcd} (\Lambda^n_0)^{s\flat} \arrow[r] \arrow[d] & X \arrow[d] \\ (\Delta^n)^{s\flat} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since the special edge $\Delta^{\{0,1\}}$ is marked in X and marked edges are p-cocartesian, we find a lift by the definition of p-cocartesian edges. To see that p satisfies the RLP with respect to the map $(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ we have to show that a composite of marked edges is again marked. This follows from the assumptions and the fact that a composite of p-cocartesian edges is again p-cocartesian Lemma 10.5. Finally we need to argue that p has the RLP wrt $K^\flat \to K^\sharp$ for any ∞-groupoid K. This follows from the fact that equivalences are p-cocartesian and thus marked by assumption on p. □

Corollary 11.6. *A map of marked simplicial sets $p\colon (X,S) \to Y^\sharp$ has the right lifting property with respect to marked left anodyne maps if and only if S equals all p-cocartesian edges and p is a cocartesian fibration.*

We find the following consequence which will be of use to us later.

Corollary 11.7. *The map $(\Lambda^2_0)^\sharp \amalg_{(\Lambda^2_0)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ is marked anodyne.*

Proof. We observe that the left hand side is the simplicial set Δ^2 in which the edges $\Delta^{\{0,1\}}$ and $\Delta^{\{0,2\}}$ are marked. We need to show that this map satisfies the LLP wrt maps $p\colon X \to Y$ satisfying the properties (1)–(3) of Proposition 11.5. So we consider a diagram

$$\begin{tikzcd} \Delta^{\{1,2\}} \arrow[r] & (\Lambda^2_0)^\sharp \amalg_{(\Lambda^2_0)^\flat} (\Delta^2)^\flat \arrow[r] \arrow[d] & X \arrow[d] \\ & (\Delta^2)^\sharp \arrow[r] & Y \end{tikzcd}$$

and need to show that in the top horizontal composite is marked. Since the marked edges in X are precisely the p-cocartesian edges this follows again from Lemma 10.5. □

Lecture 9 – 03.06.2019.

Lemma 11.8. *The pushout product of a marked left anodyne map with any monomorphism is again marked left anodyne.*

Proof. We refer to [Lur09, Prop. 3.1.2.3.] for a full proof of this result. It is in spirit very similar to the arguments we have given when showing that a left/right/inner anodyne map pushout product with a monomorphism is again left/right/inner anodyne: The first thing to observe are the following

(a) The set of monomorphisms such that the conclusion holds is saturated,
(b) The set of marked anodyne maps for which the conclusion holds is saturated,
(c) The monomorphisms in marked simplicial sets are generated by the maps $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ and the map $(\Delta^1)^\flat \to (\Delta^1)^\sharp$.

We will prove (c): It is clear that the boundary inclusions as described generate monomorphisms of the form $K^\flat \to L^\flat$. For every marked simplicial set (K,S) and monomorphism of simplicial sets $K \to L$, the following is a pushout of marked simplicial sets

$$\begin{tikzcd} K^\flat \arrow[r] \arrow[d] & (K,S) \arrow[d] \\ L^\flat \arrow[r] & (L,S) \end{tikzcd}$$

so that the right vertical map is generated by the boundary inclusions. To show that a general monomorphism of marked simplicial set is generated by the above it suffices to see that $(L,S) \to (L,S')$ is generated by the above for $S \subseteq S'$. For this we observe that there is a pushout

$$\begin{tikzcd} \coprod\limits_{s' \in S' \setminus S} (\Delta^1)^\flat \arrow[r] \arrow[d] & (L,S) \arrow[d] \\ \coprod\limits_{s' \in S' \setminus S} (\Delta^1)^\sharp \arrow[r] & (L,S') \end{tikzcd}$$

so that the claim is proven.

It hence now suffices to show that the pushout product of a map of the kind (1)–(4) of Definition 11.4 with a map of the kind appearing in (c) above is marked anodyne. There are thus eight cases to consider.

(1) We consider the pushout product of $(\Lambda^n_i)^\flat \subseteq (\Delta^n)^\flat$ with $(\partial\Delta^n)^\flat \subseteq (\Delta^n)^\flat$. Since $(-)^\flat$ preserves colimits the pushout product map is given by applying $(-)^\flat$ to the pushout product of the underlying maps of simplicial sets. This is again inner anodyne, so that it becomes marked anodyne upon applying $(-)^\flat$.

(2) The pushout product of $(\Lambda^n_i)^\flat \subseteq (\Delta^n)^\flat$ with $(\Delta^1)^\flat \to (\Delta^1)^\sharp$ is an isomorphism and thus marked anodyne: This uses that $n \geq 2$ so that the map $\Lambda^n_i \to \Delta^n$ is an isomorphism on vertices.

(3) The pushout product of $K^\flat \to K^\sharp$ with $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ is an isomorphism if $n > 0$ and equals the map $K^\flat \to K^\sharp$ for $n = 0$; in either case it is marked anodyne.

(4) The pushout product of $K^\flat \to K^\sharp$ with $(\Delta^1)^\flat \to (\Delta^1)^\sharp$ is the map $(K \times \Delta^1, S) \to (K \times \Delta^1)^\sharp$ where S is given by the pairs (a,b) where either a or b is degenerate. Since every 1-simplex (a,b) in $(K \times \Delta^1)$ is a composite of (a,id) and (id,b) and identities are degenerate, the map we are interested in is marked anodyne because adding a composite of marked edges is marked anodyne.

(5) The pushout product of $(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ with $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ is an isomorphism for $n \geq 1$ and the given map for $n = 0$ and hence is marked anodyne in any case.

(6) The pushout product of $(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$ with $(\Delta^1)^\flat \to (\Delta^1)^\sharp$ is almost an isomorphism: the only edge which is not marked in the domain (in the target all edges are marked) is the edge $(0 \to 2, 0 \to 1)$. It is however a composite of marked edges, so that the needed map is again marked anodyne.

The remaining cases are easiest to to argue when using the alternative generating set of marked left anodyne maps, i.e. working with the set $(3')$ instead of (3). The key point is that the saturated set generated by the maps

$$(\Delta^1)^\sharp \times (\partial\Delta^n)^\flat \cup \{0\} \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$$

is the same as that for the maps

$$(\Delta^1)^\sharp \times A^\flat \cup (\Delta^1)^\flat \times B^\flat \to (\Delta^1)^\sharp \times B^\flat$$

for monomorphisms $A \to B$.

(7) The pushout product of $(\Delta^1)^\sharp \times (\partial\Delta^n)^\flat \cup \{0\} \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$ with $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ is of the latter kind: This follows from the associativity of pushout products: It is given by the pushout product of $\{0\} \to (\Delta^1)^\sharp$ with the pushout product of $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$ with itself, which is clearly of the form $A^\flat \to B^\flat$ for a monomorphism $A \to B$.

(8) The pushout product of $(\Delta^1)^\sharp \times (\partial\Delta^n)^\flat \cup \{0\} \times (\Delta^n)^\flat \to (\Delta^1)^\sharp \times (\Delta^n)^\flat$ with $(\Delta^1)^\flat \to (\Delta^1)^\sharp$ is, as before, an isomorphism if $n > 0$. For $n = 0$, the first pushout product is simply $\{0\} \to (\Delta^1)^\sharp$, so we obtain the map $(\Delta^1 \times \Delta^1, S) \to (\Delta^1 \times \Delta^1)^\sharp$ where S consists of the degenerate edges and the edges $\{0\} \times \Delta^1$, and $\Delta^1 \times \{\varepsilon\}$ for $\varepsilon = 0, 1$. Using property (3) and Corollary 11.7 this is a composition of marked left anodyne maps.

□

We can then deduce the following result.

Proposition 11.9. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and let K be a simplicial set. Then $p_*\colon \mathcal{E}^K \to \mathcal{C}^K$ is again a cocartesian fibration, and an edge is p_*-cocartesian if and only if its image in $\mathcal{E}$ under the restriction along any object of K is p-cocartesian.*

Proof. As a special case of Lemma 11.8, we find that for any marked left anodyne map $A \to B$, the map $A \times K^\flat \to B \times K^\flat$ is also marked left anodyne. Using Proposition 11.5 one can solve any lifting problem

$$\begin{tikzcd} A \times K^\flat \arrow[r] \arrow[d] & \mathcal{E}^\natural \arrow[d] \\ B \times K^\flat \arrow[r] \arrow[ur, dashed] & \mathcal{C}^\sharp \end{tikzcd}$$

Since $(-)^\flat$ is left adjoint to the forgetful functor, this means that $(\mathcal{E}^K, S) \to (\mathcal{C}^K)^\sharp$ has the right lifting property wrt marked anodyne maps, where S consists of those edges whose restriction to any object of K become p-cocartesian. By Corollary 11.6, S constists precisely of the p_* cocartesian edges and p_* is a cocartesian fibration. □

Definition 11.10. Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and K a marked simplicial set. We denote by $\mathrm{Fun}^{\mathrm{mcc}}(K, \mathcal{E})$ the full subcategory of $\mathrm{Fun}(K, \mathcal{E})$ on functors which send all morphisms of K to p-cocartesian morphisms in $\mathcal{E}$. If K is equipped with a map $f\colon K \to \mathcal{C}^\sharp$, we denote by $\mathrm{Fun}^{\mathrm{mcc}}_f(K, \mathcal{E})$ the pullback

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{mcc}}_f(K, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{mcc}}(K, \mathcal{E}) \arrow[d, "p_*"] \\ \Delta^0 \arrow[r, "f"] & \mathrm{Fun}(K, \mathcal{C}) \end{tikzcd}$$

If K is an ordinary simplicial set, we will write $\mathrm{Fun}^{\mathrm{cc}}(K,\mathcal{E})$, respectively $\mathrm{Fun}^{\mathrm{cc}}_f(K,\mathcal{E})$ for $\mathrm{Fun}^{\mathrm{mcc}}(K^\sharp,\mathcal{E})$, respectively $\mathrm{Fun}^{\mathrm{mcc}}_f(K^\sharp,\mathcal{E})$.

Remark. In [Lur09] what we denote by $\mathrm{Fun}^{\mathrm{mcc}}(K,\mathcal{E})$ is written as $\mathrm{Map}^\flat(K,\mathcal{E}^\natural)$. Likewise, what we denote by $\mathrm{Fun}^{\mathrm{mcc}}_f(K,\mathcal{E})$ is denoted by $\mathrm{Map}^\flat_K(K,\mathcal{E}^\natural)$ in loc. cit.

The reason for this notation is the following: The category of marked simplicial sets is cartesian closed: For every marked simplicial set K, the functor $K\times -$ admits a right adjoint, denoted by $X\mapsto X^K$. We define a simplicial set $\mathrm{Map}^\flat(K,X)$ as follows. Its n-simplices are given by $\mathrm{Hom}_{\mathrm{sSet}^+}((\Delta^n)^\flat\times K,X)$. Likewise, we define a simplicial set $\mathrm{Map}^\sharp(K,X)$ whose n-simplices are given by $\mathrm{Hom}_{\mathrm{sSet}_+}((\Delta^n)^\sharp\times K,X)$. With our previous notation we have that $u(X^K)=\mathrm{Map}^\flat(K,X)$ and $m(X^K)=\mathrm{Map}^\sharp(K,X)$.

We then find the following.

Proposition 11.11. *Let $p\colon\mathcal{E}\to\mathcal{C}$ be a cocartesian fibration, let $i\colon K\to L$ be a marked anodyne map and $f\colon L\to\mathcal{C}^\sharp$ a morphism. Then the induced map*

$$\mathrm{Fun}^{\mathrm{mcc}}_f(L,\mathcal{E})\longrightarrow\mathrm{Fun}^{\mathrm{mcc}}_{fi}(K,\mathcal{E})$$

is a trivial fibration.

Proof. We need to show that any lifting problem

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{mcc}}_f(L,\mathcal{E}) \arrow[d] \\
T \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}^{\mathrm{mcc}}_{fi}(K,\mathcal{E})
\end{tikzcd}$$

can be solved if $S\to T$ is a monomorphism of simplicial sets. Unravelling definitions, this is the case if and only if the lifting problem

$$\begin{tikzcd}
S^\flat\times L\amalg_{S^\flat\times K}T^\flat\times K \arrow[r] \arrow[d] & \mathcal{E}^\natural \arrow[d] \\
T^\flat\times L \arrow[r] \arrow[ur, dashed] & \mathcal{C}^\sharp
\end{tikzcd}$$

can be solved. By Lemma 11.8, the left vertical map is marked anodyne, so the claim follows. $\square$

There is the following important lemma for us. We learned it from Hoang Kim Nguyen's thesis, [Ngu18, Lemma 3.2.3].

Lemma 11.12. *Let $K\to L$ be a left anodyne map. Then the map $K^\sharp\to L^\sharp$ of marked simplicial sets is marked anodyne.*

Proof. First we claim that the set of monomorphisms $K\to L$ of simplicial sets such that $K^\sharp\to L^\sharp$ is marked anodyne is saturated. It hence suffices to show that for $0\le i<n$, the map $(\Lambda^n_0)^\sharp\to(\Delta^n)^\sharp$ is marked anodyne. We observe that $\mathrm{sk}_1(\Lambda^n_i)=\mathrm{sk}_1(\Delta^n)$ once n is at

least 3. Thus for $n \geq 3$ and $0 < i < n$ we have a pushout

$$\begin{tikzcd} (\Lambda^n_i)^\flat \arrow[r] \arrow[d] & (\Lambda^n_i)^\sharp \arrow[d] \\ (\Delta^n)^\flat \arrow[r] & (\Delta^n)^\sharp \end{tikzcd}$$

which shows that the right vertical map is marked anodyne. Likewise, there is a pushout

$$\begin{tikzcd} (\Lambda^n_0)^{s\flat} \arrow[r] \arrow[d] & (\Lambda^n_0)^\sharp \arrow[d] \\ (\Delta^n)^{s\flat} \arrow[r] & (\Delta^n)^\sharp \end{tikzcd}$$

so that the right vertical map is again marked anodyne. It remains to treat the cases $n < 3$. The case $n = 1$ is clear, so it remains to treat the case $n = 2$, in which we need to discuss the cases $i = 0$ and $i = 1$. There are pushouts

$$\begin{tikzcd} (\Lambda^2_0)^\flat \arrow[r] \arrow[d] & (\Lambda^2_0)^{s\flat} \arrow[r] \arrow[d] & (\Lambda^2_0)^\sharp \arrow[d] \\ (\Delta^2)^\flat \arrow[r] & (\Delta^2)^{s\flat} \arrow[r] & (\Delta^2)^\flat \amalg_{(\Lambda^2_0)^\flat} (\Lambda^2_0)^\sharp \end{tikzcd}$$

so that the very right vertical map is marked anodyne. By Corollary 11.7 the further map

$$(\Delta^2)^\flat \amalg_{(\Lambda^2_0)^\flat} (\Lambda^2_0)^\sharp \to (\Delta^2)^\sharp$$

is also marked anodyne, so that the map $(\Lambda^2_0)^\sharp \to (\Delta^2)^\sharp$ is also marked anodyne. For the remaining case, we have the pushout

$$\begin{tikzcd} (\Lambda^2_1)^\flat \arrow[r] \arrow[d] & (\Lambda^2_1)^\sharp \arrow[d] \\ (\Delta^2)^\flat \arrow[r] & (\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \end{tikzcd}$$

so that the right vertical map is marked anodyne. By definition, also the map

$$(\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\flat \to (\Delta^2)^\sharp$$

is marked anodyne so the lemma is proven. □

With this at hand we have the following immediate consequence which will be very important for us later.

Corollary 11.13. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, $i\colon K \to L$ a left anodyne map of simplicial sets and $f\colon L \to \mathcal{C}$ a morphism. Then the induced map*

$$\mathrm{Fun}^{\mathrm{cc}}_f(L, \mathcal{E}) \longrightarrow \mathrm{Fun}^{\mathrm{cc}}_{fi}(K, \mathcal{E})$$

is a trivial fibration.

Proof. This is the special case of Proposition 11.11 where the marked left anodyne map is $i^\sharp\colon K^\sharp \to L^\sharp$, using Lemma 11.12. □

If $f\colon \Delta^1 \to \mathcal{C}$ is a morphism, then the ∞-category $\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E})$ parametrizes all p-cocartesian lifts of a given morphism in $\mathcal{C}$. We thus find the following.

Corollary 11.14. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and $f\colon \Delta^1 \to \mathcal{C}$ a morphism from x to y in $\mathcal{C}$. Then the map $\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E}) \to \mathcal{E}_x$ given by evaluating at $\{0\}$ is a trivial fibration. In particular, the simplicial set*

$$\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E}) \times_{\mathcal{E}_x} \{z\}$$

is a contractible Kan complex for every object z of $\mathcal{E}_x$.

Proof. We consider the pullback diagram

$$\begin{tikzcd}
\mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E}) \times_{\mathcal{E}_x} \{z\} \arrow[r] \arrow[d] & \mathrm{Fun}_f^{\mathrm{cc}}(\Delta^1, \mathcal{E}) \arrow[d] \\
\Delta^0 \arrow[r, "z"] & \mathrm{Fun}_x^{\mathrm{cc}}(\Delta^0, \mathcal{E}) \simeq \mathcal{E}_x
\end{tikzcd}$$

where the right vertical map is a trivial fibration because it is obtained by restriction along $\{0\} \to \Delta^1$ which is left anodyne so that we may apply Corollary 11.13. Thus also the left vertical map is a trivial fibration. $\square$

11.1. **Digression – marked simplicial sets and localizations.** In this section, we want to indicate how one can make use of marked simplicial sets to study Dwyer–Kan localizations. We start out with the following definition.

In what follows we will always view ∞-categories as a cocartesian fibration over Δ^0, so that $\mathcal{C}^\natural$ denotes the marked simplicial set $\mathcal{C}$ with all equivalences marked.

Definition 11.15. Let X, Y be marked simplicial sets. A morphism $f\colon X \to Y$ is called a marked equivalence if for any ∞-category, the induced map

$$\mathrm{Fun}^{\mathrm{mcc}}(Y, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{mcc}}(X, \mathcal{C})$$

is an equivalence of ∞-categories.

Example 11.16. Let $\mathcal{C}$ be an ∞-category and S a set of morphisms containing all equivalences. Let $\mathcal{C}[S^{-1}]$ be a localization. Then the map $(\mathcal{C}, S) \to \mathcal{C}[S^{-1}]^\natural$ is a marked equivalence.

Lecture 10 – 05.06.2019.

Example 11.17. Let $A \to B$ be a marked left anodyne map. Then $A \to B$ is a marked equivalence. This is a special case of Proposition 11.11: We need to show that for every ∞-category $\mathcal{E}$, the restriction functor

$$\mathrm{Fun}^{\mathrm{mcc}}(B, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{mcc}}(A, \mathcal{E})$$

is a Joyal equivalence. In fact it is a trivial fibration, because $\mathrm{Fun}^{\mathrm{mcc}}(B, \mathcal{E}) = \mathrm{Fun}_*^{\mathrm{mcc}}(B, \mathcal{E})$ for the cocartesian fibration $\mathcal{E} \to \Delta^0$ and the canonical map $*\colon B \to \Delta^0$, likewise for A in place of B.

As in Section 3, we expect to be able to factor any map as a marked anodyne map followed by a map which satisfies the RLP wrt marked anodyne maps. For maps of the form $X \to \Delta^0$, we find that the resulting map $\mathcal{C} \to \Delta^0$ is an inner fibration, and the marked edges of $\mathcal{C}$ are precisely the equivalences.

Theorem 11.18. *There exists a model structure on marked simplicial sets whose cofibrations are monomorphisms, whose equivalences are marked equivalences, and where fibrant objects are precisely ∞-categories with all equivalences marked.*

Corollary 11.19. *A Dwyer–Kan localization of $\mathcal{C}$ along S may thus be thought of as a fibrant replacement of $(\mathcal{C}, S)$ in this model structure on marked simplicial sets.*

Lemma 11.20. *Let $(\mathcal{C}, S)$ be a marked ∞-category and let $\mathcal{E}$ be an ∞-category. Then there is an equivalence $\mathrm{Fun}^{\mathrm{mcc}}((\mathcal{C}, S), \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{E})$ of ∞-categories.*

Proof. This follows immediately from the definitions of localizations. $\square$

More generally, we have:

Lemma 11.21. *Let $(\mathcal{C}, S)$ and $(\mathcal{D}, T)$ be ∞-categories equipped with sets of maps, viewed as marked simplicial sets. A map $f\colon (\mathcal{C}, S) \to (\mathcal{D}, T)$ of marked simplicial sets is a marked equivalence if and only if the induced map*

$$\bar{f}\colon \mathcal{C}[S^{-1}] \xrightarrow{\simeq} \mathcal{D}[T^{-1}]$$

on Dwyer–Kan localizations is an equivalence.

Proof. Let $\mathcal{E}$ be an auxiliary ∞-category. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{mcc}}((\mathcal{D}, T), \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{mcc}}((\mathcal{C}, S), \mathcal{E}) \arrow[d] \\
\mathrm{Fun}(\mathcal{D}[T^{-1}], \mathcal{E}) \arrow[r] & \mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{E})
\end{tikzcd}$$

in which the vertical maps are equivalences by Lemma 11.20. Hence the upper horizontal map is an equivalence if and only if the lower one is. Since $\mathcal{E}$ is an arbitrary ∞-category, the lemma follows. $\square$

We wish to use this to obtain concrete examples of Dwyer–Kan localizations. The following proposition will take care of this, see [Lur17, 1.3.4.7]. Its proof uses some techniques which we develop at most later, and is beyond the scope of these notes. The statement and conclusion, however, are easy to understand so we want to explain it here. We begin with a definition:

Definition 11.22. Let $\mathcal{C}$ be a simplicial category and x an object of $\mathcal{C}$. An interval object for x consists of the following data:

(1) an object Ix of $\mathcal{C}$, equipped with
(2) a map $h\colon \Delta^1 \to \mathrm{Hom}_{\mathcal{C}}(x, Ix)$,

satisfying the following universal property: For every object y of $\mathcal{C}$, composition with h determines an isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(Ix, y) \to \underline{\mathrm{Hom}}(\Delta^1, \mathrm{Hom}_{\mathcal{C}}(x, y)).$$

More concretely the map is given as follows. We consider the adjoint of the composition map

$$\mathrm{Hom}_{\mathcal{C}}(Ix, y) \to \underline{\mathrm{Hom}}(\mathrm{Hom}_{\mathcal{C}}(x, Ix), \mathrm{Hom}_{\mathcal{C}}(x, y))$$

and compose this wit the canonical restriction along h map

$$\underline{\mathrm{Hom}}(\mathrm{Hom}_{\mathcal{C}}(x, Ix), \mathrm{Hom}_{\mathcal{C}}(x, y)) \to \underline{\mathrm{Hom}}(\Delta^1, \mathrm{Hom}_{\mathcal{C}}(x, y)).$$

Observation 11.23. An interval object Ix for x comes equipped with a canonical map $Ix \to x$: We have to exhibit a 0-simplex of $\mathrm{Hom}_{\mathcal{C}}(Ix, x)$ which by assumption is isomorphic to $\underline{\mathrm{Hom}}(\Delta^1, \mathrm{Hom}_{\mathcal{C}}(x, x))$ where we can take the map which is constant at the identity of x.

Proposition 11.24. *Let $\mathcal{C}$ be a sinplicially enriched category and S a set of morphisms in $\mathcal{C}$. Let $u\mathcal{C}$ be the underlying ordinary category of $\mathcal{C}$. Assume that the following conditions are satisfied:*

(1) *Isomorphisms belong to S,*
(2) *S satisfies the 3-for-2 property*
(3) *For every object x of $\mathcal{C}$, there exists an interval object Ix for x,*
(4) *the canonical maps $Ix \to x$ belong to S for all x of $\mathcal{C}$.*

Let $\mathcal{C}'$ be a Kan enriched category, equipped with a weak equivalence $\mathcal{C} \to \mathcal{C}'$. We denote by S' be image of S under this functor. Then the canonical map $(\mathrm{N}(u\mathcal{C}), S) \to (\mathrm{N}(\mathcal{C}'), S')$ is a marked equivalence.

Corollary 11.25. *Suppose that every morphism in S is sent to an equivalence in $\mathrm{N}(\mathcal{C}')$. Then $\mathrm{N}(u\mathcal{C})[S^{-1}]$ is equivalent to $\mathrm{N}(\mathcal{C}')$.*

Corollary 11.26. *The canonical functor $\mathrm{sSet}[we^{-1}] \to \mathrm{An}$ is an equivalence of ∞-categories.*

Proof. We consider the simplicial category sSet. It has interval objects given by $\Delta^1 \times X$ for every X: We need to specify a map $\Delta^1 \to \underline{\mathrm{Hom}}(X, \Delta^1 \times X)$ and we choose the identity. We observe that sSet satisfy the assumptions of Proposition 11.24 where S is given by the set of weak equivalences: We only need to recall that the map $\Delta^1 \times X \to X$ is a weak equivalence because Δ^1 is contractible and geometric realization commutes with products. The small object argument provides a functor sSet $\to$ Kan which is a weak equivalence: It is weakly essentially surjective because every simplicial set is weakly equivalent to a Kan complex. To see weakly fully faithfulness, we consider two simplicial sets A and B, and denote their associated Kan complexes by X_A and X_B. Recall that there are anodyne maps $A \to X_A$ and $B \to X_B$. The map

$$\underline{\mathrm{Hom}}(X_A, X_B) \to \underline{\mathrm{Hom}}(A, X_B)$$

is a trivial fibration and hence an equivalence. Furthermore, the composite

$$\underline{\mathrm{Hom}}(A, B) \to \underline{\mathrm{Hom}}(X_A, X_B) \to \underline{\mathrm{Hom}}(A, X_B)$$

is given by postcomposition with the map $B \to X_B$ which is anodyne and hence a weak equivalence. It hence suffices to show that postcomposition with a weak equivalence is itself a weak equivalence. This follows from Exercise 107. We therefore apply Corollary 11.25: $\mathcal{C}'$ can be chosen to be the Kan enriched category Kan, S is the set of weak equivalences. Under the functor $\mathcal{C} \to \mathrm{N}(\mathcal{C}')$ weak equivalences are sent to equivalences. The claim thus follows. □

Corollary 11.27. *The functor $\mathrm{sSet}[Joy^{-1}] \to \mathrm{Cat}_\infty$ is an equivalence of ∞-categories.*

Proof. We consider first the category sSet_+ with simplicial enrichment given by $\mathrm{Map}^\sharp(X, Y)$. The fibrant replacement functor gives a simplicial functor $\mathrm{sSet}_+ \to \mathrm{Cat}^1_\infty$: We have to argue that for ∞-categories $\mathcal{C}$ and $\mathcal{D}$, there is a canonical map $\mathrm{Map}^\sharp(\mathcal{C}^\natural, \mathcal{D}^\natural) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})^\simeq$ which is a weak equivalence. In fact, we claim that these simplicial sets are isomorphic: In both cases they are a subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, namely on those n-simplices $\mathcal{C} \times \Delta^n \to \mathcal{D}$, all of whose edges are pointwise marked in $\mathcal{D}^\natural$, respectively equivalences in $\mathcal{D}$ – this uses again the pointwise criterion for natural equivalences we have developed in Theorem 6.1. In particular, we find that $\mathcal{C}'$ can be chosen to be Cat^1_∞ with its canonical Kan enrichment as discussed previously. We claim that the functor $\mathrm{sSet}_+ \to \mathrm{Cat}^1_\infty$ sends marked equivalences to Joyal equivalences: This is because marked (left) anodyne maps are marked equivalences and marked equivalences satisfy the 3-for-2 property:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \bar{X} \arrow[d] \\ Y \arrow[r] & \bar{Y} \end{tikzcd}$$

where the horizontal maps are marked left anodyne, and the map $X \to Y$ is a marked equivalence by assumption. We then let S be the set of marked equivalences, which satisfies (1) and (2) of Proposition 11.24. We define an interval object by $(\Delta^1)^\sharp \times X$, it is straightforward to see that this is in fact an interval object according to **??**. We claim that the canonical map $(\Delta^1)^\sharp \times X \to X$ is a marked equivalence: By exponentiating this follows from the fact that $(\Delta^1)^\sharp \to \Delta^0$ is a marked equivalence, thanks to Lemma 11.21.

We may thus apply Proposition 11.24 and Corollary 11.25 and obtain that the functor $\mathrm{N}(u(\mathrm{sSet}_+))[me^{-1}] \to \mathrm{Cat}_\infty$ is an equivalence of ∞-categories. It remains then to show that

$$\mathrm{N}(\mathrm{sSet}_+)[me^{-1}] \simeq \mathrm{N}(\mathrm{sSet})[Joy^{-1}].$$

This follows again because the latter can be identified with the localization of the subcategory of fibrant objects using an argument similar to what we have seen earlier in Lemma 8.9. □

12. Straightening-Unstraightening

Lecture 11 – 12.06.2019. In this section we want to formulate and discuss in parts an important correspondence: The Grothendieck construction. We begin with an informal construction. Consider a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$. We observe that we can extract the following data from this:

(1) for each object x of $\mathcal{C}$, we have the ∞-category $\mathcal{E}_x$,
(2) for each morphism $f\colon x \to y$ in $\mathcal{C}$, and an object z in $\mathcal{E}_x$, we can choose a p-cocartesian lift $z \to w$ of f, we will denote $w = f_!(z)$
(3) given a further object z' in $\mathcal{E}_x$ and a morphism $\alpha\colon z \to z'$, we can choose another p-cocartesian lift $z' \to w' = f_!(z')$ and obtain a diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & z' \arrow[d] \\ f_!(z) \arrow[r, dashed] & f_!(z') \end{tikzcd}$$

and since $z \to f_!(z)$ is p-cocartesian, the space of dashed arrows making the diagram commutative is contractible. We will denote any such dashed arrow by $f_!(\alpha)$

Summarizing, associated to a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$, we wish to find a functor $\mathcal{C} \to \mathrm{Cat}_\infty$, sending an object to the fibre of p, and sending a morphism $f\colon x \to y$, the "functor" $f_!$.

Proposition 12.1. *Given a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$ and a morphism $f\colon \Delta^1 \to \mathcal{C}$ from x to y, there exists a functor $\mathcal{E}_x \times \Delta^1 \to \mathcal{E}$ whose restriction to every object z of $\mathcal{E}_x$ provides a p-cocartesian morphism $\alpha\colon z \to z'$ over f. Restricting this functor to $\mathcal{E}_x \times \{1\}$ gives a functor $f_!\colon \mathcal{E}_x \to \mathcal{E}_y$.*

Proof. We begin by constructing for each cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$ and each morphism $f\colon \Delta^1 \to \mathcal{C}$ from x to y a functor $f_!\colon \mathcal{E}_x \to \mathcal{E}_y$.

We recall that in this situation, the canonical map $\mathrm{Fun}^{\mathrm{cc}}_f(\Delta^1, \mathcal{E}) \to \mathcal{E}_x$ given by taking the source of a morphism is a trivial fibration. Choosing a section of this trivial fibration produces the composite

$$f_!\colon \mathcal{E}_x \to \mathrm{Fun}^{\mathrm{cc}}_f(\Delta^1, \mathcal{E}) \to \mathcal{E}_y$$

where the latter map is given by taking the target of a morphism. Furthermore, we find that the first map is adjoint to a map

$$\mathcal{E}_x \times \Delta^1 \to \mathcal{E}$$

with the following properties: Its restriction to $\mathcal{E}_x \times \{1\}$ is given by $f_!$ and it makes the diagram

$$\begin{tikzcd}
\mathcal{E}_x \times \Delta^1 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
\Delta^1 \arrow[r, "f"] & \mathcal{C}
\end{tikzcd}$$

commute and furthermore, for each object z in $\mathcal{E}_x$, the resulting morphism $\Delta^1 \to \mathcal{E}$ is a p-cocartesian morphism with source equal to z and target equal to $f_!(z)$. $\square$

We wish to show that the association $f \mapsto f_!$ is "functorial in f". For this we consider a 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$ inside $\mathcal{C}$, which exhibits h as a composition of f and g. We then consider the diagram

$$\begin{tikzcd}
 & \mathrm{Fun}^{\mathrm{cc}}_f(\Delta^{\{0,1\}}, \mathcal{E}) \arrow[r] \arrow[dl, "\simeq"] & \mathcal{E}_y \arrow[dd, bend left=60, "g_!"] \\
\mathcal{E}_x \arrow[urr, bend left=60, "f_!"] \arrow[drr, bend right=60, "h_!"'] & \mathrm{Fun}^{\mathrm{cc}}_\sigma(\Delta^2, \mathcal{E}) \arrow[r] \arrow[l, "\simeq"] \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \mathrm{Fun}^{\mathrm{cc}}_g(\Delta^{\{1,2\}}, \mathcal{E}) \arrow[u, "\simeq"] \arrow[d] \\
 & \mathrm{Fun}^{\mathrm{cc}}_h(\Delta^{\{0,2\}}, \mathcal{E}) \arrow[r] \arrow[ul, "\simeq"] & \mathcal{E}_z
\end{tikzcd}$$

The maps labelled with a $\simeq$ are trivial fibrations because they arise as restrictions along left anodyne maps. This shows that there is a natural isomorphism between $g_! f_!$ and $h_!$.

We now explain the general construction. For this, we fix a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$.

Lemma 12.2. *Associating to $\sigma\colon \Delta^n \to \mathcal{C}$ the ∞-category $\mathrm{Fun}^{\mathrm{cc}}_\sigma(\Delta^n, \mathcal{E})$ extends to a functor $\Theta(p)\colon \Delta^{\mathrm{op}}_{/\mathcal{C}} \to \mathrm{sSet}$.*

Proof. We need to show that a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "f"] \arrow[dr, "\sigma"'] & & \Delta^m \arrow[dl, "\tau"] \\ & \mathcal{C} & \end{tikzcd}$$

induces a well-defined a functorial map $\mathrm{Fun}^{\mathrm{cc}}_{\tau}(\Delta^m, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{cc}}_{\sigma}(\Delta^n, \mathcal{E})$ which is clear from the definition. ☐

Definition 12.3. Let X be a simplicial set. We denote by W_X the set of all morphisms $f\colon [n] \to [m]$ in $\Delta_{/X}$ such that $f(0) = 0$.

Lemma 12.4. *For a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$, the functor $\Theta(p)$ sends any morphism in $W_{\mathcal{C}}$ to a Joyal equivalence.*

Proof. Consider a morphism f in $\Delta_{/\mathcal{C}}$, represented by the composite $\Delta^n \xrightarrow{f} \Delta^m \xrightarrow{\sigma} \mathcal{C}$. We will write $\tau = \sigma f$. By assumption, the composite

$$\Delta^{\{0\}} \to \Delta^n \xrightarrow{f} \Delta^m$$

picks out the object 0 in Δ^m. We thus have a commutative diagram

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{cc}}_{\sigma}(\Delta^m, \mathcal{E}) \arrow[rr, "f^*"] \arrow[dr, "\simeq"'] & & \mathrm{Fun}^{\mathrm{cc}}_{\tau}(\Delta^n, \mathcal{E}) \arrow[dl, "\simeq"] \\ & \mathrm{Fun}^{\mathrm{cc}}_{\{0\}}(\Delta^0, \mathcal{E}) & \end{tikzcd}$$

in which both diagonal maps are trivial fibrations by Corollary 11.13 because for any $k \geq 0$, the inclusion $\Delta^{\{0\}} \to \Delta^k$ is left anodyne. Hence also the map f^* is a Joyal equivalence as claimed. ☐

Corollary 12.5. *For every cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$, we obtain a functor*

$$\Theta(p)\colon \mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_{\mathcal{C}}^{-1}] \longrightarrow \mathrm{Cat}_\infty.$$

Proof. By the previous lemma, $\Theta(p)$ induces a functor between the localizations

$$\mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_{\mathcal{C}}^{-1}] \longrightarrow \mathrm{sSet}[\mathrm{Joy}^{-1}]$$

and the latter admits a further functor to Cat_∞ (which is in fact an equivalence by Corollary 11.27). ☐

Lemma 12.6. *Let X be a simplicial set. There is a canonical map of simplicial sets $\mathrm{N}(\Delta^{\mathrm{op}}_{/X}) \to X$ called the initial vertex map.*

Proof. Recall that a k-simplex of the nerve is given by a sequence

$$[n_0] \xrightarrow{\alpha_1} [n_1] \xrightarrow{\alpha_2} [n_2] \xrightarrow{\alpha_0} \cdots \xrightarrow{\alpha_k} [n_k]$$

together with a map $\sigma\colon \Delta^{n_k} \to X$. We observe that the association $\alpha\colon [k] \to [n_k]$ given by sending 0 to 0 and i to $\alpha_k \circ \cdots \circ \alpha_{k-i+1}(0)$ is a map of linearly ordered sets. We hence obtain a k-simplex of X by the composite $\Delta^k \xrightarrow{\alpha} \Delta^{n_k} \xrightarrow{\sigma} X$. It is straightforward to check that this is compatible with the simplicial structure maps. ☐

Lemma 12.7. *The initial vertex map sends all morphisms in W_X to degenerate edges of X.*

Proof. Recall that a 1-simplex in W_X is represented by the composite $[n] \xrightarrow{f} [m] \xrightarrow{\sigma} X$ where the map f sends 0 to 0. The resulting 1-simplex of X is given by restricting the map σ along the map $[1] \to [m]$ given by sending 0 to 0 and 1 to $f(0) = 0$. This is a degenerate edge in $[m]$ and thus remains degenerate after applying the map σ. □

We will now need the following result, which is due to Joyal, Dwyer-Kan, and has also been proved by Stevenson [Ste17, Theorem 1.3].

Theorem 12.8. *For every ∞-category $\mathcal{C}$, the initial vertex map induces a Joyal equivalence*

$$\mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_{\mathcal{C}}^{-1}] \xrightarrow{\simeq} \mathcal{C}.$$

Proof. The proof will consist of two steps: First, one shows that one can reduce the claim to showing it only for $\mathcal{C} = \Delta^n$, and then one has to show the claim in this case.

As a first step, we need a slightly more general version of the above: We want to show that these maps make sense for an arbitrary simplicial set X in place of $\mathcal{C}$: Clearly, the initial vertex map defines a map $\Delta^{\mathrm{op}}_{/X} \to X$. Now, for the moment, let us define for a marked simplicial set (X, S) a simplicial set $L(X, S)$ by the pushout

$$\begin{tikzcd} \coprod\limits_{f \in S} \Delta^1 \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod\limits_{f \in S} J \arrow[r] & L(X,S) \end{tikzcd}$$

and observe that if $X = \mathcal{C}$ is an ∞-category, then $L(X, S)$ is Joyal equivalent to $\mathcal{C}[S^{-1}]$, which was defined by choosing an inner anodyne map $L(\mathcal{C}, S) \to \mathcal{C}[S^{-1}]$ to obtain an ∞-category. The initial vertex map takes a morphism in W_X to a degenerate edge in X, so one can clearly extend the corresponding map $\Delta^1 \to X$ over the inclusion $\Delta^1 \to J$. In particular, we obtain an induced map $L(\mathrm{N}(\Delta^{\mathrm{op}}_{/X}), W_X) \to X$ and we claim that this is a Joyal equivalence for every simplicial set X. Once this is shown, so is the theorem, by the above observation.

We denote the functor from simplicial sets to marked simplicial sets, sending X to $(\mathrm{N}(\Delta^{\mathrm{op}}_{/X}), W_X)$ by F. We will use the following properties, whose verification we leave as an exercise:

(1) The functor LF preserves colimits,
(2) the functor LF preserves monomorphisms,
(3) the initial vertex maps assemble into a natural transformation $LF \Rightarrow \mathrm{id}$.

Let us now suppose that the theorem is shown for $\mathcal{C} = \Delta^n$ and let X be an arbitrary simplicial set. We write X as the colimit over its skeleta $\mathrm{sk}_n(X)$ and obtain the map

$$L(\mathrm{N}(\Delta^{\mathrm{op}}_{/X}), W_X) \cong \operatorname*{colim}_n L(\mathrm{N}(\Delta^{\mathrm{op}}_{/\mathrm{sk}_n(X)}), W_{\mathrm{sk}_n(X)}) \to \operatorname*{colim}_n \mathrm{sk}_n(X) \cong X$$

where we use the initial vertex map at each step. The fact that the needed diagrams commute is an exercise. If we can show that each initial vertex map

$$L(\mathrm{N}(\Delta^{\mathrm{op}}_{/\mathrm{sk}_n(X)}), W_{\mathrm{sk}_n(X)}) \to \mathrm{sk}_n(X)$$

is a Joyal equivalence, then so is the above map by yet another exercise.

We then perform an induction over the dimension n. The induction start forces X to be a disjoint union of Δ^0's, and since the initial vertex map commutes with disjoint union, this map is a Joyal equivalence by assumption.

For the induction step, we consider the pushout

$$\begin{tikzcd} \coprod_{J_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X) \arrow[d] \\ \coprod_{J_n} \Delta^n \arrow[r] & \mathrm{sk}_n(X) \end{tikzcd}$$

and recall that the initial vertex map commutes with colimits. By induction and assumption, the initial vertex map is a Joyal equivalence on the corners except a priori the lower right corner. However, since the functor $L(\mathrm{N}(\Delta^{\mathrm{op}}_{/(-)}), W_{(-)})$ preserves colimits and monomorphisms, it follows from Exercise 101 that this map is also a Joyal equivalence.

Lecture 12 – 24.06.2019. We hence now need to show the statement of the theorem for Δ^n. We observe that in this case, the initial vertex map

$$\mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n}) \to \Delta^n$$

is the map induced on nerves of the functor $\Delta^{\mathrm{op}}/\Delta^n \to [n]$ sending $f\colon [m] \to [n]$ to $f(0)$: Its effect on morphisms is given by the following: Suppose given a composite $[k] \to [m] \to [n]$ where the composite is g and the latter map is f. We then need to find a morphism in $[n]$ from $f(0)$ to $g(0)$. In other words we need to show that $f(0) \leq g(0)$. But we have $g(0) = f(h(0)) \geq f(0)$ because f is monotone increasing and h is also monotone increasing.

We now construct a functor in the other direction: $[n] \to \Delta^{\mathrm{op}}_{/\Delta^n}$ given as follows: The object i of $[n]$ is sent to the map $\Delta^{\{i,\dots,n\}} \to \Delta^n$. Clearly, if $i \leq j$, there is a commutative diagram

$$\begin{tikzcd} \Delta^{\{i,\dots,n\}} \arrow[r] & \Delta^n \\ \Delta^{\{j,\dots,n\}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

so this gives a functor as desired. Its composition with the initial vertex map is given by the identity, as one checks immediately. We now consider the composite

$$\Delta^{\mathrm{op}}_{/\Delta^n} \xrightarrow{IV} [n] \xrightarrow{i} \Delta^{\mathrm{op}}_{/\Delta^n}.$$

We claim that there is a canonical natural transformation from this composite $i \circ IV$ to the identity functor. Indeed, the composite is given by sending $f\colon [m] \to [n]$ to the canonical inclusion $\{f(0), \dots, n\} \to [n]$. The canonical commutative triangle

$$\begin{tikzcd} \{0,\dots,m\} \arrow[r, "f"] \arrow[d, "f"] & \{0,\dots,n\} \\ \{f(0),\dots,n\} \arrow[ur] & \end{tikzcd}$$

gives the components of this natural transformation (the left vertical map). We observe that these components are all contained in the set W_{Δ^n}.

This construction provides a map of simplicial sets

$$\Delta^1 \to \mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n}), \mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n}))$$

restricting to $i \circ IV$ on 0 and to the identity on 1. Postcomposing with the localization map $\mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n}) \to \mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n})[W^{-1}_{\Delta^n}]$, and recalling that $i \circ IV$ sends W_{Δ^n} to equivalences, we obtain a map

$$\Delta^1 \to \mathrm{Fun}(\mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n})[W^{-1}_{\Delta^n}], \mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n})[W^{-1}_{\Delta^n}]).$$

We claim that this is a natural equivalence, which follows from the fact that the components of the transformation above are contained in the set W_{Δ^n}. We have thus constructed functors

$$\mathrm{N}([n]) \to \mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n})[W^{-1}_{\Delta^n}] \to \mathrm{N}([n]) \to \mathrm{N}(\Delta^{\mathrm{op}}_{/\Delta^n})[W^{-1}_{\Delta^n}]$$

such that both composites are naturally equivalent to the identity functor. Hence, the initial vertex map is a Joyal equivalence as desired. □

Corollary 12.9. *Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration. Inverting the above equivalence, we obtain a functor*

$$\mathcal{C} \xleftarrow{\simeq} \mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}})[W_{\mathcal{C}}^{-1}] \longrightarrow \mathrm{Cat}_\infty.$$

We call this the straightening of the cocartesian fibration p.

We end this section with the straightening-unstraightening equivalence of Lurie. Informally, it says that the straightening construction of Corollary 12.9 induces an equivalence of suitable ∞-categories. To state it precisely, let us denote by $\mathrm{CoCart}(\mathcal{C})$ the subcategory of the slice $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ on objects which are cocartesian fibrations $\mathcal{E} \to \mathcal{C}$ and whose morphisms are the morphisms of cocartesian fibrations according to Definition 10.19.

Theorem 12.10. *For every ∞-category $\mathcal{C}$, there is an equivalence of ∞-categories*

$$\mathrm{CoCart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty).$$

On objects, this equivalence implements our previous construction.

Definition 12.11. Consider the ∞-category $\mathcal{C} = \mathrm{Cat}_\infty$ and the identity functor. By Theorem 12.10, this corresponds to a cocartesian fibration over Cat_∞, called *the universal cocartesian fibration.* This is a functor $(\mathrm{Cat}_\infty)_{*/\!/} \to \mathrm{Cat}_\infty$, and $(\mathrm{Cat}_\infty)_{*/\!/}$ is an ∞-category whose objects are pairs $(\mathcal{C}, x)$ where x is an object of $\mathcal{C}$, and morphisms from $(\mathcal{C}, x)$ to $(\mathcal{D}, y)$ consist of pairs (F, α) where $F\colon \mathcal{C} \to \mathcal{D}$ and $\alpha\colon y \to Fx$ is a morphism in $\mathcal{D}$.

Remark. Constructing the ∞-category $(\mathrm{Cat}_\infty)_{*/\!/}$ is not easy: It involves the composition in an ∞-category which is not strict. There are ways to work around this, but we will not get into the details here, see [RV18, Remark 6.1.19]. The idea is to consider the coherent nerve of the simplicial category of ∞-categories $\mathrm{N}(\mathrm{Cat}^1_\infty)$ without passing to the groupoids of the functor categories. This is a simplicial set, and one can form its slice under the point. This gives a map of simplicial sets $\mathrm{N}(\mathrm{Cat}^1_\infty)_{\Delta^0/\!/} \to \mathrm{N}(\mathrm{Cat}^1_\infty)$. By construction, there is also a functor $\mathrm{Cat}_\infty \to \mathrm{N}(\mathrm{Cat}^1_\infty)$, and the pullback of the slice projection turns out to be a cocartesian fibration.

Remark. In general, one would like to have for each cocartesian fibration $\mathcal{E} \to \mathcal{D}$, and each ∞-category $\mathcal{C}$ a functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{CoCart}(\mathcal{C})$ given on objects by pulling back the given cocartesian fibration. The statement that $\mathcal{E} \to \mathcal{D}$ is universal then translates to the property that this functor is an equivalence of ∞-categories. Such a construction is done in [RV18, Theorem 6.1.13].

Remark. By means of the universal cocartesian fibration we can also say what the the equivalence

$$\mathrm{CoCart}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$$

does to a functor $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$. It sends it to the pulled pack cocartesian fibration $F^*p\colon F^*(\mathrm{Cat}_\infty)_{*/\!/} \to \mathcal{C}$.

We will now need the following lemma.

Lemma 12.12. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor between ∞-categories. Then for any simplicial set K, the induced functor $\mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{D})$ is again fully faithful.*

Proof. We first observe that a functor f is fully faithful if and only if the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{D}) \arrow[d] \\
\mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{D}
\end{tikzcd}$$

induces Joyal equivalences on all fibres over points of $\mathcal{C} \times \mathcal{C}$. We can apply the functor $\mathrm{Fun}(K, -)$ to this diagram. It preserves fibres, and Joyal equivalences so that also the resulting diagram has the property that it induces Joyal equivalences on vertical fibres. Using the equivalence $\mathrm{Fun}(K, \mathrm{Fun}(\Delta^1, \mathcal{C})) \cong \mathrm{Fun}(\Delta^1, \mathrm{Fun}(K, \mathcal{C}))$ the lemma follows. $\square$

Recall from Corollary 9.34 that the canonical functor $\mathrm{An} \to \mathrm{Cat}_\infty$ is fully faithful. It follows from Lemma 12.12 that for any ∞-category $\mathcal{C}$, the functor

$$\mathrm{Fun}(\mathcal{C}, \mathrm{An}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$$

is also fully faithful. In particular, under the above equivalence, the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathrm{An})$ must correspond to some full subcategory of $\mathrm{CoCart}(\mathcal{C})$. This is given by the following:

Theorem 12.13. *For every ∞-category $\mathcal{C}$, the straightening-unstraightening equivalence restricts to an equivalence*

$$\mathrm{LFib}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{An})$$

where $\mathrm{LFib}(\mathcal{C})$ denotes the full subcategory of the slice $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ on left fibrations.

Proof. Under the straightening-unstraightening equivalence, the functor $\mathcal{C} \to \mathrm{An} \to \mathrm{Cat}_\infty$ corresponds to a cocartesian fibration $\mathcal{E} \to \mathcal{C}$ whose fibres are ∞-groupoids. Hence $\mathcal{E} \to \mathcal{C}$ is a left fibration by the dual version of Proposition 10.17. Since any morphism $\mathcal{E} \to \mathcal{E}'$ over $\mathcal{C}$ preserves cocartesian edges, as every edge is cocartesian by Lemma 10.16, this is in fact the *full* subcategory of the slice category as claimed. $\square$

The following is almost a corollary of the above.

Theorem 12.14. *Let X be an ∞-groupoid. Then the straightening-unstraightening equivalence restricts to an equivalence*

$$\mathrm{An}_{/X} \simeq \mathrm{Fun}(X, \mathrm{An})$$

Proof. By Theorem 12.13, we need to show that there is a canonical equivalence $\mathrm{LFib}(X) \simeq \mathrm{An}_{/X}$. Since X is an ∞-groupoid any left fibration $\mathcal{E} \to X$ is in fact a Kan fibration. In particular, $\mathcal{E}$ is itself a Kan complex. We thus find that the category $\mathrm{LFib}(X)$ is the full subcategory of the slice $(\mathrm{Cat}_\infty)_{/X}$ whose objects consist of Kan fibrations. We obtain the following diagram

$$\begin{tikzcd} \mathrm{LFib}(X) \arrow[r] \arrow[dr] & \mathrm{An}_{/X} \arrow[d] \\ & (\mathrm{Cat}_\infty)_{/X} \end{tikzcd}$$

and claim that the functor $\mathrm{An}_{/X} \to (\mathrm{Cat}_\infty)_{/X}$ is fully faithful because the functor $\mathrm{An} \to \mathrm{Cat}_\infty$ is, see **??**. It follows that the functor $\mathrm{LFib}(X) \to \mathrm{An}_{/X}$ is also fully faithful Furthermore, any map $Y \to X$ between ∞-groupoids is equivalent to a Kan fibration. This implies that the inclusion $\mathrm{LFib}(X) \to \mathrm{An}_{/X}$ is essentially surjective and fully faithful and thus an equivalence as needed. □

To finish the proof, we will need the following analysis of the mapping anima in slice ∞-categories. This will become important again later. The dual version is [Lur09, Lemma 5.5.5.12].

Proposition 12.15. *Let $\mathcal{C}$ be an ∞-category and $f\colon x \to z$ and $g\colon y \to z$ be morphisms in $\mathcal{C}$, viewed as objects of $\mathcal{C}_{/z}$. Then the diagram*

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}_{/z}}(f,g) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(x,y) \arrow[d, "g_*"] \\ \Delta^0 \arrow[r, "f"] & \mathrm{map}_{\mathcal{C}}(x,z) \end{tikzcd}$$

is homotopy cartesian.

Proof. Recall that the map $g_*\colon \mathrm{map}_{\mathcal{C}}(x,y) \to \mathrm{map}_{\mathcal{C}}(y,z)$ is constructed as follows. We have the two canonical restriction functors

$$\mathcal{C}_{/y} \xleftarrow{\simeq} \mathcal{C}_{/g} \to \mathcal{C}_{/z}$$

the first of which is an equivalence. Inverting this equivalence, and taking fibres over x in $\mathcal{C}$, we obtain

$$\mathcal{C}_{/y} \times_{\mathcal{C}} \{x\} \simeq \mathcal{C}_{/g} \times_{\mathcal{C}} \{x\} \to \mathcal{C}_{/z} \times_{\mathcal{C}} \{x\}$$

where the first and last term are given by $\mathrm{map}^R_{\mathcal{C}}(x,y)$ and $\mathrm{map}^R_{\mathcal{C}}(x,z)$ respectively. We then consider the diagram

$$\begin{tikzcd} \mathcal{C}_{/g} \times_{\mathcal{C}_{/z}} \{f\} \arrow[r] \arrow[d] & \mathcal{C}_{/g} \times_{\mathcal{C}} \{x\} \arrow[r] \arrow[d] & \mathcal{C}_{/g} \arrow[d] \\ \Delta^0 \arrow[r, "f"] & \mathcal{C}_{/z} \times_{\mathcal{C}} \{x\} \arrow[r] & \mathcal{C}_{/z} \end{tikzcd}$$

in which both the right square and the big square are pullbacks. Hence all squares are pullbacks. Furthermore, the very right vertical map is a right fibration, hence the middle vertical map is a right fibration whose target is an ∞-groupoid. Hence the middle vertical map is a Kan fibration and models the map given by postcomposition with g. It hence remains to show that there is a canonical equivalence

$$\mathrm{map}_{\mathcal{C}_{/z}}(f,g) \simeq \mathcal{C}_{/g} \times_{\mathcal{C}_{/z}} \times \{f\}.$$

For this we observe that $(\mathcal{C}_{/z})_{/g} \cong \mathcal{C}_{/g}$, so the claim follows. □

Lecture 13 – 26.06.2019.

Corollary 12.16. *Let $\mathcal{C} \subseteq \mathcal{D}$ be a full subcategory and let z be an object of $\mathcal{C}$. Then the canonical functor $\mathcal{C}_{/z} \to \mathcal{D}_{/z}$ is again fully faithful.*

Proof. Let $f\colon x \to z$ and $g\colon y \to z$ be objects of $\mathcal{C}_{/z}$. We wish to show that the map

$$\mathrm{map}_{\mathcal{C}_{/z}}(f,g) \to \mathrm{map}_{\mathcal{D}_{/z}}(f,g)$$

is a homotopy equivalence. By Proposition 12.15 it therefore suffices to prove that in the diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{D}}(x,y) \arrow[d] \\ \mathrm{map}_{\mathcal{C}}(x,z) \arrow[r] & \mathrm{map}_{\mathcal{D}}(x,z) \end{tikzcd}$$

both horizontal maps are equivalences. This follows from the assumption that f is fully faithful. □

13. Terminal and initial objects

Definition 13.1. Let $\mathcal{C}$ be an ∞-category. An object x is said to be initial if for all objects y of $\mathcal{C}$, the mapping space $\mathrm{map}_{\mathcal{C}}(x,y)$ is contractible. Likewise x is said to be terminal if it is initial in $\mathcal{C}^{\mathrm{op}}$, i.e. if for all other objects y, the mapping space $\mathrm{map}_{\mathcal{C}}(y,x)$ is contractible.

It will be useful to have the following characterizations:

Lemma 13.2. *Let $\mathcal{C}$ be an ∞-category and x in $\mathcal{C}$ an object. Then the following conditions are equivalent.*

1. *x is terminal,*
2. *the functor $\mathcal{C}_{/x} \to \mathcal{C}$ is a trivial fibration, and*
3. *for every $n \geq 1$, every lifting problem*

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

admits a solution.

Proof. To show that (1) and (2) are equivalent, we consider the following diagram

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[rr] \arrow[dr] & & \mathcal{C} \arrow[dl, equal] \\ & \mathcal{C} & \end{tikzcd}$$

and we wish to show that the horizontal map is a trivial fibration. We already know that it is a right fibration, so it suffices to show that it is a Joyal equivalence if and only if x is terminal. By Theorem 10.21 this map is a Joyal equivalence if and only if it is a Joyal equivalence fibrewise, which amounts to saying that for all objects y of $\mathcal{C}$ the canonical map $\mathrm{map}^R_{\mathcal{C}}(y,x) \to \Delta^0$ is a Joyal equivalence.

To see that (2) and (3) are equivalent, we observe that the map

$$\partial\Delta^{n-1} \star \Delta^0 \amalg_{\partial\Delta^{n-1}\star\emptyset} \Delta^{n-1} \star \emptyset \longrightarrow \Delta^{n-1} \star \Delta^0$$

is isomorphic to the map

$$\partial\Delta^n \to \Delta^n.$$

Hence the lifting problem

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

is equivalent to the lifting problem

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d] & \mathcal{C}_{/x} \arrow[d] \\ \Delta^{n-1} \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

so the lemma follows. $\square$

The following tells us that initial and terminal objects, if they exist, are unique up to contractible choices.

Proposition 13.3. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_{\mathrm{term}}$ be the full subcategory spanned by all terminal objects. Then $\mathcal{C}_{\mathrm{term}}$ is either empty or a contractible Kan complex.*

Proof. Suppose that $\mathcal{C}_{\mathrm{term}}$ is not empty. We need to show that any lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{\mathrm{term}} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. If $n = 0$, this exists by the assumption that $\mathcal{C}_{\mathrm{term}}$ is not empty. If $n \geq 1$, we use (3) of Lemma 13.2 which is possible as in particular the object $\Delta^{\{n\}}$ of $\partial\Delta^n$ is mapped to a terminal object. $\square$

Lemma 13.4. *Let $\mathcal{C}$ be an ∞-category. Then an object x of $\mathcal{C}$ is initial if and only if the map $x\colon \Delta^0 \to \mathcal{C}$ is left anodyne. Dually x is terminal if the map $\Delta^0 \to \mathcal{C}$ is right anodyne.*

Proof. We prove the statement for initial objects. The case for terminal objects is obtained by passing to opposite categories. We first observe that for any monomorphism $S \to T$, the map $S \star \Delta^0 \to T \star \Delta^0$ is right anodyne, and the map $\Delta^0 \star S \to \Delta^0 \star T$ is left anodyne. To see this it suffices to treat the case where $S \to T$ is a boundary inclusion $\partial\Delta^n \to \Delta^n$ in which case the maps in question become $\Lambda^{n+1}_{n+1} \to \Delta^{n+1}$ and $\Lambda^{n+1}_0 \to \Delta^{n+1}$. Now let us assume that

x is initial. By the version of Lemma 13.2 for initial objects, we find that $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial fibration. We choose a section $s\colon \mathcal{C} \to \mathcal{C}_{x/}$ and consider the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "x"] & \Delta^0 \star \Delta^0 \arrow[r] \arrow[d, "\Delta^0 \star x"] & \Delta^0 \arrow[d, "x"] \\ \mathcal{C} \arrow[r] & \Delta^0 \star \mathcal{C} \arrow[r, "\hat{s}"] & \mathcal{C} \end{tikzcd}$$

where $\hat{s}$ is the adjoint map of $s\colon \mathcal{C} \to \mathcal{C}_{x/}$. It follows that both horizontal composites are the identity. We thus find that the map $\Delta^0 \xrightarrow{x} \mathcal{C}$ is a retract of the map $\Delta^0 \star x$ which is left anodyne by our first observation.

Conversely, assume that $\Delta^0 \xrightarrow{x} \mathcal{C}$ is left anodyne. We can consider the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "\mathrm{id}_x"] \arrow[d, "x"] & \mathcal{C}_{x/} \arrow[d] \\ \mathcal{C} \arrow[r, equal] \arrow[ur, dashed, "s"] & \mathcal{C} \end{tikzcd}$$

and find a dashed arrow s making the diagram commute. We will show that x is initial by establishing (3) of Lemma 13.2. We thus consider a diagram

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[r, "s"] & \mathcal{C}_{x/} \arrow[r] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] \arrow[urr, dotted] & & & \end{tikzcd}$$

By construction $s(x) = \mathrm{id}_x$ and is thus an initial object of $\mathcal{C}_{x/}$ by Exercise 133. Hence the dotted arrow exists and thus also a dashed arrow. □

The following is also useful to know.

Proposition 13.5. *Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. Suppose given a functor $F\colon K \to \mathcal{C}$ such that for all objects x of K, the object $F(x)$ is initial, respectively terminal, in $\mathcal{C}$. Then F is initial, respectively terminal, in* $\mathrm{Fun}(K, \mathcal{C})$.

Proof. We show the case of terminal objects. We need to prove that for all $n \geq 1$, any lifting problem

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "F"] & \partial\Delta^n \arrow[r, "\hat{F}"] \arrow[d] & \mathrm{Fun}(K, \mathcal{C}) \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

admits a solution. By adjunction, this corresponds to the lifting problem

$$\begin{tikzcd}
K \times \Delta^{\{n\}} \arrow[r] \arrow[rr, bend left, "F"] & K \times \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& K \times \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

where now by assumption the further restriction of the top horizontal composite along any object $x\colon \Delta^0 \to K$ is a terminal object of $\mathcal{C}$. We consider the filtration $F_k(K) = \mathrm{sk}_k(K \times \Delta^n) \cup K \times \partial\Delta^n$ and inductively wish to solve the extension problem

$$\begin{tikzcd}
\coprod\limits_{i \in I(k)} \partial\Delta^k \arrow[r, "a_i"] \arrow[d] & F_{k-1}(K) \arrow[r, "f_{k-1}"] \arrow[d] & \mathcal{C} \\
\coprod\limits_{i \in I(k)} \Delta^k \arrow[r] & F_k(K) \arrow[ur, dashed] &
\end{tikzcd}$$

For this it suffices to observe that for all $i \in I(k)$ the composite $f_{k-1} \circ a_i$ sends the vertex $\{k\}$ to a terminal object in $\mathcal{C}$. By definition $I(k)$ consists of those non-degenerate k-simplices of $K \times \Delta^n$ which are not contained in $K \times \partial\Delta^n$, in other words consists in particular of pairs $(\alpha_i, \beta_i) \in K_k \times \Delta^n_k$ such that $b\colon [k] \to [n]$ is surjective. In particular, β_i sends the object $\{k\}$ to $\{n\}$. Hence $f_{k-1}(a_i(\{k\})) = \hat{F}(\alpha_i(\{k\}, \{n\}))$ which is a terminal object by assumption. Hence the dashed arrow exists. Passing to the colimit over k then gives the proposition. $\square$

The converse of Proposition 13.5 is almost true.

Lemma 13.6. *Suppose that $\mathcal{C}_{\mathrm{term}}$ is not empty. Then any terminal object of* $\mathrm{Fun}(K, \mathcal{C})$ *takes values in $\mathcal{C}_{\mathrm{term}}$.*

Proof. Let x be a terminal object and consider the constant functor c_x with value x. By Proposition 13.5, c_x is a terminal object of $\mathrm{Fun}(K, \mathcal{C})$. By Proposition 13.3, any other terminal object of $\mathrm{Fun}(K, \mathcal{C})$ is equivalent to c_x. In particular, any other terminal object T evaluates on an object k to $T(k)$ which is equivalent to $c_x(k) = x$ and is thus terminal. $\square$

14. The Yoneda lemma

Lecture 14 – 01.07.2019. We now wish to show that for every ∞-category $\mathcal{C}$, there is a Yoneda functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ which should send an object x of $\mathcal{C}$ to the "functor" $y \mapsto \mathrm{map}_{\mathcal{C}}(y, x)$. Then we will establish that this functor is fully faithful, which is the ∞-categorical version of the Yoneda lemma. The fact that $\mathrm{map}_{\mathcal{C}}(-, x)$ is a functor for every single x is something we now know.

Definition 14.1. Let x be an object of an ∞-category $\mathcal{C}$. Then the functor $\mathcal{C}^{/x} \to \mathcal{C}$ is a right fibration and hence by Theorem 12.13 (applied to $\mathcal{C}^{\mathrm{op}}$) is equivalently given by a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ sending y to $\mathcal{C}^{/x} \times_{\mathcal{C}} \{y\} \simeq \mathrm{map}_{\mathcal{C}}(y, x)$. We shall denote this functor by $\mathrm{map}_{\mathcal{C}}(-, x)$.

The task now is to make precise that the functors $\mathrm{map}_{\mathcal{C}}(-, x)$ in turn are functorial in x. We begin with a construction.

Lemma 14.2. *The association $[n] \mapsto [n] \star [n]^{\mathrm{op}} \cong [2n+1]$ extends to a functor $\Delta \to \Delta$. In particular, sending $[n]$ to $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ is a cosimplicial object in simplicial sets, i.e. a functor $\Delta \to \mathrm{sSet}$.*

Definition 14.3. Let $\mathcal{C}$ be an ∞-category. We define its twisted arrow category $\mathrm{Tw}(\mathcal{C})$ to be the following simplicial set

$$\mathrm{Tw}(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star (\Delta^n)^{\mathrm{op}}, \mathcal{C})$$

where the simplicial structure comes from the cosimplicial object $[n] \mapsto \Delta^n \star (\Delta^n)^{\mathrm{op}}$. The inclusions $\Delta^n \to \Delta^n \star (\Delta^n)^{\mathrm{op}} \leftarrow (\Delta^n)^{\mathrm{op}}$ determine a functor

$$\mathrm{Tw}(\mathcal{C}) \longrightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

The following proof is taken from [Lur17, Proposition 5.2.1.3].

Proposition 14.4. *For an ∞-category $\mathcal{C}$, the functor*

$$\mathrm{Tw}(\mathcal{C}) \longrightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$$

is a right fibration. In particular, $\mathrm{Tw}(\mathcal{C})$ is again an ∞-category.

Proof. Let $0 < k \leq n$, and consider a lifting problem

$$\begin{tikzcd} \Lambda^k_n \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{tikzcd}$$

Unravelling the definition of the twisted arrow category, this corresponds to the following lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^{2n+1} \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

where K is the subsimplicial set of Δ^{2n+1} consisting of those faces σ satisfying either of the following three properties:

(1) σ is contained in $\Delta^{\{0,\dots,n\}} \subseteq \Delta^{2n+1}$,
(2) σ is contained in $\Delta^{\{n+1,\dots,2n+1\}} \subseteq \Delta^{2n+1}$,
(3) There exists $j \neq k$, with $0 \leq j \leq n$ such that neither j nor $2n+1-j$ is a vertex of σ.

Since $\mathcal{C}$ is an ∞-category, it suffices to show that the inclusion $K \to \Delta^{2n+1}$ is inner anodyne. We call a simplex σ primary, if it is not contained in K and its vertices are contained in the set $\{k, \dots, 2n+1\}$. We call σ secondary if it is not contained in K and not primary. We let S be the set containing the following simplices τ of Δ^{2n+1}:

(1) τ is primary and k is not a vertex of τ,
(2) τ is secondary and $2n+1-k$ is not a vertex of τ.

Given a simplex τ in S, we let τ' be the simplex obtained by adding the vertex k if τ is primary, and by adding $2n+1-k$ is τ is secondary. We observe that each simplex of Δ^{2n+1} is either contained in K, in S, or is of the form τ' for a unique τ in S: If it is neither contained in K or S, then it must be either primary and contain k as vertex, or be secondary and contain $2n+1-k$ as vertex. In either case, one can remove the vertex k or $2n+1-k$ and obtain a primary and secondary simplex as needed.

We now choose an ordering $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ of S with the following two properties:

(1) If $p \leq q$, then $\dim(\sigma_p) \leq \dim(\sigma_q)$,
(2) If $p \leq q$ and $\dim(\sigma_p) = \dim(\sigma_q)$ and σ_q is primary, then also σ_p is primary.

For $0 \leq q \leq m$, we let K_q be the sub simplicial set of Δ^{2n+1} obtained from K by adding the simplices σ_p and σ'_p for $1 \leq p \leq q$. Clearly $K_m = \Delta^{2n+1}$, so we obtain a filtration

$$K \to K_1 \to K_2 \to \cdots \to K_m = \Delta^{2n+1}$$

and it will suffices to show that for each q, the map $K_{q-1} \to K_q$ is inner anodyne. Since K_q is obtained from K_{q-1} by adding the simplices σ_q and σ'_q, and σ'_q contains σ_q, it suffices to show that there is a pushout

$$\begin{tikzcd} \Lambda^d_j \arrow[r] \arrow[d] & K_{q-1} \arrow[d] \\ \Delta^d \arrow[r, "\sigma'_q"] & K_q \end{tikzcd}$$

where d is the dimension of σ'_q and $0 < j < d$. For this, we need to see which of the faces of σ'_q are already contained in K_{q-1}, and argue that precisely one inner face is not contained in K_{q-1}. $\square$

For every ∞-category, we denote by $\mathrm{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ the functor associated with the right fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Remark. We could also go a different route here: If we only wanted to construct a mapping functor $\mathrm{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$, we can observe that by definition $\mathrm{An} = \mathrm{N}(\mathrm{Kan})$, so such a functor is equivalently given by a simplicial functor $\mathfrak{C}[\mathcal{C}^{\mathrm{op}} \times \mathcal{C}] \to \mathrm{Kan}$. We can consider the composite

$$\mathfrak{C}[\mathcal{C}^{\mathrm{op}} \times \mathcal{C}] \to \mathfrak{C}[\mathcal{C}]^{\mathrm{op}} \times \mathfrak{C}[\mathcal{C}] \to \mathrm{Kan}$$

where the first is given by the canonical map and the second by the simplicial mapping space followed by a functorial Kan-replacement. The fact that this is (at least pointwise) equivalent to our approach is [Lur09, Theorem 1.1.5.13].

The following is then contained in the proof of [Lur17, Proposition 5.2.1.10].

Lemma 14.5. *For an ∞-category $\mathcal{C}$ and every object x of $\mathcal{C}$, there is a canonical commutative diagram*

$$\begin{tikzcd} \mathcal{C}_{/x} \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d] \\ \mathcal{C} \times \{x\} \arrow[r] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{tikzcd}$$

This diagram is homotopy cartesian. In other words, the induced map $\mathcal{C}_{/x} \to \mathrm{Tw}(\mathcal{C})_x$ is a Joyal equivalence between right fibrations over $\mathcal{C}$.

Proof. We recall that the n-simplices of $\mathrm{Tw}(\mathcal{C})_x$ are given by those maps $\Delta^n \star (\Delta^n)^{\mathrm{op}} \to \mathcal{C}$ whose restriction along the inclusion $(\Delta^n)^{\mathrm{op}} \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ are the map which is constant at the object x of $\mathcal{C}$. One defines an auxiliary simplicial set $\mathcal{M}$, whose n-simplices are given by maps $\Delta^n \star \Delta^0 \star (\Delta^n)^{\mathrm{op}} \to \mathcal{C}$, whose restriction along $\emptyset \star \Delta^0 \star (\Delta^n)^{\mathrm{op}} \to \Delta^n \star \Delta^0 \star (\Delta^n)^{\mathrm{op}}$ are constant at x. The obvious inclusions define maps

$$\mathcal{C}_{/x} \leftarrow \mathcal{M} \to \mathrm{Tw}(\mathcal{C})_x$$

and the left map admits a section induced from the map $\Delta^0\star(\Delta^n)^{\mathrm{op}} \to \Delta^0$. It hence suffices to show that both of the above maps are Joyal equivalences. This is done by showing that both are trivial fibrations, by explicitly exhibiting the lifting property with respect to boundary inclusions. Let us treat the case $\mathcal{M} \to \mathcal{C}_{/x}$. We consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{/x} \end{tikzcd}$$

and unravel definitions to see that this is equivalently given by a lifting problem

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \star \Delta^0 \star \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

where K denotes the smallest sub simplicial set containing $\Delta^n \star \Delta^0 \star \emptyset$, $\emptyset \star \Delta^0 \star \Delta^n$, and $\Delta^I \star \Delta^0 \star (\Delta^I)^{\mathrm{op}}$, for every proper subset $I \subseteq [n]$. By Theorem 9.13, it suffices to show that the inclusion $K \to \Delta^n \star \Delta^0 \star \Delta^n$ is a Joyal equivalence. First, we claim that the composite

$$\Delta^n \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \to K \to \Delta^n \star \Delta^0 \star \Delta^n$$

is a Joyal equivalence. To see this we consider the diagram

$$\begin{tikzcd} I^{n+1} \amalg_{\Delta^0} I^{n+1} \arrow[r] \arrow[d] & I^{2n+2} \arrow[d] \\ \Delta^{n+1} \amalg_{\Delta^0} \Delta^{n+1} \arrow[r] & \Delta^{2n+2} \end{tikzcd}$$

in which the upper horizontal map is an isomorphism and the lower horizontal map is the map under investigation. It hence suffices to show that the vertical maps are Joyal equivalences. For the right hand this follows since the spine is inclusion is inner anodyne by Proposition 3.17 and thus a Joyal equivalence by Corollary 6.13, and for the left vertical map we argue likewise for the maps $I^{n+1} \to \Delta^{n+1}$ and then use Corollary 9.7 to conclude that the map on pushouts is also a Joyal equivalence. It now suffices to show that the map

$$\Delta^n \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \to K$$

is a Joyal equivalence. We denote by K_0 the sub simplicial set of K spanned by the simplices of the form $\Delta^I \star \Delta^0 \star (\Delta^I)^{\mathrm{op}}$ for $I \subseteq [n]$ a proper subset. The following diagram

$$\begin{tikzcd} \partial\Delta^n \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \partial\Delta^n \arrow[r, "j"] \arrow[d] & K_0 \arrow[d] \\ \Delta^n \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star \Delta^n \arrow[r] & K \end{tikzcd}$$

is a pushout, so it suffices to show that the map j is a Joyal equivalence. This map is a colimit of maps of the form

$$\Delta^I \star \Delta^0 \amalg_{\Delta^0} \Delta^0 \star (\Delta^I)^{\mathrm{op}} \to \Delta^I \star \Delta^0(\Delta^I)^{\mathrm{op}}$$

which we have just argued to be Joyal equivalences. We thus conclude that the map $\mathcal{M} \to \mathcal{C}_{/x}$ is a Joyal equivalence.

It remains to prove that the map $\mathcal{M} \to \mathrm{Tw}(\mathcal{C})_x$ is also a Joyal equivalence. The proof is very similar in spirit to the one of Proposition 14.4, so we will refrain from spelling out the details and refer to [Lur17, Proposition 5.2.1.10] instead. □

Corollary 14.6. *For every object x, the composite $\mathcal{C} \times \{x\} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ is equivalent to the functor $\mathrm{map}_{\mathcal{C}}(-, x)$.*

Proof. The first functor is the one associated to the right fibration $\mathrm{Tw}(\mathcal{C})_x \to \mathcal{C}$, whereas the other one is associated to the right fibration $\mathcal{C}^{/x} \to \mathcal{C}$. By Lemma 14.5 and Proposition 9.25, these right fibrations are equivalent. Hence the claim follows from the straightening-unstraightening equivalence Theorem 12.13. □

Definition 14.7. Let $\mathcal{C}$ be an ∞-category. The functor $\mathrm{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ is adjoint to a functor $Y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ which we call the Yoneda functor.

The following is the ∞-categorical version of the Yoneda lemma.

Proposition 14.8. *Let $F\colon \mathcal{C} \to \mathrm{An}$ be a functor and let x be an object of $\mathcal{C}$. Then the canonical map*

$$\mathrm{map}_{\mathrm{Fun}(\mathcal{C},\mathrm{An})}(\mathrm{map}_{\mathcal{C}}(x,-), F) \to F(x)$$

given by evaluation at the identity is an equivalence.

Proof. We let $p\colon \mathcal{E} \to \mathcal{C}$ be the left fibration corresponding to the functor F. Under the straightening-unstraightening equivalence Theorem 12.13

$$\mathrm{Fun}(\mathcal{C}, \mathrm{An}) \simeq \mathrm{LFib}(\mathcal{C}) \overset{\text{full}}{\subseteq} \mathrm{Cat}_{\infty/\mathcal{C}},$$

the left hand mapping anima corresponds to $\mathrm{map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\mathcal{C}_{x/}, \mathcal{E})$ and we claim that there is a canonical equivalence

$$\mathrm{map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\mathcal{C}_{x/}, \mathcal{E}) \simeq \mathrm{Fun}_q(\mathcal{C}_{x/}, \mathcal{E})$$

where $q\colon \mathcal{C}_{x/} \to \mathcal{C}$ is the canonical forgetful functor and the latter is as in Definition 11.10 without the superscripts.

Taking this for granted for the moment, we need to show that the map

$$\mathrm{Fun}_q(\mathcal{C}_{x/}, \mathcal{E}) \to \mathrm{Fun}_{\mathrm{id}_x}(\Delta^0, \mathcal{E}) \cong \mathcal{E}_x$$

is an equivalence: Recall that by construction of the straightening-unstraightening equivalence, there is a canonical equivalence $\mathcal{E}_x \simeq F(x)$. Here, the map is induced by the canonical map $\Delta^0 \to \mathcal{C}_{x/}$ specifying the identity of x. By Exercise 133 the identity of x is an initial object of $\mathcal{C}_{x/}$ so that the map $\Delta^0 \to \mathcal{C}_{x/}$ is left anodyne by Lemma 13.4. It then follows from Corollary 11.13 that the map in question is a trivial fibration and thus a Joyal equivalence: Since $p\colon \mathcal{E} \to \mathcal{C}$ is a left fibration we find that $\mathrm{Fun}_q(\mathcal{C}_{x/}, \mathcal{E}) = \mathrm{Fun}_q^{\mathrm{cc}}(\mathcal{C}_{x/}, \mathcal{E})$, and likewise for Δ^0 in place of $\mathcal{C}_{x/}$.

It remains to prove the claim about the mapping anima in $\mathrm{Cat}_{\infty/\mathcal{C}}$. For this we invoke Proposition 12.15, which says that the following diagram is homotopy cartesian

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(q,p) \arrow[r] \arrow[d] & \mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{C}_{x/}, \mathcal{E}) \arrow[d, "p_*"] \\
\Delta^0 \arrow[r, "q"] & \mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{C}_{x/}, \mathcal{C})
\end{tikzcd}$$

We now recall from Theorem 9.31 that the mapping anima in Cat_∞ are canonically equivalent to the groupoid cores of the functor categories. Hence the above right vertical map identifies with the left vertical map in the diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}_{x/}, \mathcal{E})^{\simeq} \arrow[r] \arrow[d, "p_*"] & \mathrm{Fun}(\mathcal{C}_{x/}, \mathcal{E}) \arrow[d, "p_*"] \\ \mathrm{Fun}(\mathcal{C}_{x/}, \mathcal{C})^{\simeq} \arrow[r] & \mathrm{Fun}(\mathcal{C}_{x/}, \mathcal{C}) \end{tikzcd}$$

Since p is a right fibration, so is p_* by Theorem 3.32. Thus, by Proposition 5.3 the map p_* is conservative, so that the diagram is cartesian by Exercise 79. Pasting together the two diagrams, we find that the square

$$\begin{tikzcd} \mathrm{map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(q, p) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}_{x/}, \mathcal{E}) \arrow[d, "p_*"] \\ \Delta^0 \arrow[r, "q"] & \mathrm{Fun}(\mathcal{C}_{x/}, \mathcal{C}) \end{tikzcd}$$

is homotopy cartesian. Since $\mathrm{Fun}_q(\mathcal{C}_{x/}, \mathcal{E})$ is the pullback of this diagram, we find that the canonical map

$$\mathrm{map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(q, p) \to \mathrm{Fun}_q(\mathcal{C}_{x/}, \mathcal{E})$$

is a Joyal equivalence as claimed. $\square$

With this we find the usual consequence of the Yoneda lemma, namely that the Yoneda functor is fully faithful. Henceforth, it will be called the Yoneda embedding.

Proposition 14.9. *Let $\mathcal{C}$ be an ∞-category. Then the Yoneda functor $Y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is fully faithful.*

Proof. We simply calculate that the evaluation map

$$\mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{An})}(\mathrm{map}_{\mathcal{C}}(-, x), \mathrm{map}_{\mathcal{C}}(-, y)) \longrightarrow \mathrm{map}_{\mathcal{C}}(x, y)$$

is an equivalence by Proposition 14.8 and that the composite

$$\mathrm{map}_{\mathcal{C}}(x, y) \xrightarrow{Y} \mathrm{map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{An})}(\mathrm{map}_{\mathcal{C}}(-, x), \mathrm{map}_{\mathcal{C}}(-, y)) \longrightarrow \mathrm{map}_{\mathcal{C}}(x, y)$$

is also an equivalence: The first map is equivalently described by the map

$$\mathrm{map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{map}_{\mathrm{Cat}_{\infty/\mathcal{C}}}(\mathcal{C}_{x/}, \mathcal{C}_{y/})$$

given by post composition on the slices. Restricting this to the identity of x clearly induces an equivalence as needed. Hence also the map induced by the Yoneda functor is an equivalence, which precisely says that the Yoneda functor is fully faithful. $\square$

15. Limits and colimits

The following definition of colimits is taken from Krause–Nikolaus.

Definition 15.1. Let $F\colon K \to \mathcal{C}$ be a functor and let x be an object of $\mathcal{C}$. We define a simplicial set $\mathrm{Map}_{\mathcal{C}}(F,x)$ by the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F,x) \arrow[r] \arrow[d] & \mathrm{Fun}(K \star \Delta^0, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(F,x)}"] & \mathrm{Fun}(K,\mathcal{C}) \times \mathcal{C} \end{tikzcd}$$

where the right vertical map is given by restriction along the canonical inclusion $K \cup \{\infty\} \subseteq K \star \Delta^0$.

Lemma 15.2. *In the above situation,* $\mathrm{Map}_{\mathcal{C}}(F,x)$ *is an* ∞*-groupoid. If* $F = y\colon \Delta^0 \to \mathcal{C}$*, then* $\mathrm{Map}_{\mathcal{C}}(y,x) = \mathrm{map}_{\mathcal{C}}(y,x)$*.*

Proof. Restriction along a monomorphism is a conservative inner fibration. Such are stable under pullbacks. The second part is clear from the definition, as $\Delta^0 \star \Delta^0 = \Delta^1$. □

Proposition 15.3. *Let* $F\colon K \to \mathcal{C}$ *be a functor and* $i\colon L \to K$ *be a map of simplicial sets. Then for all objects* x *of* $\mathcal{C}$*,* i *induces a map* $\mathrm{Map}_{\mathcal{C}}(F,x) \to \mathrm{Map}_{\mathcal{C}}(Fi,x)$*. If* i *is right anodyne, this map is a homotopy equivalence.*

Proof. To see the first statement, we observe that there is a commutative diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "{(F,x)}"] \arrow[d, equal] & \mathrm{Fun}(K,\mathcal{C}) \times \mathcal{C} \arrow[d] & \mathrm{Fun}(K \star \Delta^0, \mathcal{C}) \arrow[l] \arrow[d] \\ \Delta^0 \arrow[r, "{(Fi,x)}"] & \mathrm{Fun}(L,\mathcal{C}) \times \mathcal{C} & \mathrm{Fun}(L \star \Delta^0, \mathcal{C}) \arrow[l] \end{tikzcd}$$

which induces the map of interest on pullbacks. To see that this map is a homotopy equivalence if i is right anodyne, we consider the following. We claim that the diagram

$$\begin{tikzcd} \mathrm{Fun}(K \star \Delta^0, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(L \star \Delta^0, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}(K \cup \{\infty\}, \mathcal{C}) \arrow[r] & \mathrm{Fun}(L \cup \{\infty\}, \mathcal{C}) \end{tikzcd}$$

is homotopy cartesian. To see this, we calculate the pullback to be given by

$$\mathrm{Fun}(L \star \Delta^0 \amalg_{L \cup \{\infty\}} K \cup \{\infty\}, \mathcal{C}).$$

The comparison map is then induced by the canonical inclusion $L \star \Delta^0 \amalg_{L\cup\{\infty\}} K \cup \{\infty\} \to K \star \Delta^0$ which is inner anodyne by **??**, so that the comparison map is a Joyal equivalence as needed. It follows that also the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F,x) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(Fi,x) \arrow[d] \\ \Delta^0 \arrow[r, equal] & \Delta^0 \end{tikzcd}$$

is homotopy cartesian, so that the upper map is a homotopy equivalence as desired. □

Definition 15.4. Let $F\colon K \to \mathcal{C}$ be a functor and $\bar{F}\colon K \star \Delta^0 \to \mathcal{C}$ a cone over F. We say that $\bar{F}$ is a colimit cone if for all objects x of $\mathcal{C}$, the canonical map

$$\mathrm{Map}_{\mathcal{C}}(\bar{F}, x) \to \mathrm{Map}_{\mathcal{C}}(F, x)$$

is a homotopy equivalence.

Remark. Since the map $\{\infty\} \to K \star \Delta^0$ is right anodyne, we find that for any cone $\bar{F}\colon K \star \Delta^0 \to \mathcal{C}$, the canonical map

$$\mathrm{Map}_{\mathcal{C}}(\bar{F}, x) \to \mathrm{Map}_{\mathcal{C}}(\bar{F}(\infty), x) = \mathrm{map}_{\mathcal{C}}(\bar{F}(\infty), x)$$

is a homotopy equivalence. Hence $\bar{F}$ is a colimit cone if and only if for all objects x of $\mathcal{C}$, the above maps assemble to a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(F, x) \simeq \mathrm{map}_{\mathcal{C}}(\bar{F}(\infty), x).$$

Definition 15.5. Dually, for a functor $F\colon K \to \mathcal{C}$, one defines a simplicial set $\mathrm{Map}_{\mathcal{C}}(x, F)$ as the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x, F) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^0 \star K, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(x,F)}"] & \mathcal{C} \times \mathrm{Fun}(K, \mathcal{C}) \end{tikzcd}$$

As before, a map $i\colon L \to K$ induces a map $\mathrm{Map}_{\mathcal{C}}(x, F) \to \mathrm{Map}_{\mathcal{C}}(x, Fi)$ which is a homotopy equivalence if i is left anodyne. A cocone $\bar{F}\colon \Delta^0 \star K \to \mathcal{C}$ of F is then called a limit cocone if the canonical map

$$\mathrm{Map}_{\mathcal{C}}(x, \bar{F}) \to \mathrm{Map}_{\mathcal{C}}(x, F)$$

is a homotopy equivalence for all x in $\mathcal{C}$.

Remark. Let us compare this definition to the definition of initial and terminal objects. The goal is to see that an initial object is a colimit of the empty functor $\emptyset \to \mathcal{C}$: A cone over the empty functor is simply a functor $y\colon \Delta^0 \to \mathcal{C}$. Furthermore, $\mathrm{Map}_{\mathcal{C}}(\emptyset, x) \cong \Delta^0$, and $\mathrm{Map}_{\mathcal{C}}(y, x) = \mathrm{map}_{\mathcal{C}}(y, x)$ by Lemma 15.2. Thus we find that an object y, viewed as cone over the empty functor, is a colimit cone if and only if for all objects x, the mapping anima $\mathrm{map}_{\mathcal{C}}(y, x)$ is Joyal equivalent to Δ^0, i.e. is contractible. Thus, a colimit cone over the empty functor is precisely an initial object. Likewise, a limit cocone over the empty functor is precisely a terminal object.

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Example 15.6.
(1) A colimit over a set (viewed as a discrete category) is called a coproduct. A limit over a set is called a product.
(2) A colimit of $\Lambda^2_0 \to \mathcal{C}$ is called a pushout. A limit of $\Lambda^2_2 \to \mathcal{C}$ is called a pullback. Notice that $\Lambda^2_0 \star \Delta^0 \cong \Delta^1 \times \Delta^1$ and likewise that $\Delta^0 \star \Lambda^2_2 \cong \Delta^1 \times \Delta^1$.

In Lemma 13.2, we characterized initial and terminal objects in terms of certain maps between slices to be trivial fibrations. We will now work towards a similar description for general colimits and limits. To get started, we have the following lemma.

Lemma 15.7. *For any two simplicial sets X and S, there is a canonical isomorphism*

$$K \diamond S \cong [(K \diamond \Delta^0) \times S] \amalg_{K \times S} K$$

compatible with the maps from K and S.

Proof. Consider the following diagram

$$\begin{tikzcd}
K \times \Delta^1 \times S \arrow[d, equal] & K \times \partial\Delta^1 \times S \arrow[l] \arrow[r] \arrow[d, equal] & K \times S \amalg S \arrow[d] \\
K \times \Delta^1 \times S & K \times \partial\Delta^1 \times S \arrow[l] \arrow[r] & K \amalg S
\end{tikzcd}$$

whose pushouts are given by $(K \diamond \Delta^0) \times S$ and $K \diamond S$, respectively. We thus find that the the right of the small squares in the diagram

$$\begin{tikzcd}
K \times S \arrow[r] \arrow[d] & K \times S \amalg S \arrow[r] \arrow[d] & (K \diamond \Delta^0) \times S \arrow[d] \\
K \arrow[r] & K \amalg S \arrow[r] & K \diamond S
\end{tikzcd}$$

is a pushout. The left square is a pushout by inspection, so also the combined square is a pushout. $\square$

Lemma 15.8. *Let $F\colon K \to \mathcal{C}$ be a functor and let x be an object of $\mathcal{C}$. Then the diagrams*

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(F, x) \arrow[r] \arrow[d] & \mathcal{C}_{F/} \arrow[d] \\
\Delta^0 \arrow[r, "x"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(x, F) \arrow[r] \arrow[d] & \mathcal{C}_{/F} \arrow[d] \\
\Delta^0 \arrow[r, "x"] & \mathcal{C}
\end{tikzcd}$$

are homotopy cartesian.

Proof. We argue for the left hand square, the other case is analogous. We first show that there is a pullback diagram

$$\begin{tikzcd}
\mathcal{C}^{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(K \diamond \Delta^0, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "F"] & \mathrm{Fun}(K, \mathcal{C})
\end{tikzcd}$$

For the time being, let us call the pullback $\Phi(F)$. For a simplicial set S, a map to $\mathcal{C}^{F/}$ corresponds to a map $K \diamond S \to \mathcal{C}$, whereas a map to $\Phi(F)$ corresponds to a map

$$(K \diamond \Delta^0) \times S \amalg_{K \times S} K \to \mathcal{C}.$$

Thus we conclude by Lemma 15.7. We then define a simplicial set $\mathcal{C}_{/\!\!/F}$ be the pullback

$$\begin{tikzcd}
\mathcal{C}_{F/\!\!/} \arrow[r] \arrow[d] & \mathrm{Fun}(K \star \Delta^0, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "F"] & \mathrm{Fun}(K, \mathcal{C})
\end{tikzcd}$$

From the canonical map $K \diamond \Delta^0 \to K \star \Delta^0$ we obtain maps

$$\mathcal{C}_{F/\!\!/} \to \mathcal{C}^{F/} \leftarrow \mathcal{C}_{F/}$$

and we claim that both are Joyal equivalences: the right map was dealt with in Proposition 9.25 and for the left map it follows again from the fact that the map $K \diamond \Delta^0 \to K \star \Delta^0$ is a Joyal equivalence, and that the pullbacks involving $\mathcal{C}^{F/}$ and $C_{F/\!/}$ are invariant under Joyal equivalences by Lemma 9.6. We then consider the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(F,x) \arrow[r] \arrow[d] & \mathcal{C}_{F/\!/} \arrow[r] \arrow[d] & \mathrm{Fun}(K\star\Delta^0,\mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r] & \mathcal{C} \arrow[r] \arrow[d] & \mathrm{Fun}(K,\mathcal{C})\times\mathcal{C} \arrow[d] \\
& \Delta^0 \arrow[r] & \mathrm{Fun}(K,\mathcal{C})
\end{tikzcd}$$

consisting of pullback diagrams. The claim then follows from the fact that there is a Joyal equivalence $\mathcal{C}_{F/\!/} \simeq \mathcal{C}^{F/}$. $\square$

Remark. One could define a variant of $\mathrm{Map}_{\mathcal{C}}(F,x)$ using the fat join instead of the ordinary join. The resulting ∞-groupoid $\widetilde{\mathrm{Map}}_{\mathcal{C}}(F,x)$ will be canonically equivalent to $\mathrm{Map}_{\mathcal{C}}(F,x)$ and we will freely exchange the two when useful. The proof of Lemma 15.8 then shows that for this variant the following diagram is a pullback.

$$\begin{tikzcd}
\widetilde{\mathrm{Map}_{\mathcal{C}}}(F,x) \arrow[r] \arrow[d] & \mathcal{C}^{F/} \arrow[d] \\
\Delta^0 \arrow[r] & \mathcal{C}
\end{tikzcd}$$

The following theorem says that our definition of limits and colimits coincides with the one given usually, for instance in [Lur09].

Theorem 15.9. *Let $F\colon K \to \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$. A cone $\bar{F}\colon K\star\Delta^0 \to \mathcal{C}$ of F is a colimit cone if and only if it is an initial object of $\mathcal{C}_{F/}$. Dually, a cocone $\bar{F}\colon \Delta^0\star K \to \mathcal{C}$ of F is a limit cocone if and only if it is a terminal object of $\mathcal{C}_{/F}$.*

Proof. A cocone $\bar{F}$ gives rise to a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{\bar{F}/} \arrow[rr] \arrow[dr] & & \mathcal{C}_{F/} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

of left fibrations. By Exercise 140, $\bar{F}$ is an initial object if and only if the horizontal map is a trivial fibration, which is the case if and only if it is a Joyal equivalence. By Theorem 10.21, this is the case if and only if the induced map on fibres over objects x of $\mathcal{C}$ is a homotopy equivalence. By Lemma 15.8, the induced map on fibres is equivalent to the map

$$\mathrm{Map}_{\mathcal{C}}(\bar{F},x) \to \mathrm{Map}_{\mathcal{C}}(F,x).$$

We thus find that the map $\mathcal{C}_{\bar{F}/} \to \mathcal{C}_{F/}$ is a Joyal equivalence if and only if $\bar{F}$ is a colimit cone of F. The argument for limits is the same, using the dual version of Lemma 15.8. $\square$

The following will be very useful later,

Proposition 15.10. *Let $\mathcal{C}$ be an ∞-category and consider a pushout of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & K \end{tikzcd}$$

in which the map i is a monomorphism. Let $F\colon K \to \mathcal{C}$ be a functor, and denote by F_A, F_B, and F_C its restriction to A, B and C. Then, for each object x of $\mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F, x) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(F_B, x) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(F_C, x) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(F_A, x) \end{tikzcd}$$

is homotopy cartesian.

Proof. To prove this, we may replace $\mathrm{Map}_{\mathcal{C}}(F, x)$ with $\widetilde{\mathrm{Map}}_{\mathcal{C}}(F, x)$, the version defined using the fat slice. We observe that the diagram is a pullback, because $\widetilde{\mathrm{Map}}_{\mathcal{C}}(F, x)$ is defined as a fibre which commutes with pullbacks, and both functors $\mathrm{Fun}(-, \mathcal{C})$ and $\mathrm{Fun}(-, \mathcal{C}) \times \mathcal{C}$ send pushouts to pullbacks. Then we can use that $- \diamond \Delta^0$ also preserves pushouts. It hence suffices to show that the map $\widetilde{\mathrm{Map}}_{\mathcal{C}}(F_B, x) \to \widetilde{\mathrm{Map}}_{\mathcal{C}}(F_A, x)$ is a Kan fibration. By a previous remark there is a pullback diagram

$$\begin{tikzcd} \widetilde{\mathrm{Map}}_{\mathcal{C}}(F_B, x) \arrow[r] \arrow[d] & \mathcal{C}^{F_B/} \arrow[d] \\ \widetilde{\mathrm{Map}}_{\mathcal{C}}(F_A, x) \arrow[r] & \mathcal{C}^{F_A/} \end{tikzcd}$$

so it suffices to recall that the map $\mathcal{C}^{F_B/} \to \mathcal{C}^{F_A/}$ is a left fibration, so that its pullback is a left fibration between Kan complexes, and thus a Kan fibration. $\square$

We again find that (co)limits, if they exist, are unique up to contractible choice.

Lemma 15.11. *Let $p\colon K \to \mathcal{C}$ be a diagram. Let $(\mathcal{C}_{p/})^{\mathrm{colim}} \subseteq \mathcal{C}_{p/}$ and $(\mathcal{C}_{/p})^{\mathrm{lim}} \subseteq \mathcal{C}_{/p}$ be the full subcategories spanned by colimit cones and limit cocones. Then $(\mathcal{C}_{p/})^{\mathrm{colim}}$, respectively $(\mathcal{C}_{/p})^{\mathrm{lim}}$, are either empty or contractible Kan complexes.*

Proof. This is merely a reformulation of the case for initial and terminal objects Proposition 13.3. $\square$

The following will be used later to show that forming colimits (if possible) is a functor.

Proposition 15.12. *Let K be a simplicial set, $F\colon K \to \mathcal{C}$ a functor, and x an object of an ∞-category $\mathcal{C}$. There is a canonical homotopy equivalence*

$$\mathrm{Map}_{\mathcal{C}}(F, x) \simeq \mathrm{map}_{\mathrm{Fun}(K, \mathcal{C})}(F, \mathrm{const}_x).$$

Proof. We claim that the following diagram is cartesian

$$\begin{tikzcd} \mathcal{C}^{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathcal{C})^{F/} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathrm{const}"] & \mathrm{Fun}(K, \mathcal{C}) \end{tikzcd}$$

where we also view F as a functor $\Delta^0 \to \mathrm{Fun}(K, \mathcal{C})$. Thus, pulling back along a map $x\colon \Delta^0 \to \mathcal{C}$, we obtain a Joyal equivalence

$$\mathrm{Map}_{\mathcal{C}}(F, x) \xrightarrow{\simeq} \mathrm{map}^L_{\mathrm{Fun}(K,\mathcal{C})}(F, \mathrm{const}_x)$$

as claimed. To see the claim, we find that a map $X \to \mathcal{C}^{F/}$ corresponds to a map $X \diamond K \to \mathcal{C}$ whose restriction to K is F. Likewise a map to the pullback of the above diagram corresponds to a map

$$(X \diamond \Delta^0) \times K \amalg_{X \times K} X \to \mathcal{C}$$

restricting also appropriately. We thus again conclude from Lemma 15.7. $\square$

Remark. In fact, the proof shows that there is an isomorphism of simplicial sets

$$\widetilde{\mathrm{Map}}_{\mathcal{C}}(F, x) \cong \mathrm{map}_{\mathrm{Fun}(K,\mathcal{C})}(F, \mathrm{const}_x).$$

Lemma 15.13. *Let $\mathcal{C}$ be an ∞-category. Let $\bar{\mathrm{id}}\colon \mathcal{C} \star \Delta^0 \to \mathcal{C}$ be a cone over the identity of $\mathcal{C}$. Then $\bar{\mathrm{id}}$ is a colimit cone if and only if $\bar{\mathrm{id}}(\infty)$ is a terminal object. In particular, $\mathcal{C}$ has a terminal object if and only if the identity functor has a colimit.*

Proof. Suppose t is a terminal object of $\mathcal{C}$. Consider the composite $\{t\} \to \mathcal{C} \to \mathcal{C}$ where the latter functor is the identity. We obtain an induced functor on slices $\mathcal{C}_{\mathrm{id}/} \to \mathcal{C}_{t/}$, which is a trivial fibration since the inclusion $\{t\} \to \mathcal{C}$ is right anodyne by Lemma 13.4. Since $\mathcal{C}_{t/}$ has an initial object, so does $\mathcal{C}_{\mathrm{id}/}$. Hence by Theorem 15.9, t is a colimit of the identity functor. Conversely, let us assume that the identity has a colimit cone $\bar{\mathrm{id}}$. We will now need to show that $x = \bar{\mathrm{id}}(\infty)$ is a terminal object. We will allude to Lemma 13.2 and consider a lifting problem

$$\begin{tikzcd}
\Delta^{\{n\}} \arrow[r] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\
& \Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

in which the upper composite is given by the object x. We apply the functor $- \star \Delta^0$ to this diagram and obtain

$$\begin{tikzcd}
\Delta^{\{n\}} \arrow[r] \arrow[d] & \Delta^{\{n,n+1\}} \arrow[d] \arrow[rrd, bend left] & & \\
\partial\Delta^n \arrow[r] \arrow[d] & \Lambda^{n+1}_{n+1} \arrow[r] \arrow[d] & \mathcal{C} \star \Delta^0 \arrow[r, "\bar{\mathrm{id}}"] & \mathcal{C} \\
\Delta^n \arrow[r] & \Delta^{n+1} \arrow[rru, dashed] & &
\end{tikzcd}$$

The composite $\Delta^{\{n,n+1\}} \to \mathcal{C}$ is morphism in $\mathcal{C}$ from x to x and we claim that this morphism is an equivalence. Hence a dashed arrow exists by Joyal's lifting theorem Theorem 5.8. We claim that this morphism extends to a morphism in $\mathcal{C}_{\mathrm{id}/}$ from $\bar{\mathrm{id}}$ to itself. Since $\bar{\mathrm{id}}$ is a colimit cone, it is an initial object of $\mathcal{C}_{\mathrm{id}/}$, so that all its endomorphisms are invertible by Proposition 13.3. To see the claim, we consider the following composite

$$\mathcal{C} \star \Delta^1 \cong \mathcal{C} \star \Delta^0 \star \Delta^0 \xrightarrow{\bar{\mathrm{id}} \star \Delta^0} \mathcal{C} \star \Delta^0 \xrightarrow{\bar{\mathrm{id}}} \mathcal{C}$$

which determines a 1-simplex in $\mathcal{C}_{\mathrm{id}}$, whose source and target are $\bar{\mathrm{id}}$. It hence suffices to show that the induced map on cone points is the one considered above. By construction, we thus have to analyze the map

$$\Delta^1 \to \mathcal{C} \star \Delta^1 \to \mathcal{C} \star \Delta^0 \to \mathcal{C}$$

and observe that the map $\Delta^1 \to \mathcal{C} \star \Delta^0$ sends 0 to $\bar{\mathrm{id}}(\infty) = x$ and 1 to the cone point ∞. It is thus given by the map

$$\Delta^1 \cong \Delta^0 \star \Delta^0 \xrightarrow{x \star \mathrm{id}} \mathcal{C} \star \Delta^0.$$

Composing this map with $\bar{\mathrm{id}}$ is precisely the bended map in the above diagram. □

Lecture 16 – 08.07.2019. We will need the following refined version of Proposition 15.10.

Proposition 15.14. *Let $\mathcal{C}$ be an ∞-category and consider a pushout of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & K \end{tikzcd}$$

in which the map i is a monomorphism. Let $F\colon K \to \mathcal{C}$ be a functor, and denote by F_A, F_B, and F_C its restriction to A, B and C. Then the square

$$\begin{tikzcd} \mathcal{C}^{F/} \arrow[r] \arrow[d] & \mathcal{C}^{F_B/} \arrow[d] \\ \mathcal{C}^{F_C/} \arrow[r] & \mathcal{C}^{F_A/} \end{tikzcd}$$

is a cartesian and homotopy cartesian square of left fibrations over $\mathcal{C}$.

Remark. Upon passing to fibres over objects x of $\mathcal{C}$ we obtain the cartesian and homotopy cartesian square of Proposition 15.10.

We now interpret the definitions and properties of colimits as follows. First we observe that the left fibration

$$\mathcal{C}_{F/} \to \mathcal{C}$$

corresponds by straightening-unstraightening to a functor $\mathcal{C} \to \mathrm{An}$. As it takes on an object x in $\mathcal{C}$ the value $\mathrm{Map}_{\mathcal{C}}(F, x)$, we simply denote this functor by $\mathrm{Map}_{\mathcal{C}}(F, -)$.

Definition 15.15. A functor $\mathcal{C} \to \mathrm{An}$ is called representable if it is equivalent to the functor $\mathrm{map}_{\mathcal{C}}(x, -)$ for some x in $\mathcal{C}$. Any such x will be called a representing object. Equivalently, a functor is representable by x, if the associated left fibration $\mathcal{E} \to \mathcal{C}$ is equivalent to the left fibration $\mathcal{C}_{x/} \to \mathcal{C}$.

Proposition 15.16. *Let $F\colon K \to \mathcal{C}$ be a functor. Then F admits a colimit if and only if the functor $\mathrm{Map}_{\mathcal{C}}(F, -)$ is representable, and any representing object is a colimit of F.*

Proof. Suppose F admits a colimit cone $\bar{F}\colon K \star \Delta^0 \to \mathcal{C}$ and let $x = \bar{F}(\infty)$. We have seen previously that this implies that both functors

$$\mathcal{C}_{F/} \leftarrow \mathcal{C}_{\bar{F}/} \to \mathcal{C}_{x/}$$

are equivalences of left fibrations over $\mathcal{C}$: For the right map this is always the case, and the left one is an equivalence if and only if $\bar{F}$ is a colimit cone.

It hence remains to show that if $\mathrm{Map}_{\mathcal{C}}(F,-)$ is representable by an object x, then F admits a colimit. By the assumption we find an equivalence

$$\mathcal{C}_{x/} \simeq \mathcal{C}_{F/}$$

of left fibrations over $\mathcal{C}$. Pick an initial object of $\mathcal{C}_{x/}$ for instance the identity of x. Under the above equivalence this is transported to an initial object of $\mathcal{C}_{F/}$. Any such is a colimit cone, whose colimit point is x by construction. □

Corollary 15.17. *Every diagram $F\colon \Delta^n \to \mathcal{C}$ admits both a limit and a colimit: A limit is given by evaluation on 0, and a colimit is given by evaluation on n.*

Proof. It suffices to recall that these inclusions induce equivalences of left, resp. right fibrations $\mathcal{C}^{F/} \simeq \mathcal{C}^{F(n)/}$ and $\mathcal{C}^{/F} \simeq \mathcal{C}^{/F(0)}$, so that both $\mathcal{C}^{F/}$ and $\mathcal{C}^{/F}$ are representable left fibrations. □

We will need the following technical proposition.

Proposition 15.18. *Let $F\colon K \to \mathcal{C}$ be a functor, and let $i\colon L \to K$ be a map of simplicial sets. Let $\bar{F}$ be a colimit cone of F, and let $\overline{Fi}$ be a colimit cone of Fi. Let $G\colon \Delta^1 \to \mathcal{C}^{Fi/}$ be a map with $G(0) = \overline{Fi}$ and $G(1) = \bar{F}i$. Let g be the composite $\Delta^1 \to \mathcal{C}^{Fi/} \to \mathcal{C}$, and let $g(0) = x$ and $g(1) = y$. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{C}^{F/} \arrow[r] & \mathcal{C}^{Fi/} & \mathcal{C}^{\overline{Fi}/} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathcal{C}^{x/} \\
\mathcal{C}^{\bar{F}/} \arrow[u, "\simeq"] \arrow[r] \arrow[d, "\simeq"] & \mathcal{C}^{\bar{F}i/} \arrow[u] \arrow[d] & (\mathcal{C}^{Fi/})^{G/} \arrow[l, "\simeq"'] \arrow[u] \arrow[d] & \\
\mathcal{C}^{y/} \arrow[r, equal] & \mathcal{C}^{y/} & \mathcal{C}^{g/} \arrow[l, "\simeq"'] \arrow[uur] &
\end{tikzcd}$$

and y is a colimit of F and x is a colimit of Fi. The resulting map

$$\mathcal{C}^{y/} \to \mathcal{C}^{x/}$$

is given by precomposition with the map g which we call the induced map $x = \mathrm{colim}_L Fi \to \mathrm{colim}_K F = y$.

Remark. The obvious dual situation for limiting cocones holds as well.

Observation 15.19. Informally, we summarize the above situation by saying that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^{F/} \arrow[r] \arrow[d, "\simeq"] & \mathcal{C}^{Fi/} \arrow[d, "\simeq"] \\
\mathcal{C}^{y/} \arrow[r, "g"] & \mathcal{C}^{x/}
\end{tikzcd}$$

where the vertical maps are the equivalences coming from the fact that y is a colimit of F and x is a colimit of Fi. Of course, this diagram does not actually commute, but it commutes up to an invertible natural transformation. On the other hand, we could also take this diagram as a definition for the morphism $g\colon x \to y$: Inverting the map $\mathcal{C}^{F/} \to \mathcal{C}^{y/}$, we obtain a functor $\mathcal{C}^{y/} \to \mathcal{C}^{x/}$ making the diagram commute. By the Yoneda lemma, such a functor is equivalently given by an object of $\mathcal{C}^{x/} \times_{\mathcal{C}} \{y\} \simeq \mathrm{map}_{\mathcal{C}}(x,y)$, this is going to be a morphism equivalent to the morphism g of the previous proposition. The advantage of the above approach is that it shows that the morphism g canonically lifts to a morphism between cones over Fi.

Lemma 15.20. *Let I be a discrete category and let $F\colon I \to \mathcal{C}$ be a diagram, i.e. a collection x_i of objects of $\mathcal{C}$. Show that there is a canonical equivalence*

$$\mathrm{Map}_{\mathcal{C}}(F, y) \simeq \prod_{i \in I} \mathrm{map}_{\mathcal{C}}(x_i, y).$$

between functors $\mathcal{C} \to \mathrm{An}$ in y.

Proof. The following diagram is a pushout:

$$\begin{tikzcd} I \times \{0\} \amalg I \times \{1\} \arrow[r] \arrow[d] & \coprod_I \Delta^1 \arrow[d] \\ I \amalg \{\infty\} \arrow[r] & I \star \Delta^0 \end{tikzcd}$$

Thus in the following diagram, both small squares are pullbacks

$$\begin{tikzcd} \mathcal{C}_{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(I \star \Delta^0, \mathcal{C}) \arrow[r] \arrow[d] & \prod_I \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "{(F, \mathrm{id})}"] & \mathrm{Fun}(I, \mathcal{C}) \times \mathcal{C} \arrow[r, "{(s, \Delta)}"] & \prod_I \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

and thus so is the big square. This should imply the lemma.

Since pullbacks commute with arbitrary products, we find that there is an isomorphism

$$\mathcal{C}_{F/} \cong \prod_I \mathcal{C}^{x_i/}$$

of left fibrations. This implies the lemma. $\square$

Corollary 15.21. *Let $F\colon \Lambda^2_0 \to \mathcal{C}$ be a diagram, depicted as*

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"] & y \\ z & \end{tikzcd}$$

and let t be another object. Then there is a homotopy cartesian square of ∞-groupoids

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F, t) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(y, t) \arrow[d] \\ \mathrm{map}_{\mathcal{C}}(z, t) \arrow[r] & \mathrm{map}_{\mathcal{C}}(x, t) \end{tikzcd}$$

which is natural in t.

Proof. By **??** and the fact that Λ^2_0 is the pushout $\Delta^1 \amalg_{\Delta^0} \Delta^1$, we find that the diagram

$$\begin{tikzcd}
\mathcal{C}^{F/} \arrow[r] \arrow[d] & \mathcal{C}^{f/} \arrow[d] \\
\mathcal{C}^{g/} \arrow[r] & \mathcal{C}^{x/}
\end{tikzcd}$$

is cartesian and homotopy cartesian and consists of left fibrations over $\mathcal{C}$. Furthermore, $\mathcal{C}^{f/} \simeq \mathcal{C}^{y/}$ and $\mathcal{C}^{g/} \simeq \mathcal{C}^{z/}$. Passing to fibres over a point t we obtain the statement. $\square$

Proposition 15.22. *Let $\mathcal{C}$ be an ∞-category and K a simplicial set, written as a pushout $B \amalg_A C$, where the map $A \to B$ is a monomorphism. Let $F \colon K \to \mathcal{C}$ be a diagram. Suppose $F_{|B}$ has a colimit y, $F_{|A}$ has a colimit x and $F_{|C}$ has a colimit y. If $\mathcal{C}$ has pushouts, then a pushout $y \amalg_x z$ is a colimit of F.*

Proof. [Lur09, 4.4.2.2]. Let $G \colon \Lambda^2_0 \to \mathcal{C}$ be the diagram given by $y \leftarrow x \to z$, where the maps come from Proposition 15.18. Let $\bar{G} \colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a colimit cone of G. We wish to show that $w = \bar{G}(1,1)$ is a colimit of $F \colon K \to \mathcal{C}$. By Proposition 15.16, we have to show that there is an equivalence of functors between $\mathrm{map}_{\mathcal{C}}(w, -)$ and $\mathrm{Map}_{\mathcal{C}}(F, -)$. Rephrasing in terms of left fibrations, we need to show that there is an equivalence of left fibrations $\mathcal{C}^{w/} \simeq \mathcal{C}^{F/}$. We have that $\mathcal{C}^{w/} \simeq \mathcal{C}^{G/}$ since w is a colimit of G. Furthermore, there is a homotopy cartesian square

$$\begin{tikzcd}
\mathcal{C}^{G/} \arrow[r] \arrow[d] & \mathcal{C}^{y/} \arrow[d] \\
\mathcal{C}^{z/} \arrow[r] & \mathcal{C}^{x/}
\end{tikzcd}$$

Since x, y, and z are themselves colimits, we find that there is commutative diagram, see Observation 15.19

$$\begin{tikzcd}
\mathcal{C}^{F_B/} \arrow[r] \arrow[d, "\simeq"] & \mathcal{C}^{F_A/} \arrow[d, "\simeq"] & \mathcal{C}^{F_C/} \arrow[l] \arrow[d, "\simeq"] \\
\mathcal{C}^{y/} \arrow[r] & \mathcal{C}^{x/} & \mathcal{C}^{z/} \arrow[l]
\end{tikzcd}$$

in which the vertical comparison maps are Joyal equivalences. We thus find that there is an induced equivalence on homotopy pullbacks

$$\mathcal{C}^{G/} \simeq \mathcal{C}^{F/}$$

so that $\mathcal{C}^{F/}$ is indeed representable, with w a representing object. $\square$

Lecture 17 – 10.07.2019.

Proposition 15.23. *Let $F \colon K \to \mathcal{C}$ be a functor, and let $K = \operatorname{colim}_{i \geq 0} K_i$ be an $\mathbb{N}$-indexed decomposition with each map $K_i \to K_{i+1}$ a monomorphism. Let F_i be the restriction of F to K_i. Suppose that for all i the functor F_i admits a colimit and that $\mathcal{C}$ admits colimits over 1-dimensional simplicial sets. Then F admits a colimit.*

Proof. First, we find that there is an isomorphism of left fibrations $\mathcal{C}^{F/} \to \lim_{i \geq 0} \mathcal{C}^{F_i/}$, because for every simplicial set X, the functor $X \diamond -$ preserves connected colimits. By assumption, all $\mathcal{C}^{F_i}$ are corpresentable left fibrations, say $\mathcal{C}^{F_i/} \simeq \mathcal{C}^{x_i/}$. We thus obtain canonical maps $\mathcal{C}^{x_{i-1}/} \to \mathcal{C}^{x_i/}$ making the comparison diagrams commute. By the Yoneda lemma, all of these maps corresponds to morphisms $\alpha_i\colon x_{i-1} \to x_i$, and they assemble into a functor $G\colon I^\infty \to \mathcal{C}$. Since $I^\infty = \operatorname{colim}_n I^n$, we find that $\mathcal{C}^{G/} \to \lim_{i\geq 0} \mathcal{C}^{G_i}$. Since I^n has a terminal object, we find that $\mathcal{C}^{G_i} \simeq \mathcal{C}^{G(i)/} = \mathcal{C}^{x_i/}$. We obtain commutative diagrams

$$\begin{tikzcd}
\mathcal{C}^{F_i/} \arrow[r] \arrow[d] & \mathcal{C}^{x_i/} \arrow[d] & \mathcal{C}^{G_i/} \arrow[l] \arrow[d] \\
\mathcal{C}^{F_{i-1}/} \arrow[r] & \mathcal{C}^{x_{i-1}/} & \mathcal{C}^{G_{i-1}/} \arrow[l]
\end{tikzcd}$$

in which all horizontal maps are Joyal equivalences, and the outer vertical maps are isofibrations. We thus find that the induced map on vertical limits is a Joyal equivalence $\mathcal{C}^{G/} \simeq \mathcal{C}^{F/}$. Since I^∞ is 1-dimensional, we find that $\mathcal{C}$ admits I^∞-indexed colimits. Hence $\mathcal{C}^{G/}$ is corepresentable, and thus so is $\mathcal{C}^{F/}$. Thus F admits a colimit. □

Proposition 15.24. *If an ∞-category $\mathcal{C}$ admits small coproducts and pushouts, then it admits all small colimits.*

Proof. [Lur09, 4.4.2.6]. We first show that $\mathcal{C}$ admits colimits indexed over finite dimensional simplicial sets K by induction over the dimension of K. If K is zero dimensional, it is simply a discrete set, so that colimits over such are coproducts and hence exist by assumption. Now suppose K is n-dimensional, and consider its skeletal pushout

$$\begin{tikzcd}
\coprod_{i\in I} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(K) \arrow[d] \\
\coprod_{i\in I} \Delta^n \arrow[r] & K
\end{tikzcd}$$

and consider a functor $F\colon K \to \mathcal{C}$. By Proposition 15.22 it suffices to argue that each restriction of F to any of the other three corners admits a colimit. For $\mathrm{sk}_{n-1}(K)$ and $\coprod_{i\in I} \partial\Delta^n$ this follows by the induction hypothesis. It remains to show that any functor $\coprod_{i\in I} \Delta^n \to \mathcal{C}$ admits a colimit. By Corollary 15.17 we know that every single functor $\Delta^n \to \mathcal{C}$ admits a colimit. By the same argument we find that the restriction along all terminal objects gives a Joyal equivalence

$$\mathcal{C}^{\coprod_{i\in I} \Delta^n/} \xrightarrow{\simeq} \mathcal{C}^{I/}$$

because the coproduct of right anodyne maps is again right anodyne. It hence suffices to argue that $\mathcal{C}^{I/}$ is equivalent to a representable left fibration, which again follows from the assumption that $\mathcal{C}$ admits coproducts.

We then use Proposition 15.23 to conclude that $\mathcal{C}$ admits K-shaped colimits for all small simplicial sets K by writing K as the colimit over its skeleta. □

In the presence of finite coproducts, having pushouts is in fact equivalent to having coequalizers.

Lemma 15.25. *If an ∞-category $\mathcal{C}$ admits finite coproducts, then it admits pushouts if and only if it admits coequalizers. In particular, $\mathcal{C}$ admits small coproducts and coequalizers, then it admits all small colimits.*

Proof. The coequalizer category is the pushout of the diagram $\Delta^1 \leftarrow \partial\Delta^1 \rightarrow \Delta^1$. Hence by Proposition 15.22 $\mathcal{C}$ admits coequalizers if it admits pushouts and finite coproducts. To see the converse, one argues similarly as in Proposition 15.22: Suppose K' is the coequalizer of two morphisms $L \to K$ of simplicial sets, and suppose $K' \to \mathcal{C}$ is a functor such that the restrictions to K and to L admit a colimit. Then the coequalizer of the colimits is a colimit of the functor $K' \to \mathcal{C}$, see [Lur09, 4.4.3.1] or Exercise 141. Having this, we observe that there is a coequalizer diagram

$$\Delta^0 \rightrightarrows \Delta^1 \amalg \Delta^1 \longrightarrow \Lambda^2_0$$

where the two maps are the two inclusions as vertex 0. We deduce that colimits over Λ^2_0 are given by a coequalizer of two maps between colimits indexed over Δ^0 and $\Delta^1 \amalg \Delta^1$ which exist if $\mathcal{C}$ admits finite coproducts. □

Definition 15.26. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and let $F\colon K \to C$ be a diagram. Suppose that F admits a colimit in $\mathcal{C}$. We say that f preserves this colimit, if for some (and hence any) colimit cone $\bar{F}\colon K \star \Delta^0 \to \mathcal{C}$, the resulting diagram $K \star \Delta^0 \to \mathcal{C} \to \mathcal{D}$ is a colimit cone over fF.

We say that F preserves K-shaped colimits, if for every functor $F\colon K \to \mathcal{C}$ which admits a colimit, F preserves this colimit.

Remark. A functor $f\colon \mathcal{C} \to \mathcal{D}$ thus preserves K-shaped colimits, if for every functor $F\colon K \to \mathcal{C}$, the induced functor $\mathcal{C}^{F/} \to \mathcal{D}^{fF/}$ preserves initial objects.

Proposition 15.27. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F preserves small colimits if and only if it preserves small coproducts and pushouts. The same holds true if one replaces pushouts by coequalizers.*

Proof. Exercise. □

We will collect the following properties of colimits, and leave proofs for later or as exercises.

Proposition 15.28. *If $\mathcal{D}$ is (co)complete and K is a small simplicial set, then* $\mathrm{Fun}(K, \mathcal{D})$ *is again (co)complete and colimits are calculated pointwise, i.e. for every object x of K, the evaluation functor* $\mathrm{Fun}(K, \mathcal{D}) \to \mathcal{D}$ *preserves (co)limits.*

Proposition 15.29. *Let $\mathcal{C}$ be a (co)complete ∞-category and $p\colon K \to \mathcal{C}$ a diagram. Then $\mathcal{C}^{/F}$ admits colimits and the functor $\mathcal{C}^{/F} \to \mathcal{C}$ preserves colimits. Dually, $\mathcal{C}^{F/}$ admits limits and $\mathcal{C}^{F/} \to \mathcal{C}$ preserves limits.*

Proof. [Lur09, 1.2.13.8]. □

Proposition 15.30. *Let $\mathcal{C}$ be a cocomplete ∞-category and let $F\colon K \to \mathcal{C}$ be a diagram. Then $\mathcal{C}^{F/}$ is again cocomplete. Dually, if $\mathcal{C}$ is complete, then $C^{/F}$ is complete. The forgetful functors, however, in general do not preserve these (co)limits.*

Proposition 15.31. *Let $f\colon \mathcal{E} \to \mathcal{C}$ be a left fibration and let K be a weakly contractible simplicial set. Then f preserves K-shaped colimits. Likewise, right fibrations preserve contractible limits.*

Proof. [Lur09, 4.4.2.8 & 4.4.2.9]. □

We will later give an independent proof of the following theorem, making use of the straightening-unstraightening equivalence. The advantage of the following proof is to see that limits and colimits are given as we expect.

Theorem 15.32. *The ∞-categories* Cat_∞ *and* An *admit all small limits and colimits.*

Proof. By Proposition 15.24 it suffices to show that these ∞-categories admit small products, coproducts, pullbacks and pushouts. Coproducts and products are quite easy, as we have seen earlier. We indicate that Cat_∞ admits pullbacks, all other cases are similar in flavour. Consider a diagram $F\colon \Lambda^2_2 \to \mathrm{Cat}_\infty$, given by two functors

$$\mathcal{D} \xrightarrow{p} \mathcal{C} \xleftarrow{f} \mathcal{C}'.$$

Without loss of generality, we may assume that $\mathcal{D} \to \mathcal{C}$ is an isofibration. We let $\mathcal{D}'$ be the pullback of the above diagram of simplicial sets and let $\bar{F}\colon \Delta^1 \times \Delta^1 \to \mathrm{Cat}_\infty$ be the whole pullback diagram. We wish to show that $\mathrm{Cat}_\infty^{/F}$ is a representable right fibration. By the dual argument of Corollary 15.21, we know that there is a cartesian and homotopy cartesian square

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{/F} \arrow[r] \arrow[d] & \mathrm{Cat}_\infty^{/f} \arrow[d] \\
\mathrm{Cat}_\infty^{/p} \arrow[r] & \mathrm{Cat}_\infty^{/\mathcal{C}}
\end{tikzcd}$$

Furthermore, there is a canonical map $\mathrm{Cat}_\infty^{/\mathcal{D}'} \simeq \mathrm{Cat}_\infty^{/\bar{F}} \to \mathrm{Cat}_\infty^{/F}$ and we wish to show that this functor is an equivalence. We will show that it is essentially surjective and fully faithful. An object of $\mathcal{C}^{/F}$ can be represented (up to equivalence) by a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\
\mathcal{C}' \arrow[r, "f"] & \mathcal{C}
\end{tikzcd}$$

by the universal property, $\mathcal{E}$ comes with a unique map to $\mathcal{D}'$ which gives the resulting object of $\mathrm{Cat}_\infty^{/\mathcal{D}'}$. To see fully faithfulness it suffices to show that for any two objects $\mathcal{E}, \mathcal{E}'$ of $\mathrm{Cat}_\infty^{/\mathcal{D}'}$ the following diagram of mapping anima is homotopy cartesian.

$$\begin{tikzcd}
\mathrm{map}_{\mathrm{Cat}_\infty^{/\mathcal{D}'}}(\mathcal{E}, \mathcal{E}') \arrow[r] \arrow[d] & \mathrm{map}_{\mathrm{Cat}_\infty^{/\mathcal{C}'}}(\mathcal{E}, \mathcal{E}') \arrow[d] \\
\mathrm{map}_{\mathrm{Cat}_\infty^{/\mathcal{D}}}(\mathcal{E}, \mathcal{E}') \arrow[r] & \mathrm{map}_{\mathrm{Cat}_\infty^{/\mathcal{C}}}(\mathcal{E}, \mathcal{E}')
\end{tikzcd}$$

This follows from the description of mapping anima in slice-categories, Proposition 12.15, the fact that $\mathcal{D}' \cong \mathcal{D} \times_{\mathcal{C}} \mathcal{C}'$, and that $\mathcal{D} \to \mathcal{C}$ is an isofibration. $\square$

16. Cofinal and coinitial functors

Definition 16.1. Let $f\colon K \to L$ be a map of simplicial sets and $p\colon X \to L$ an inner fibration. As before, we define an ∞-category by the pullback

$$\begin{tikzcd} \mathrm{Fun}_f(K,X) \arrow[r] \arrow[d] & \mathrm{Fun}(K,X) \arrow[d, "p_*"] \\ \Delta^0 \arrow[r, "f"] & \mathrm{Fun}(K,L) \end{tikzcd}$$

If $f = \mathrm{id}\colon L \to L$ we simply write $\mathrm{Fun}_L(L,X)$ instead of $\mathrm{Fun}_{\mathrm{id}}(K,X)$.

Definition 16.2. Let $f\colon K \to L$ be a map of simplicial sets. Then f is called cofinal if for all right fibrations $p\colon X \to Y$, the canonical map

$$\mathrm{Fun}_L(L,X) \to \mathrm{Fun}_f(K,X)$$

induced by f is a Joyal equivalence. Likewise, it is called coinitial, if for all left fibrations $p\colon X \to Y$, the canonical map

$$\mathrm{Fun}_L(L,X) \to \mathrm{Fun}_f(K,X)$$

is a Joyal equivalence.

Remark. By construction, $\mathrm{Fun}_f(K,X)$ is an ∞-groupoid if $p\colon X \to L$ is a right or left fibration, as then its canonical map to Δ^0 is a right or left fibration and hence a Kan fibration. In previous notation, if $p\colon \mathcal{E} \to \mathcal{C}$ is a left fibration between ∞-categories, this was written as $\mathrm{Fun}^{\mathrm{cc}}_f(\mathcal{C}, \mathcal{E}^\natural)$ because for left fibrations, any morphism in $\mathcal{E}$ is p-cocartesian.

Proposition 16.3. *Let $f\colon K \to L$ and $g\colon L \to M$ be maps of simplicial sets.*

(1) *If f is cofinal, then gf is cofinal if and only if g is.*
(2) *If f is cofinal, then f is a weak equivalence.*
(3) *If f is a monomorphism, then f is cofinal if and only if it is right anodyne.*

Proof. We prove (1) first and consider a right fibration $\mathcal{D} \to M$ and the diagram

$$\begin{tikzcd} \mathrm{Fun}_M(M,\mathcal{D}) \arrow[r] \arrow[dr] & \mathrm{Fun}_g(L,\mathcal{D}) \arrow[d] & \mathrm{Fun}_L(L, g^*\mathcal{D}) \arrow[l, "\cong"'] \arrow[d] \\ & \mathrm{Fun}_{gf}(K,\mathcal{D}) & \mathrm{Fun}_f(K, g^*(\mathcal{D})) \arrow[l, "\cong"'] \end{tikzcd}$$

in which the right horizontal maps are isomorphisms, and the right most vertical map is a Joyal equivalence by the assumption that f is cofinal and the fact that pullbacks of right fibrations are right fibrations. We thus conclude the statement by the 3-for-2 property. To show (2) it suffices to prove that for any Kan complex X, the canonical map $\mathrm{Fun}(L,X) \to \mathrm{Fun}(K,X)$ is a homotopy equivalence. Consider the map $L \times X \to L$ which is a Kan fibration, and in particular a right fibration. We find that $\mathrm{Fun}_L(L, L\times X) \cong \mathrm{Fun}(L,X)$ and that $\mathrm{Fun}_f(K, L\times X) \cong \mathrm{Fun}(K,X)$ as needed. One direction of (3) has been done previously: A right anodyne map is cofinal, in fact the restriction map one has to analyze is a trivial fibration: By adjunction it suffices to prove that for any monomorphism $S \to T$, the induced map

$$S \times L \cup T \times K \to T \times L$$

is right anodyne as well. Conversely, suppose that f is a cofinal monomorphism and let $X \to Y$ be a right fibration, and consider a lifting problem to see whether f is right anodyne. By pulling back, we may assume that $Y = L$ and get a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d] \\ L \arrow[r, equal] \arrow[ur, dashed] & L \end{tikzcd}$$

and wish to show the existence of the dashed arrow. By assumption we know that the morphism

$$\mathrm{Fun}_L(L, X) \to \mathrm{Fun}_f(K, X)$$

is a Joyal equivalence, and we wish to show that it is in fact a trivial fibration (as then it is surjective on 0-simplices). It hence suffices to show that it is an isofibration. To see that it is an inner fibration, we again use that the pushout product of an inner anodyne map with a monomorphism is inner anodyne, and that right fibrations are in particular inner fibrations. For the remaining property, we have to show that any diagram

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \mathrm{Fun}_L(L, X) \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}_f(K, X) \end{tikzcd}$$

admits a dashed arrow: This is because the right vertical map is conservative since it is a functor between ∞-groupoids. Now we use that the pushout product of a right anodyne map and a monomorphism is again right anodyne, so that a lift exists since $X \to L$ is a right fibration. $\square$

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Corollary 16.4. *Let $f\colon K \to L$ and $g\colon L \to M$ be maps of simplicial sets.*

(1) *If f is coinitial, then gf is coinitial if and only if g is.*
(2) *If f is a monomorphism, then f is coinitial if and only if f is left anodyne*

Proof. This follows immediately from Exercise 142 $\square$

Corollary 16.5. *Among monomorphisms, the left and right anodyne maps satisfy the right cancellation property: If f and g are composable morphisms, and both f and gf are left, resp. right anodyne, then so is g.*

Next we want to prove an important characterization of cofinal maps, which builds on the following lemma.

Lemma 16.6. *Let $F\colon L \to \mathcal{C}$ be a diagram and x an object of $\mathcal{C}$. Then there is a canonical cartesian (and homotopy cartesian) diagram as follows.*

$$\begin{tikzcd} \widetilde{\mathrm{Map}}_{\mathcal{C}}(F, x) \arrow[r] \arrow[d] & \mathrm{Fun}(L, \mathcal{C}^{/x}) \arrow[d] \\ \Delta^0 \arrow[r, "F"] & \mathrm{Fun}(L, \mathcal{C}) \end{tikzcd}$$

Proof. Again, for the time of the proof, let us call the pullback $\Phi(F)$. We claim that a map from a simplicial set X to $\Phi(F)$ corresponds to a map

$$[(X \times L) \diamond \Delta^0] \amalg_{X \times L} L \to \mathcal{C}$$

whose restriction to Δ^0 is given by x and whose restriction to L is F. On the other hand, a map from X to $\widetilde{\mathrm{Map}}_{\mathcal{C}}(F, x)$ corresponds to a map

$$[X \times (L \diamond \Delta^0)] \amalg_{X \times (L \cup \{\infty\})} L \cup \{\infty\} \to \mathcal{C}$$

whose restriction to $L \cup \{\infty\}$ is the pair (F, x). We claim that there is an isomorphism of simplicial sets

$$[(X \times L) \diamond \Delta^0] \amalg_{X \times L} L \cong [X \times (L \diamond \Delta^0)] \amalg_{X \times (L \cup \{\infty\})} L \cup \{\infty\}.$$

For this we calculate as follows:

$$\begin{aligned}
[(X \times L) \diamond \Delta^0] \amalg_{X \times L} L &\cong \Big[(X \times L \times \Delta^1) \amalg_{X \times L \times \partial\Delta^1} (X \times L) \amalg \Delta^0\Big] \amalg_{X \times L} L \\
&\cong (X \times L \times \Delta^1) \amalg_{X \times L \times \partial\Delta^1} \big((X \times L) \amalg \Delta^0 \amalg_{X \times L} L\big) \\
&\cong (X \times L \times \Delta^1) \amalg_{X \times L \times \partial\Delta^1} L \amalg \Delta^0 \\
&\cong (X \times L \times \Delta^1) \amalg_{X \times L \times \partial\Delta^1} \big(X \times (L \amalg \Delta^0) \amalg_{X \times (L \amalg \Delta^0)} L \amalg \Delta^0\big) \\
&\cong \Big((X \times L \times \Delta^1) \amalg_{X \times L \times \partial\Delta^1} X \times (L \amalg \Delta^0)\big) \amalg_{X \times (L \amalg \Delta^0)} L \amalg \Delta^0 \\
&\cong X \times (L \diamond \Delta^0) \amalg_{X \times (L \amalg \Delta^0)} L \amalg \Delta^0
\end{aligned}$$

which shows the lemma, once we convince ourselves that the inclusions of Δ^0 and L correspond to each other which is a simple matter of checking the maps. □

Theorem 16.7. *Let $f\colon K \to L$ be a map of simplicial sets. Then f is cofinal if and only if for each ∞-category $\mathcal{C}$ and each diagram $p\colon L \to \mathcal{C}$, the induced map $\mathcal{C}^{p/} \to \mathcal{C}^{pf/}$ is a Joyal equivalence.*

Proof. First we assume that f is cofinal and show that then the map $\mathcal{C}^{p/} \to \mathcal{C}^{pf/}$ is a Joyal equivalence. Since this is a map of left fibrations over $\mathcal{C}$ it suffices to show that the induced map on fibres over object x of $\mathcal{C}$ is an equivalence. By Lemma 16.6, this map identifies up to homotopy equivalence with the map $\mathrm{Fun}_p(L, \mathcal{C}^{/x}) \to \mathrm{Fun}_{pf}(K, \mathcal{C}^{/x})$ which is an equivalence because $\mathcal{C}^{/x} \to \mathcal{C}$ is a right fibration and f is cofinal. Conversely, assume that $\mathcal{C}^{p/} \to \mathcal{C}^{pf/}$ is a Joyal equivalence for any diagram $p\colon L \to \mathcal{C}$ and let $X \to L$ be a right fibration. By the straightening-unstraightening equivalence, there is a functor $p\colon L \to \mathrm{An}^{\mathrm{op}}$ whose pullback of the universal right fibration is equivalent to $X \to L$. Now we use that the universal right fibration is given by $(\mathrm{An}_{*/})^{\mathrm{op}} \to \mathrm{An}^{\mathrm{op}}$ which is a representable right fibration since $(\mathrm{An}_{*/})^{\mathrm{op}} \simeq (\mathrm{An}^{\mathrm{op}})_{/*}$. Thus, using Lemma 16.6, we find

$$\mathrm{Fun}_L(L, X) \simeq \mathrm{Fun}_p(L, (\mathrm{An}_{*/})^{\mathrm{op}}) \simeq \mathrm{Map}_{\mathrm{An}^{\mathrm{op}}}(p, *) \simeq (\mathrm{An}^{\mathrm{op}})^{p/} \times_{\mathrm{An}^{\mathrm{op}}} \{*\}$$

and likewise that

$$\mathrm{Fun}_f(K, X) \simeq (\mathrm{An}^{\mathrm{op}})^{pf/} \times_{\mathrm{An}^{\mathrm{op}}} \{*\}.$$

The map we have to investigate is the map induced by the map $(\mathrm{An}^{\mathrm{op}})^{p/} \to (\mathrm{An}^{\mathrm{op}})^{pf/}$ of left fibrations over $\mathrm{An}^{\mathrm{op}}$ by taking the fibre over $* \in \mathrm{An}^{\mathrm{op}}$. By assumption this map is a Joyal equivalence, and thus so is the induced map on fibres. □

Corollary 16.8. *Any Joyal equivalence $f\colon K \to L$ is cofinal.*

Proof. By Theorem 16.7 it suffices to show that for each ∞-category $\mathcal{C}$ and each diagram, the induced map $\mathcal{C}^{p/} \to \mathcal{C}^{pf/}$ is a Joyal equivalence. For this it suffices to show that for each further ∞-category $\mathcal{D}$, the induced map

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}^{p/}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}^{pf/})$$

is an equivalence. By adjunction, these are isomorphic to

$$\mathrm{Fun}_p(\mathcal{D} \diamond L, \mathcal{C}) \to \mathrm{Fun}_{pf}(\mathcal{D} \diamond K, \mathcal{C})$$

which are in turn given by the pullbacks in the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, "p"] \arrow[d, equal] & \mathrm{Fun}(L, \mathcal{C}) \arrow[d] & \mathrm{Fun}(\mathcal{D} \diamond L, \mathcal{C}) \arrow[l] \arrow[d] \\
\Delta^0 \arrow[r, "pf"] & \mathrm{Fun}(K, \mathcal{C}) & \mathrm{Fun}(\mathcal{D} \diamond K, \mathcal{C}) \arrow[l]
\end{tikzcd}$$

in which the right horizontal maps are isofibrations and all vertical maps are Joyal equivalences: For the right hand side this follows from the fact that $\mathcal{D} \diamond K \to \mathcal{D} \diamond L$ is again a Joyal equivalence by Corollary 9.16. Thus the induced map on pullbacks is a Joyal equivalence as well by Lemma 9.6. □

Corollary 16.9. *Let $f\colon K \to L$ be a cofinal map and let $p\colon L \to \mathcal{C}$ be a diagram with $\mathcal{C}$ an ∞-category. Then f admits a colimit if and only if pf admits a colimit, and in either case, f preserves this colimit.*

Lemma 16.10. *Let $f\colon K \to L$ be a map of simplicial sets. Then f is a trivial fibration if and only if it is a cofinal right fibration.*

Proof. The only if follows from the fact that trivial fibrations are right fibrations and Joyal equivalences. To see the converse, we will show that the fibres are contractible and allude to the dual version of Exercise 112. So let $K \to L$ be a cofinal right fibration. We obtain that the map

$$\mathrm{Fun}_L(L, K) \to \mathrm{Fun}_f(K, K)$$

is an equivalence. The right hand side contains the functor $\mathrm{id}_K\colon K \to K$, so there exists an object $\varphi\colon \Delta^0 \to \mathrm{Fun}_L(L, X)$ whose image in $\mathrm{Fun}_f(K, K)$ is equivalent to id_K. Spelling this out, we obtain that $f\varphi = \mathrm{id}_L$ and that there exists a 1-simplex $\Delta^1 \to \mathrm{Fun}_f(K, K)$ connecting id_K to φf. This corresponds to a commutative diagram

$$\begin{tikzcd}
\Delta^1 \times K \arrow[r, "h"] \arrow[d, "\mathrm{pr}"] & K \arrow[d, "f"] \\
K \arrow[r, "f"] & L
\end{tikzcd}$$

One can then restrict the map h to $\Delta^1 \times K_x$ for any 0-simplex x of L. The resulting map is easily seen to give a homotopy between id_{K_x} and φf restricted to K_x. The latter map is constant, since f is constant on the fibres. Thus each fibre K_x is a contractible Kan complex. □

Proposition 16.11. *A map is cofinal if and only if it is a composite of a right anodyne map followed by a trivial fibration.*

Proof. The if case is clear: Both right anodyne maps and trivial fibrations are cofinal, and compositions of cofinal maps are cofinal. Conversely, given a cofinal map $f\colon K \to L$, we may factor it as a right anodyne map $K \to K'$ followed by a right fibration $K' \to L$. Since right anodyne maps are cofinal, we find that the right fibration $K' \to L$ is cofinal by Proposition 16.3 part (1). We then conclude the proposition from Lemma 16.10. $\square$

Definition 16.12. Let $p\colon Y \to X$ be a map of simplicial sets. We call p *smooth*, if for every pullback diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

where i is cofinal, the map j is again cofinal. Dually, it is called proper if for every such pullback diagram where i is coinitial, the map j is again coinitial.

Definition 16.13. A map of simplicial sets $p\colon Y \to X$ is called universally smooth if the pullback along any map $X' \to X$ is smooth. Likewise, it is called universally proper if the pullback along any map is proper.

Remark. A word of warning is in order. In [Lur09] what we call universally smooth is simply called smooth, and likewise for proper and universally proper. The reason to favour universally proper over what we call proper ist that universally proper maps are closed under pullback, whereas proper maps are not closed under pullback, see Exercise 146. In [Cis19, Ngu18] what we call universally smooth is called proper, and what we call universally proper is called smooth. The terminology in fact breaks a little earlier: What we call cofinal is called final in loc. cit, and what we call coinitial is called cofinal. What we call smooth or proper does not have a separate name in loc. cit.

For the following proposition, we follow the proof given in [Ngu18, 2.3.23] and [Cis19].

Proposition 16.14. *Consider a pullback diagram*

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

where p is a left fibration and i is right anodyne. Then the map j is again right anodyne.

Proof. The first step is to see that it suffices to prove the claim for i contained in a generating set of right anodyne maps. To see this, we claim that the set S of right anodyne maps satisfying the conclusion of the lemma is saturated, see Exercise 144. For compositions and retracts this is easy, one needs to work a little harder to see that these maps are closed under pushouts. It hence suffices to show that the maps $\{1\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n \to \Delta^1 \times \Delta^n$ are contained in this set by Corollary 3.31. We observe also that the set S has the right cancellation property, since right anodyne maps have this property, see again Exercise 144. Thus it suffices to show that for any simplicial set K, the map

$$\{1\} \times K \to \Delta^1 \times K$$

is contained in S: Consider the diagram

$$\begin{tikzcd}
\{1\} \times \partial\Delta^n \arrow[r] \arrow[d] & \Delta^1 \times \partial\Delta^n \arrow[d] \arrow[dr, bend left] & \\
\{1\} \times \Delta^n \arrow[r] \arrow[rr, bend right] & \{1\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n \arrow[r] & \Delta^1 \times \Delta^n
\end{tikzcd}$$

Using the claim, we find that the top horizontal map is contained in S, hence as S is closed under pushouts, so is the lower horizontal map. Also, the the lower bended map is contained in S. Since S satisfies the right cancellation property, also the diagonal map is contained in S.

Now we observe that the maps $\{1\} \times K \to \Delta^1 \times K$ are particular instances of right anodyne extensions: They are right deformation retracts. Such are closed under pullbacks along left fibrations, as we show in Exercise 145. ☐

Remark. By applying the opposite functor, we find that if p is a right fibration and i is left anodyne, then j is again left anodyne.

Remark. The conclusion of Proposition 16.14 holds more generally for cocartesian fibrations $p\colon Y \to X$, see [Lur09, Proposition 4.1.2.15].

Corollary 16.15. *Left fibrations are universally smooth. In fact, cocartesian fibrations are universally smooth. Right fibrations, in fact cartesian fibrations, are universally proper.*

Proof. Since left and right fibrations are closed under pullbacks, it suffices to show that a left fibration is smooth. Consider a diagram as in the definition of smooth maps. Factor the map i as

$$A \xrightarrow{i'} A' \xrightarrow{p} X$$

with i' right anodyne and p a trivial fibration and consider the enlarged diagram

$$\begin{tikzcd}
B \arrow[r, "j'"] \arrow[d] & B' \arrow[r, "q"] \arrow[d] & Y \arrow[d] \\
A \arrow[r] & A' \arrow[r] & X
\end{tikzcd}$$

We find that q is a trivial fibration and that j' is right anodyne by Proposition 16.14. Thus j is cofinal as a composition of cofinal maps. ☐

Using that left fibrations are smooth and right fibrations are proper, we obtain a nice proof of an ∞-categorical version of Quillen's Theorem A. This is yet another characterization of cofinality for maps whose target is an ∞-category.

Theorem 16.16. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map of simplicial sets with $\mathcal{D}$ an ∞-category. Then f is cofinal if and only if for all objects d of $\mathcal{D}$, the simplicial set $\mathcal{C}_{d/}$ is weakly contractible.*

Proof. First assume that f is cofinal. In the pullback diagram

$$\begin{tikzcd} \mathcal{C}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

the right vertical map is a left fibration, and thus smooth. It follows that $\mathcal{C}_{d/} \to \mathcal{D}_{d/}$ is cofinal and thus a weak equivalence. Since $\mathcal{D}_{d/}$ is weakly contractible (it has an initial object) so is $\mathcal{C}_{d/}$. Conversely, we consider a factorization of f as

$$\mathcal{C} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{D}$$

where i is right anodyne and p is a right fibration. We aim to show that p is a trivial fibration, so that f is cofinal by Proposition 16.11. Consider the diagram

$$\begin{tikzcd} \mathcal{C}_{d/} \arrow[r] \arrow[d] & \mathcal{E}_{d/} \arrow[r] \arrow[d] & \mathcal{D}_{d/} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{D} \end{tikzcd}$$

In which all squares are pullbacks. Since the very right vertical map is a left fibration, so is the middle vertical map. By Proposition 16.14, the map $\mathcal{C}_{d/} \to \mathcal{E}_{d/}$ is again right anodyne and hence a weak equivalence, and hence $\mathcal{E}_{d/}$ is weakly contractible.

Now consider the diagram

$$\begin{tikzcd} \mathcal{E}_{d} \arrow[r] \arrow[d] & \mathcal{E}_{d/} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \Delta^0 \arrow[r, "\mathrm{id}_d"] & \mathcal{D}_{d/} \arrow[r] & \mathcal{D} \end{tikzcd}$$

Again, all squares are pullbacks. This time the right most vertical map is a right fibration, hence so is the middle vertical map. Furthermore, the map $\Delta^0 \to \mathcal{D}_{d/}$ is left anodyne by Lemma 13.4 as id_d is an initial object of $\mathcal{D}_{d/}$, see Exercise 133. Hence, by the dual version of Proposition 16.14, the map $\mathcal{E}_d \to \mathcal{E}_{d/}$ is left anodyne and thus a weak equivalence. Since $\mathcal{E}_{d/}$ is weakly contractible by the first step and $\mathcal{E}_d$ is an ∞-groupoid, this means that the fibres $\mathcal{E}_d$ of the right fibration $p\colon \mathcal{E} \to \mathcal{D}$ are contractible. Hence p is a trivial fibration, for instance by Exercise 112. ☐

Here is the actual statement Quillen proved:

Corollary 16.17. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. If all slices $\mathcal{C}_{d/}$ are weakly contractible, then the functor* $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ *is a weak equivalence.*

Proof. By Theorem 16.16, the functor $\mathrm{N}(F)\colon \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ is cofinal, and hence a weak equivalence by Proposition 16.3. ☐

17. Adjunctions

Definition 17.1. An adjunction is a bicartesian fibration $\mathcal{E} \to \Delta^1$, i.e. a functor which is both a cartesian and a cocartesian fibration.

Definition 17.2. Given an adjunction, we can use the straightening theorem to obtain a functor $f\colon \mathcal{E}_0 \to \mathcal{E}_1$, classified by the cocartesian fibration, and also to obtain a functor $g\colon \mathcal{E}_1 \to \mathcal{E}_0$, classified by the underlying cartesian fibration of the adjunction. We refer to f as the left adjoint and to g as the right adjoint of the adjunction.

Remark. We say that a functor $f\colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint, if there exists an adjunction $\mathcal{E} \to \Delta^1$, whose associated functor is equivalent to f, i.e. where one specifies equivalences $\mathcal{C} \simeq \mathcal{E}_0$ and $\mathcal{D} \simeq \mathcal{E}_1$, such that the composite $\mathcal{C} \simeq \mathcal{E}_0 \to \mathcal{E}_1 \simeq \mathcal{D}$ is equivalent to f. In general, an adjunction between two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ hence consists of a bicartesian fibration $\mathcal{E} \to \Delta^1$ together with specified equivalences $\mathcal{C} \simeq \mathcal{E}_0$ and $\mathcal{D} \simeq \mathcal{E}_1$.

Remark. We directly want to show that an adjunction $\mathcal{E} \to \Delta^1$ gives rise to an equivalence of anima

$$\mathrm{map}_{\mathcal{E}_1}(f(x), z) \simeq \mathrm{map}_{\mathcal{E}_0}(x, g(z))$$

if x is an object of $\mathcal{E}_0$ and z is an object of $\mathcal{E}_1$. For this we consider the anima $\mathrm{map}_{\mathcal{E}}(x,z)$. Choosing a cartesian lift of the unique map $0 \to 1$ with target z, Corollary 10.12 shows that there is a fibre sequence

$$\mathrm{map}_{\mathcal{E}_0}(x, g(z)) \to \mathrm{map}_{\mathcal{E}}(x,z) \to \mathrm{map}_{\Delta^1}(0,1) \simeq *$$

so that the first map is a homotopy equivalence. Likewise, choosing a cocartesian lift with domain x, we obtain a fibre sequence

$$\mathrm{map}_{\mathcal{E}_1}(f(x), z) \to \mathrm{map}_{\mathcal{E}}(x,z) \to \mathrm{map}_{\Delta^1}(0,1) \simeq *$$

so we find the desired equivalence as the zig-zag

$$\mathrm{map}_{\mathcal{E}_1}(f(x), z) \xrightarrow{\simeq} \mathrm{map}_{\mathcal{E}}(x,z) \xleftarrow{\simeq} \mathrm{map}_{\mathcal{E}_0}(x, g(z)).$$

We now want to promote this to a natural equivalence of functors $\mathcal{E}_0^{\mathrm{op}} \times \mathcal{E}_1 \to \mathrm{An}$.

Proposition 17.3. *Let $\mathcal{E} \to \Delta^1$ be an adjunction, and $f\colon \mathcal{E}_0 \to \mathcal{E}_1$ and $g\colon \mathcal{E}_1 \to \mathcal{E}_0$ the associated functors. Then there is a natural equivalence of functors*

$$\mathrm{map}_{\mathcal{E}_0}(-, g(-)) \simeq \mathrm{map}_{\mathcal{E}_1}(f(-), -).$$

Proof. We claim that both functors are equivalent to the composite

$$\mathcal{E}_0^{\mathrm{op}} \times \mathcal{E}_1 \to \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathrm{An}$$

where the latter is the bivariant mapping anima functor for the ∞-category $\mathcal{E}$. We notice that there is a natural transformation of functors $\tau_g\colon i_0 \circ g \to i_1$ and $\tau_f\colon i_0 \to i_1 \circ f$ which picks out the required (co)cartesian maps: One considers the diagrams

$$\begin{tikzcd} \mathcal{E}_0 \times \{0\} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}_0 \times \Delta^1 \arrow[r] \arrow[ur, dashed, "\tau_f"] & \Delta^1 \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}_1 \times \{1\} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}_1 \times \Delta^1 \arrow[r] \arrow[ur, dashed, "\tau_g"] & \Delta^1 \end{tikzcd}$$

where the dashed arrows exist because $\mathcal{E}^{\mathcal{E}_i} \to \Delta^{\mathcal{E}_i}$ is again bicartesian so that we find such lifts as desired. We then consider the composite

$$\mathcal{E}_0^{\mathrm{op}} \times \mathcal{E}_1 \times \Delta^1 \to \mathcal{E}_0^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathrm{An}$$

which is a natural transformation from $\mathrm{map}_{\mathcal{E}}(-, g(-))$ to $\mathrm{map}_{\mathcal{E}}(-, -)$ which is pointwise an equivalence, since τ_g is pointwise a cartesian edge. Using that $\mathcal{E}_0 \to \mathcal{E}$ is fully faithful (Δ^1 has trivial anima of self-maps) and natural transformations which are pointwise equivalences are themselves equivalences, Corollary 6.2, we conclude that τ_g induces a natural equivalence

$$\mathrm{map}_{\mathcal{E}_0}(-, g(-)) \simeq \mathrm{map}_{\mathcal{E}}(-, -) \colon \mathcal{E}_0^{\mathrm{op}} \times \mathcal{E}_1 \to \mathrm{An}.$$

Likewise, τ_f induces a natural equivalence

$$\mathrm{map}_{\mathcal{E}_1}(f(-), -) \simeq \mathrm{map}_{\mathcal{E}}(-, -) \colon \mathcal{E}_0^{\mathrm{op}} \times \mathcal{E}_1 \to \mathrm{An}$$

which shows the proposition. □

Next we wish to define unit and counit transformations associated to an adjunction. As in the proof of Proposition 17.3 we consider the transformations τ_f and τ_g and observe that the two maps

$$\begin{tikzcd} \mathcal{E}_1 \times \Delta^1 \arrow[r, "\tau_g"] \arrow[d, "g\times\mathrm{id}"'] & \mathcal{E} \\ \mathcal{E}_0 \times \Delta^1 \arrow[ur, "\tau_f"'] & \end{tikzcd}$$

agree when restricted to $\mathcal{E}_1 \times \{0\}$: By definition, the horizontal map is the functor $i_0 \circ g$, whereas the composite is the composite of g with the inclusion i_0.

Hence these two maps combine to a map $\mathcal{E}_1 \times \Lambda^2_0 \to \mathcal{E}$ such that the restriction to $\mathcal{E}_1 \times \Delta^{\{0,1\}}$ is given by $\tau_f \circ (g \times \mathrm{id})$, and the restriction to $\mathcal{E}_1 \times \Delta^{\{0,2\}}$ is given by τ_g. This gives the top horizontal map in the diagram

$$\begin{tikzcd} \mathcal{E}_1 \times \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}_1 \times \Delta^2 \arrow[r] \arrow[ur, dashed] & \Delta^1 \end{tikzcd}$$

and the lower horizontal map is given by the composite $\mathcal{E}_1 \times \Delta^2 \to \Delta^2 \to \Delta^1$ in which the latter map sends 0 to 0 and both 1 and 2 to 1. To see that the diagram commutes, it suffices to recall that both $p \circ \tau_f$ and $p \circ \tau_g$ are the projections. We now find that this lifting problem can be solved, since the composite

$$\mathcal{E}_1 \times \Delta^{\{0,1\}} \to \mathcal{E}_1 \times \Lambda^2_0 \to \mathcal{E}$$

is pointwise cocartesian. For an object z of $\mathcal{E}_1$, the resulting 2-simplex of $\mathcal{E}$ is given by

$$\begin{tikzcd} & f(g(z)) \arrow[dr, dashed] & \\ g(z) \arrow[ur, "\mathrm{cocart}"] \arrow[rr, "\mathrm{cart}"] & & z \end{tikzcd}$$

Finally, we see that the restriction $\mathcal{E}_1 \times \Delta^{\{1,2\}} \to \mathcal{E}_1 \times \Delta^2 \to \mathcal{E}$ factors through the inclusion $\mathcal{E}_1 \to \mathcal{E}$ by construction.

A similar constructions provides a functor $\mathcal{E}_0 \times \Delta^2 \to \mathcal{E}_0$ which can be depicted as follows

$$\begin{tikzcd} & g(f(x)) \arrow[dr, "\mathrm{cart}"] & \\ x \arrow[ur, dashed] \arrow[rr, "\mathrm{cocart}"] & & f(x) \end{tikzcd}$$

Definition 17.4. We refer to the resulting functor $\varepsilon\colon \mathcal{E}_1 \times \Delta^1 \to \mathcal{E}_1$ as the counit of the adjunction. Dually, we refer to the resulting functor $\eta\colon \mathcal{E}_0 \times \Delta^1 \to \mathcal{E}_0$ as the unit of the adjunction.

Remark. We notice that in an adjunction $\mathcal{E} \to \Delta^1$, the unique morphism $0 \to 1$ has both a cartesian and a cocartesian lift. However, in general a cartesian lift need not be cocartesian and vice versa. In fact, this can be controlled very nicely.

Proposition 17.5. *Let $p\colon \mathcal{E} \to \Delta^1$ be an adjunction, with left adjoint $f\colon \mathcal{E}_0 \to \mathcal{E}_1$ and right adjoint $g\colon \mathcal{E}_1 \to \mathcal{E}_0$. Then p-cartesian edges are p-cocartesian if and only if g is fully faithful. Conversely, p-cocartesian edges are p-cartesian if and only if f is fully faithful.*

In particular, f and g are mutually inverse equivalences if and only if the set of p-cartesian edges equals the set of p-cocartesian edges.

Proof. We first show that cartesian edges are cocartesian if and only if the counit is an equivalence. Consider the 2-simplex as above

$$\begin{tikzcd} & f(g(z)) \arrow[dr, dashed] & \\ g(z) \arrow[ur, "\text{cocart}"] \arrow[rr, "\text{cart}"] & & z \end{tikzcd}$$

and assume that cartesian edges are cocartesian. By the dual version of Lemma 10.5 we find that the counit map $f(g(z)) \to z$ also cocartesian and thus is an equivalence for every z because its image is invertible in Δ^1. Hence the counit is a natural equivalence. Conversely, if the counit is an equivalence, then it is also cocartesian so that the cartesian edge $g(z) \to z$ is also cocartesian as a composition of such.

Now we show that g is fully faithful if and only if the counit is an equivalence. By construction, we have that the diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{E}_1}(z,w) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{E}_0}(g(z),g(w)) \arrow[d] \\ \mathrm{map}_{\mathcal{E}_1}(f(g(z)),w) \arrow[r] & \mathrm{map}_{\mathcal{E}}(g(z),w) \end{tikzcd}$$

commutes. The lower horizontal map and the right vertical map are equivalences because they are induced by post composition with a cartesian edge, respectively with pre composition with a cocartesian edge. Hence g is fully faithful if and only if the counit is an equivalence.

The argument for the unit is similar and the in particular follows since f and g are mutually inverse if and only if both are fully faithful. □

Definition 17.6. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then f is said to admit a right adjoint if the cocartesian fibration $\mathcal{E} \to \Delta^1$ associated to f is cartesian. Conversely, $g\colon \mathcal{D} \to \mathcal{C}$ is said to admit a left adjoint if the cartesian fibration associated to g is cocartesian.

The following is a useful way of constructing functors:

Proposition 17.7. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Specify for each object x of $\mathcal{D}$ an object gx of $\mathcal{C}$ and maps $f(gx) \to x$ in $\mathcal{D}$. If the induced composite*

$$\mathrm{map}_{\mathcal{C}}(z,gx) \xrightarrow{f} \mathrm{map}_{\mathcal{D}}(f(z),f(gx)) \xrightarrow{\varepsilon} \mathrm{map}_{\mathcal{D}}(f(z),x)$$

is an equivalence, then there exists a functor $g\colon \mathcal{D} \to \mathcal{C}$ sending x to gx which is right adjoint to f. Furthermore, the counit of the adjunction is then equivalent to the chosen map $f(gx) \to x$.

Proof. Let $p\colon \mathcal{E} \to \Delta^1$ be the cocartesian fibration associated to the functor f. We aim to show that p is cartesian. In other words, we must specify for each object x of $\mathcal{E}_1 \simeq \mathcal{D}$ a p-cartesian morphism over the unique non-identity morphism of Δ^1. We consider the object gx of $\mathcal{E}_0 \simeq \mathcal{C}$ and choose a p-cocartesian morphism $gx \to f(gx)$. Composing this with the specified morphism $f(gx) \to x$ we obtain a map $gx \to x$ over the unique non-identity morphism of Δ^1, and we wish to show that this map is p-cartesian, as then the first part of the proposition follows. We recall that a morphism $\alpha\colon u \to v$ in $\mathcal{E}$ over $0 \to 1$ is p-cartesian if and only if the map

$$\mathrm{map}_{\mathcal{E}}(w,u) \xrightarrow{\alpha_*} \mathrm{map}_{\mathcal{E}}(w,v)$$

is a homotopy equivalence for all $w \in \mathcal{E}_0$ (Exercise). In other words, we must show that the composite

$$\mathrm{map}_{\mathcal{E}}(z,gx) \to \mathrm{map}_{\mathcal{E}}(z,f(gx)) \to \mathrm{map}_{\mathcal{E}}(z,x)$$

is an equivalence for all $z \in \mathcal{E}_0$. For this we choose a p-cocartesian edge $z \to f(z)$ and consider the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{E}}(z,gx) \arrow[r] \arrow[d, equal] & \mathrm{map}_{\mathcal{E}}(z,f(gx)) \arrow[r] & \mathrm{map}_{\mathcal{E}}(z,x) \\
\mathrm{map}_{\mathcal{E}}(z,gx) \arrow[r, "f"] & \mathrm{map}_{\mathcal{E}}(f(z),f(gx)) \arrow[r] \arrow[u, "\simeq"] & \mathrm{map}_{\mathcal{E}}(f(z),x) \arrow[u, "\simeq"]
\end{tikzcd}$$

in which the middle and right vertical maps are equivalences since $z \to f(z)$ is p-cocartesian and x and $f(gx)$ are objects of $\mathcal{E}_1$. The lower composite is an equivalence by assumption, and hence so is the upper composite. We conclude that the above constructed map $gx \to x$ is p-cartesian.

It remains to prove the claim about the counit of the adjunction, but this follows from the construction: We need to see that there is a 2-simplex in $\mathcal{E}$

$$\begin{tikzcd}
& f(gx) \arrow[dr, "\varepsilon"] & \\
gx \arrow[ur, "\text{cocart}"] \arrow[rr, "\text{cart}"] & & x
\end{tikzcd}$$

and we have just verified that any composite is a cartesian edge $gx \to x$. $\square$

Remark. Likewise, specifying gx for each object x of $\mathcal{D}$ and maps $x \to f(gx)$ such that the composite

$$\mathrm{map}_{\mathcal{C}}(gx,z) \to \mathrm{map}_{\mathcal{D}}(f(gx),f(z)) \to \mathrm{map}_{\mathcal{D}}(x,f(z))$$

is an equivalence for all $z \in \mathcal{C}$ gives a functor g which is left adjoin to f and the specified map being equivalent to the unit of the adjunction.

Proposition 17.8. *The association $\mathcal{C} \mapsto \mathcal{C}[W^{-1}]$ where W consists of all morphisms of $\mathcal{C}$ extends to a left adjoint of the inclusion $\mathrm{An} \to \mathrm{Cat}_\infty$.*

Proof. By Proposition 17.7 and its variant for the existence of left adjoints it suffices to specify for each ∞-category $\mathcal{C}$ the ∞-groupoid $\mathcal{C}[W^{-1}]$, together with the map $\mathcal{C} \to \mathcal{C}[W^{-1}]$ and show that the for each ∞-groupoid X, the composite

$$\mathrm{map}_{\mathrm{An}}(\mathcal{C}[W^{-1}],X) \to \mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{C}[W^{-1}],X) \to \mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{C},X)$$

is an equivalence. As we have already seen, the functor $\mathrm{An} \to \mathrm{Cat}_\infty$ is fully faithful, so the first map is an equivalence. Then we recall that for arbitrary ∞-categories $\mathcal{C}$ and $\mathcal{D}$ we have

$$\mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}.$$

However, if $\mathcal{D}$ is an ∞-groupoid, so is $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ so that we it suffices to show that the map

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], X) \simeq \mathrm{Fun}(\mathcal{C}, X)$$

is an equivalence for all ∞-groupoids. The universal property of the localization shows that the former is canonically equivalent to $\mathrm{Fun}^{\simeq}(\mathcal{C}, X)$, the full subcategory of $\mathrm{Fun}(\mathcal{C}, X)$ on functors sending all morphisms to equivalences. Since every morphism in X is an equivalence, the inclusion $\mathrm{Fun}^{\simeq}(\mathcal{C}, X) \to \mathrm{Fun}(\mathcal{C}, X)$ is an equivalence, so the proposition is shown. □

The next proposition makes sure that if we already have a candidate right adjoint functor, then it is really a right adjoint.

Proposition 17.9. *Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ be functors and let $\varepsilon\colon fg \to \mathrm{id}$ be a natural transformation such that the induced map*

$$\mathrm{map}(x, g(y)) \to \mathrm{map}(f(x), f(g(y))) \to \mathrm{map}(f(x), y)$$

is an equivalence for all x and y. Then g is right adjoint to f.

Proof. By Proposition 17.7 there exists a functor g' which is right adjoint to f, which is pointwise equivalent to g and such that the counit map is equivalent to the chosen map. We now need to show that g' is equivalent to g. We first construct a natural transformation $g \to g'$ as follows: We recall that the functor $\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C}))$, given by postcomposition with the Yoneda functor, is fully faithful. It hence suffices to construct an equivalence between the images of g and g'. These images are given by the two functors

$$d \mapsto \begin{cases} \mathrm{map}_{\mathcal{D}}(f(-), d) & \text{for } g' \\ \mathrm{map}_{\mathcal{C}}(-, g(d)) & \text{for } g \end{cases}$$

We find that these two functors are equivalent by Exercise 1, and hence deduce that g and g' are also equivalent. □

Exercise 1. Let $f\colon \mathcal{C} \to \mathcal{D}$ and let $g\colon \mathcal{D} \to \mathcal{C}$ be functors. Suppose that $\varepsilon\colon fg \to \mathrm{id}$ is a natural transformation. Show that the map

$$\mathrm{map}(x, g(y)) \to \mathrm{map}(f(x), f(g(y)) \to \mathrm{map}(f(x), y)$$

is natural in x.

Corollary 17.10. *Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ be functors. Then g is right adjoint to f if and only if there exist unit and counit transformations that satisfy the snake identities.*

Proof. The fact that the snake identities are satisfied for an adjunction is left as an exercise. The converse follows from the previous proposition, as satisfying the snake identities implies that the canonical map

$$\mathrm{map}(x, g(y)) \to \mathrm{map}(f(x), f(g(y))) \to \mathrm{map}(f(x), y)$$

is an equivalence: An inverse is given by the composite

$$\mathrm{map}(fx, y) \to \mathrm{map}(gfx, gy) \to \mathrm{map}(x, gy).$$

□

Having this we can prove the following result about the compatibility of adjunctions with Dwyer–Kan localizations and functor categories.

Proposition 17.11. *Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$. Suppose $\mathcal{C}$ is equipped with a set S of morphisms and $\mathcal{D}$ is equipped with a set T of morphisms. If $f(S) \subseteq T$ and $g(T) \subseteq S$, then there are induced functors $F\colon \mathcal{C}[S^{-1}] \to \mathcal{D}[T^{-1}]$ and $G\colon \mathcal{D}[T^{-1}] \to \mathcal{C}[S^{-1}]$. If furthermore f is left adjoint to g, then F is left adjoint to G.*

Proof. By Corollary 17.10 it suffices to construct transformations $FG \to \mathrm{id}$ and $\mathrm{id} \to GF$ satsifying the snake identities. We construct first the map $\mathrm{id} \to GF$, which is a 1-simplex of $\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{C}[S^{-1}])$ from id to the composition GF. We know that the restriction functor

$$\ell^*\colon \mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{C}[S^{-1}]) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}[S^{-1}])$$

is fully faithful, so it suffices to construct the desired 1-simplex in the latter category, namely from ℓ to $GF \circ \ell$. There is also a functor

$$\mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}[S^{-1}])$$

given by postcomposition with the localization map $\ell\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$. The unit of the adjunction η is a 1-simplex from id to gf in the former category, hence this functor gives rise to a 1-simplex in $\mathrm{Fun}(\mathcal{C}, \mathcal{C}[S^{-1}])$ from ℓ to $\ell \circ gf$. By definition of G and F, there is an equivalence $\ell \circ gf \simeq GF \circ \ell$. Hence we find a transformation from ℓ to $GF \circ \ell$ as needed. Likewise, one obtains the counit transformation $FG \to \mathrm{id}$. To see that the snake identities are fulfilled, we consider the 2-simplex whitnessing the snake identity for f in the diagram

$$\Delta^2 \to \mathrm{map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(f,f) \to \mathrm{map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D}[T^{-1}])}(\ell_{\mathcal{D}} \circ f, \ell_{\mathcal{D}} \circ f) \xleftarrow{\simeq} \mathrm{map}_{\mathrm{Fun}(\mathcal{C}[S^{-1}],\mathcal{D}[T^{-1}])}(F,F)$$

Then we observe that the resulting 2-simplex of $\mathrm{map}(F,F)$ whitnesses the sake identity for F. The argument for G is similar. $\square$

Proposition 17.12. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor and let K be a simplicial set and $\mathcal{E}$ an auxiliary ∞-category. If f admits a right, resp. a left adjoint, then so does $f_*\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{D})$. If f admits a right, resp. a left adjoint, then $f^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ admits a left, resp. a right adjoint.*

Proof. Let g be a right adjoint of f. We prove that g_* is right adjoint to f_* and that g^* is left adjoint to f^*. The other cases are similar. Let $\varepsilon\colon fg \to \mathrm{id}$ be the counit and $\eta\colon \mathrm{id} \to gf$ be the unit of the adjunction, viewed as morphisms

$$\Delta^1 \xrightarrow{\varepsilon} \mathrm{Fun}(\mathcal{D}, \mathcal{D}) \qquad \Delta^1 \xrightarrow{\eta} \mathrm{Fun}(\mathcal{C}, \mathcal{C})$$

we can then postcompose with the canonical functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^K, \mathcal{D}^K)$ and obtain new transformations $\varepsilon_*\colon f_*g_* \to \mathrm{id}_*$ and $\eta_*\colon \mathrm{id}_* \to g_*f_*$. These are easily checked to satisfy the snake identities, and so form an adjunction. Likewise, one can compose with the functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}^{\mathcal{D}}, \mathcal{E}^{\mathcal{C}})$ and obtain transformations $\varepsilon^*\colon (fg)^* = g^*f^* \to \mathrm{id}^*$ and $\eta^*\colon \mathrm{id}^* \to (gf)^* = f^*g^*$. Again, these satisfy the snake identities, and thus g^* is left adjoint to f^*. $\square$

Corollary 17.13. *The functor $\mathrm{Cat}_\infty \to \mathrm{An}$ given by taking the maximal sub-∞-groupoid is a right adjoint of the inclusion $\mathrm{An} \to \mathrm{Cat}_\infty$.*

Proof. The inclusion functor $\mathrm{Kan} \to \mathrm{Cat}^1_\infty$ has a right adjoint given by $\mathrm{Cat}^1_\infty \to \mathrm{Kan}$ sending $\mathcal{C}$ to $\mathcal{C}^{\simeq}$. The inclusion sends homotopy equivalences to Joyal equivalences, and the maximal subgroupoid functor sends Joyal equivalences to homotopy equivalences. Hence we may apply Proposition 17.11. □

Remark. One can of course also prove Corollary 17.13 using Proposition 17.9. Then one has to show that for an ∞-groupoid X and an ∞-category $\mathcal{C}$, the canonical map

$$\mathrm{Fun}(X, \mathcal{C}^{\simeq}) \to \mathrm{Fun}(X, \mathcal{C})^{\simeq}$$

is a homotopy equivalence.

Proposition 17.14. *Let $\ell\colon \mathcal{C} \to \mathcal{D}$ be a Dwyer-Kan localization. Suppose that ℓ admits a right adjoint r. Then r is fully faithful.*

Proof. We claim that the there is a commutative diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "r"] \arrow[d, "y_{\mathcal{D}}"] & \mathcal{C} \arrow[d, "y_{\mathcal{C}}"] \\ \mathcal{P}(\mathcal{D}) \arrow[r, "\ell^*"] & \mathcal{P}(\mathcal{C}) \end{tikzcd}$$

for which we have to see that there is a natural equivalence between the functors

$$d \mapsto \begin{cases} c \mapsto \mathrm{map}_{\mathcal{C}}(c, r(d)) \\ c \mapsto \mathrm{map}_{\mathcal{D}}(\ell(c), d) \end{cases}$$

which is a consequence of the fact that r is right adjoint to ℓ. Since ℓ is a localization, the functor ℓ^* is fully faithful. The proposition thus follows from the fact that the Yoneda functors are fully faithful. □

Definition 17.15. A Dwyer-Kan localization which admits a right adjoint is called a Bousfield localization.

Corollary 17.16. *Let $\mathcal{C}$ be a locally small ∞-category and let $\mathcal{C} \to \mathcal{D}$ be a Bousfield localization. Then $\mathcal{D}$ is locally small.*

Proof. By Proposition 17.14 $\mathcal{D}$ identifies with a full subcategory of $\mathcal{C}$ and is thus also locally small. □

Proposition 17.17. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be any functor which has a fully faithful right adjoint r. Then f is a Bousfield localization.*

Proof. By Proposition 17.14 it suffices to show that f is a Dwyer-Kan localization. Let $\mathcal{E}$ be an auxiliary ∞-category. We need to show that the functor $f^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is fully faithful, and characterize the essental image. To see that f^* is fully faithful, we observe that r^* is a left adjoint to f^* by **??**. Furthermore, by assumption $fr \to \mathrm{id}$ is an equivalence. From the construction, we find that also $r^*f^* \to \mathrm{id}$ is an equivalence, so that f^* is fully faithful. It remains to show that f is a Dwyer–Kan localization. If this is the case, then it must be a Dwyer–Kan localization along the set of morphisms S which are those morphisms which become equivalences after applying f. We thus need to consider a functor $a\colon \mathcal{C} \to \mathcal{E}$ with the property that it sends f-equivalences to equivalences and show that this equivalent to a composite $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ for some functor $b\colon \mathcal{D} \to \mathcal{E}$. We claim that $b = ar$ works: We have to show that there is an equivalence between a and $bf = arf$. The unit of the adjunction is a map $\mathrm{id} \to rf$, which we claim to consist of f-equivalences: applying f to the map $x \to rf(x)$

gives a map $fx \to frf(x)$, we may postcompose with the counit to obtain the composite $fx \to frf(x) \to f(x)$. The snake identity says that the composite is an equivalence, and the fact that r is fully faithful says that the counit is an equivalence. Hence the unit is an f-equivalence. It follows then from the fact that a sends f-equivalences to equivalences, that the canonical map $a \to arf$ is an equivalence as needed.

□

Example 17.18. Consider the functor $\mathrm{Cat}_\infty \to \mathrm{An}$ given by inverting all morphisms. It has a fully faithful right adjoint given by the inclusion $\mathrm{An} \to \mathrm{Cat}_\infty$ and is hence a Bousfield localization.

Proposition 17.19. *Let $\mathcal{C}$ be an ∞-category and K a simplicial set. If $\mathcal{C}$ admits K-indexed colimits, then the formation of such assembles into a functor $\operatorname{colim}_K\colon \operatorname{Fun}(K, \mathcal{C}) \to \mathcal{C}$ which is left adjoint to the constant functor. Conversely, if the constant functor $\operatorname{const}\colon \mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ admits a left adjoint F, then $F(p)$ is a colimit of p for any diagram $p\colon K \to \mathcal{C}$.*

Proof. We employ Proposition 17.7, rather the remark thereafter, for the existence of a left adjoint. Thus, first we have to specify for each object p of $\operatorname{Fun}(K, \mathcal{C})$ an object t of $\mathcal{C}$ and a map $p \to \operatorname{const}(t)$. As object we choose a colimit $\operatorname{colim}_K p$. We thus need to construct a morphism $p \to \operatorname{const}\operatorname{colim}_K p$ in $\operatorname{Fun}(K, \mathcal{C})$. By adjunction, such a morphism is a map $K \times \Delta^1 \to \mathcal{C}$. Choosing a colimit cone $\bar{p}\colon K \diamond \Delta^0 \to \mathcal{C}$, we can restrict it along the canonical map $K \times \Delta^1 \to K \diamond \Delta^0$. By construction, restricted to $K \times \{0\}$ one obtains p, and restricted to $K \times \{1\}$ one obtains the constant functor with value $\bar{p}(\infty)$ as needed.

We then have to see that the composite as in ... is an equivalence. For this we consider the commutative diagram, which shows, together with Proposition 15.12, that the map in question is in fact an equivalence.

Conversely, assume that the constant functor admits a left adjoint $F\colon \operatorname{Fun}(K, \mathcal{C}) \to \mathcal{C}$ and consider a functor $p\colon K \to \mathcal{C}$. We wish to show that $F(p)$ is a colimit of p. The unit of the adjunction gives a map $p \to \operatorname{const}F(p)$ in $\operatorname{Fun}(K, \mathcal{C})$. This map is adjoint to a map $K \times \Delta^1 \to \mathcal{C}$, and as before, one checks that this map factors through the projection $K \times \Delta^1 \to K \diamond \Delta^0$. The resulting map $\bar{p}\colon K \diamond \Delta^0 \to \mathcal{C}$ is a cone over p, and it remains to show that it is an initial cone. This is the case if and only if the canonical map

$$\operatorname{Map}_{\mathcal{C}}(\bar{p}, x) \to \operatorname{Map}_{\mathcal{C}}(p, x) \simeq \operatorname{map}_{\mathcal{C}^K}(p, \operatorname{const}_x) \simeq \operatorname{map}_{\mathcal{C}}(F(p), x)$$

is an equivalence. Again, we claim that the composite is equivalent to the canonical map given by restriction along the inclusion $\{\infty\} \to K \diamond \Delta^0$, which is an equivalence. □

Remark. Dually, the same statement holds for limits: It is then a right adjoint to the constant functor.

Proposition 17.20. *Let $\mathcal{C}$ be a cocomplete, resp. a complete ∞-category and K a simplicial set. Then $\operatorname{Fun}(K, \mathcal{C})$ is again cocomplete, resp. complete.*

Proof. We claim that the composite

$$\operatorname{Fun}(L, \operatorname{Fun}(K, \mathcal{C})) \simeq \operatorname{Fun}(K, \operatorname{Fun}(L, \mathcal{C})) \to \operatorname{Fun}(K, \mathcal{C}),$$

where the latter functor is post composition with the colimit functor $\operatorname{colim}_L\colon \operatorname{Fun}(L, \mathcal{C}) \to \mathcal{C}$, is left adjoint to the constant functor. This follows immediately from the fact that $(\operatorname{colim}_L)_*$ is right adjoint to const_* by Proposition 17.12. □

Corollary 17.21. *The constant functor* $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ *preserves limits and colimits.*

Proof. We show this only in the case where $\mathcal{C}$ admits all small limits, resp. colimits. We wish to show that if $p: L \to \mathcal{C}$ is a diagram with colimit x, then the constant functor with value x is a colimit of the diagram $L \to \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$.

We first observe that for any functor $\varphi: \mathcal{D} \to \mathcal{E}$ and any simplicial set S, the diagram

$$\begin{tikzcd} \mathrm{Fun}(S, \mathcal{D}) \arrow[r, "\varphi_*"] & \mathrm{Fun}(S, \mathcal{E}) \\ \mathcal{D} \arrow[u, "\mathrm{const}"] \arrow[r, "\varphi"] & \mathcal{E} \arrow[u, "\mathrm{const}"'] \end{tikzcd}$$

commutes. Applying this for the functor $\mathrm{colim}_L: \mathrm{Fun}(L, \mathcal{C}) \to \mathcal{C}$, and the simplicial set K, we obtain the following commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(L, \mathrm{Fun}(K, \mathcal{C})) \arrow[dr, bend left, "\mathrm{colim}_L"] & \\ \mathrm{Fun}(K, \mathrm{Fun}(L, \mathcal{C})) \arrow[u, "\simeq"] \arrow[r, "(\mathrm{colim}_L)_*"] & \mathrm{Fun}(K, \mathcal{C}) \\ \mathrm{Fun}(L, \mathcal{C}) \arrow[u, "\mathrm{const}"] \arrow[r, "\mathrm{colim}_L"] & \mathcal{C} \arrow[u, "\mathrm{const}"'] \end{tikzcd}$$

where the left vertical composite is given by post composition with the constant functor $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$. Notice that the upper triangle commutes by the proof of Proposition 17.20. The commutativity of the diagram then implies the statement of the corollary. $\square$

Proposition 17.22. *Let* $f: \mathcal{C} \to \mathcal{D}$ *be a left adjoint. Then* f *preserves colimits. Likewise, right adjoints preserve limits.*

Proof. We prove that left adjoints preserve colimits. The other case follows by passing to opposite categories. So let $F: K \to \mathcal{C}$ be a diagram and $\bar{F}$ a colimit cone. We wish to show that $f\bar{F}$ is a colimit cone of fF. For this we consider an object z of $\mathcal{D}$ and need to show that the canonical map

$$\mathrm{Map}_{\mathcal{D}}(f\bar{F}, z) \to \mathrm{Map}_{\mathcal{D}}(fF, z)$$

is an equivalence. To prove this, we claim that for any functor $G: L \to \mathcal{C}$, there is a canonical equivalence

$$\mathrm{Map}_{\mathcal{D}}(fG, z) \simeq \mathrm{Map}_{\mathcal{C}}(G, gz)$$

where g is the right adjoint of f. Taking this claim for granted for the moment, we then consider the commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{D}}(f\bar{F}, z) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{D}}(fF, z) \arrow[d, "\simeq"] \\ \mathrm{Map}_{\mathcal{C}}(\bar{F}, gz) \arrow[r, "\simeq"] & \mathrm{Map}_{\mathcal{C}}(F, gz) \end{tikzcd}$$

in which the vertical maps are equivalences by the claim, and where the lower horizontal map is an equivalence by the assumption that $\bar{F}$ is a colimit cone. Hence also the upper map is an equivalence as needed.

It remains to prove the claim. For this we consider the following chain of equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{D}}(fG, z) &\simeq \mathrm{map}_{\mathcal{D}^K}(f_*(G), \mathrm{const}_z) && \text{by Proposition 15.12} \\ &\simeq \mathrm{map}_{\mathcal{C}^K}(G, g_*(\mathrm{const}_z)) && \text{by Proposition 17.12} \\ &\simeq \mathrm{map}_{\mathcal{C}^K}(G, \mathrm{const}_{gz}) && \text{by inspection} \\ &\simeq \mathrm{Map}_{\mathcal{C}}(G, gz) && \text{by Proposition 15.12} \end{aligned}$$

where the inspection is to observe that $g_*(\mathrm{const}_z) = \mathrm{const}_{gz}$. □

Finally, we need the following proposition.

Proposition 17.23. *Consider a pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "q"] & \mathcal{C}' \arrow[d, "p"] \\ \mathcal{D} \arrow[r, "g"] & \mathcal{D}' \end{tikzcd}$$

in which the map p is an isofibration. Suppose that p preserves colimits and let $F\colon K \to \mathcal{C}$ be a diagram.

1. *A cone $\bar{F}\colon K \diamond \Delta^0 \to \mathcal{C}$ is a colimit cone if its image under f and q is a colimit cone.*
2. *If $\mathcal{C}'$ and $\mathcal{D}$ are cocomplete and g preserves colimits, then $\mathcal{C}$ is also cocomplete. Furthermore f and q preserve colimits.*

Proof. The first thing we observe is that for any object x of $\mathcal{C}$, there is a homotopy cartesian diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F, x) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}'}(fF, fx) \arrow[d] \\ \mathrm{Map}_{\mathcal{D}}(qF, qx) \arrow[r] & \mathrm{Map}_{\mathcal{D}'}(pfF, pfx) \end{tikzcd}$$

This follows from Proposition 15.12 and the fact that the mapping anima in a pullback are given by the pullback of the mapping anima. Note that we use here that applying the functor $\mathrm{Fun}(K, -)$ to the above diagram again gives a pullback diagram where one leg is an isofibration.

We now wish to analyze whether $\bar{F}$ is a colimit cone. For this we consider the above squares for $\bar{F}$ and F, and obtain a canonical commutative cube. The assumption that $f\bar{F}$ and $q\bar{F}$ are colimit cones implies that the comparison maps are equivalences on the left lower and right upper corner. Using that p preserves colimits, we find that $pf\bar{F}$ is also a colimit cone, so that the comparison map is also an equivalence there. We find that the comparison map is an equivalence also in the upper left corner, so that $\bar{F}$ is a colimit cone. This proves (1).

To prove (2) it will suffice to show that any diagram $F\colon K \to \mathcal{C}$ admits a cone $\bar{F}$ whose image in $\mathcal{D}$ and $\mathcal{C}'$ is a colimit cone. Then we apply (1) to see that $\bar{F}$ is a colimit cone, and by construction q and f send $\bar{F}$ again to a colimit cone.

To do this, we consider the composite $F_1\colon K \to \mathcal{C} \to \mathcal{C}'$ and choose a colimit cone $\bar{F}_1\colon K \diamond \Delta^0 \to \mathcal{C}'$. Likewise, we consider the composite $F_2\colon K \to \mathcal{C} \to \mathcal{D}$ and choose a colimit cone $\bar{F}_2\colon K \diamond \Delta^0 \to \mathcal{D}$. The images $g\bar{F}_2$ and $p\bar{F}_1$ are then also colimit cones by the assumption that both p and g preserve colimits. Hence there is an equivalence τ between these two cones, say

τ is an equivalence from $p\bar{F}_1$ to $g\bar{F}_2$. We thus obtain a lifting problem

$$\begin{tikzcd} \Delta^0 \arrow[r, "\bar{F}_1"] \arrow[d] & \mathcal{C}'^{fF/} \arrow[d] \\ \Delta^1 \arrow[r, "\tau"] \arrow[ur, dashed, "\hat{\tau}"] & \mathcal{D}'^{pfF/} \end{tikzcd}$$

which admits a solution since the right vertical map is an isofibration, since p is an isofibration. Furthermore, the dashed arrow is again an equivalence in $\mathcal{C}'$ and hence $\hat{\tau}(1)$ is another cocone of fF. Unravelling the definitions, we obtain a commutative diagram

$$\begin{tikzcd} K \diamond \Delta^0 \arrow[r, "\hat{\tau}(1)"] \arrow[d, "\bar{F}_2"] & \mathcal{C}' \arrow[d, "p"] \\ \mathcal{D} \arrow[r, "g"] & \mathcal{D}' \end{tikzcd}$$

so this gives a unique map $K \diamond \Delta^0 \to \mathcal{C}$ which is a cone over F. We are now in the situation that we may apply (1) and deduce the proposition. $\square$

18. An adjoint functor theorem

The goal of this section is to prove a general adjoint functor theorem, following the argument given in [NRC19].

Definition 18.1. A full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ of an ∞-category $\mathcal{C}$ is called colimit dense, if every object of $\mathcal{C}$ can be written as a colimit of a diagram $p\colon K \to \mathcal{C}_0 \subseteq \mathcal{C}$.

Theorem 18.2. *Let $\mathcal{C}$ be a locally small ∞-category which is cocomplete and contains an essentially small, colimit dense full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$. Let $\mathcal{D}$ be a locally small ∞-category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if F preserves colimits.*

Remark. Particular examples of locally small ∞-categories which admit a small colimit dense full subcategory are *accessible* ∞-categories. An ∞-category is called accessible if it is κ-accessible for some regular cardinal κ. A κ-accessible ∞-category is a locally small ∞-category $\mathcal{C}$ which admits κ-filtered colimits, and which contains an essentially small subcategory $\mathcal{C}_0$ such that every object of $\mathcal{C}_0$ is κ-compact and such that every object of $\mathcal{C}$ is a κ-filtered colimit of objects in $\mathcal{C}_0$. An accessible ∞-category which is in addition cocomplete is called *presentable*. The above theorem can hence be applied to functors between presentable ∞-categories, so that any colimit preserving functor between presentable categories admits a right adjoint.

Corollary 18.3. *Let $\mathcal{C}$ be a locally small ∞-category which is cocomplete and contains an essentially small colimit dense full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$. Then $\mathcal{C}$ is complete.*

Proof. Let K be a simplicial set. Consider the functor const: $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$. It preserves colimits, see Corollary 17.21, moreover $\mathrm{Fun}(K, \mathcal{C})$ is again locally small: this needs justification, but see [Lur09, Example 5.4.1.8]. Hence, by Theorem 18.2, the constant functor admits a right adjoint. By Proposition 17.19 this functor takes a diagram $p\colon K \to \mathcal{C}$ to a limit of p. $\square$

To prove the theorem we need some preliminaries. To obtain the right adjoint we will employ the following criterion.

Proposition 18.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if for all objects d of $\mathcal{D}$, the ∞-category $\mathcal{C}_{/d}$ admits a terminal object.*

Proof. Let d be an object of $\mathcal{D}$ and let consider a terminal object of $\mathcal{C}_{/d}$. This is given by a pair (Gd, f) where f is a morphism $FGd \to d$ in $\mathcal{D}$. We wish to use Proposition 17.7 to show that F admits a right adjoint. We must thus consider the lower composite in the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}_{/d}}((c,\alpha),(Gd,f)) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{D}_{/d}}(\alpha,f) \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "\alpha"] \\
\mathrm{map}_{\mathcal{C}}(c,Gd) \arrow[r] & \mathrm{map}_{\mathcal{D}}(Fc,FGd) \arrow[r, "f_*"] & \mathrm{map}_{\mathcal{D}}(Fc,d)
\end{tikzcd}$$

We observe that the left square is the pullback of mapping spaces induced from the pullback

$$\begin{tikzcd}
\mathcal{C}_{/d} \arrow[r] \arrow[d] & \mathcal{D}_{/d} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

of ∞-categories. Furthermore, the right square is a homotopy pullback by Proposition 12.15. Thus, the big square is a homotopy pullback as well, and the upper composite is an equivalence by the assumption that (Gd, f) is a terminal object of $\mathcal{C}_{/d}$. Hence the lower composite is also an equivalence so that we conclude the proposition from Proposition 17.7. □

Remark. Likewise, a functor admits a left adjoint if and only if for all objects d of $\mathcal{D}$, the category $\mathcal{C}_{d/}$ admits an initial object.

We thus need to find criteria that ensure that specific categories admit terminal objects. For this we will make use of the notion of weakly terminal sets:

Definition 18.5. Let $\mathcal{C}$ be an ∞-category and S a (small) set of objects. S is said to be weakly terminal, if for every object x of $\mathcal{C}$, there exists an object s in S such that the anima $\mathrm{map}_{\mathcal{C}}(x, s)$ is not empty. An object t is called weakly terminal if the set $\{t\}$ is a weakly terminal set.

Lemma 18.6. *Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be an essentially small, full subcategory of a cocomplete category which is colimit dense. Then $\mathcal{C}$ has a weakly terminal object.*

Proof. Consider the functor $\mathcal{C}_0 \to \mathcal{C}$ and pick a Joyal equivalence $\mathcal{C}' \simeq \mathcal{C}_0$ with $\mathcal{C}'$ a small simplicial set. As Joyal equivalences are cofinal by Corollary 16.8 and $\mathcal{C}$ is cocomplete, we find that the functor $\mathcal{C}_0 \to \mathcal{C}$ admits a colimit t. We claim that t is a weakly terminal object. To see this, let x be another object of $\mathcal{C}$. By assumption, there is a functor $K \to \mathcal{C}_0$ such that the colimit over the composite $K \to \mathcal{C}_0 \to \mathcal{C}$ is given by x. We obtain a canonical map x to t on colimits. In particular, the anima of maps from x to t is not empty. □

Proposition 18.7. *Let $\mathcal{C}$ be a locally small and cocomplete ∞-category and let S be a weakly terminal set. Let $\mathcal{C}_0$ be the full subcategory spanned by S. Then $\mathcal{C}_0 \to \mathcal{C}$ is cofinal.*

Proof. By Theorem 16.16 it will suffice to show that for any object x of $\mathcal{C}$, the slice $(\mathcal{C}_0)_{x/}$ is weakly contractible. We will show that for any small simplicial set K, any functor $K \to (\mathcal{C}_0)_{x/}$ factors through the inclusion $K \star \Delta^0$ which is contractible. It follows that $(\mathcal{C}_0)_{x/}$ is weakly contractible as needed. So consider a functor $K \to (\mathcal{C}_0)_{x/}$ and the composite

$$K \to (\mathcal{C}_0)_{x/} \to \mathcal{C}_{x/}.$$

As $\mathcal{C}$ is cocomplete, so is $\mathcal{C}_{x/}$ by Proposition 15.30, so we may choose a colimit cone of the above functor

$$\begin{tikzcd} K \arrow[r] \arrow[d] & (\mathcal{C}_0)_{x/} \arrow[r] & \mathcal{C}_{x/} \\ K \star \Delta^0 \arrow[urr, "\mu"'] & & \end{tikzcd}$$

and consider $\mu(\infty)\colon x \to t$. Pick an object s in S for which there exists a map $t \to s$, and pick such a map. Choosing a composite of $x \to t$ and $t \to s$ gives a 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$, which is adjoint to a map $\Delta^1 \to \mathcal{C}^{x/}$. We then consider the following lifting problem

$$\begin{tikzcd} K \star \Delta^0 \amalg_{\Delta^0} \Delta^1 \arrow[r, "(\mu,\sigma)"] \arrow[d] & \mathcal{C}_{x/} \\ K \star \Delta^1 \arrow[ur, dashed, "\mu'"] & \end{tikzcd}$$

which can be solved as the vertical map is inner anodyne by Lemma 4.22 and $\mathcal{C}_{x/}$ is an ∞-category. Restricting μ' along the inclusion $K \star \Delta^{\{1\}} \to K \star \Delta^1$ gives a functor $K \star \Delta^0 \to \mathcal{C}_{x/}$ which factors through $(\mathcal{C}_0)_{x/}$: Since $(\mathcal{C}_0)_{x/} \subseteq \mathcal{C}_{x/}$ is a full subcategory, it suffices to see that all objects of $K \star \Delta^0$ go to $(\mathcal{C}_0)_{x/}$: On K it is true by assumption and on the cone point $\{\infty\}$, by construction, one obtains the map $x \to s$ which is in $(\mathcal{C}_0)_{x/}$, again by construction. Hence the proposition is proven. □

Corollary 18.8. *Let $\mathcal{C}$ be a locally small ∞-category which is cocomplete. Assume that there exists a weakly terminal set. Then $\mathcal{C}$ admits a terminal object.*

Proof. Let S be a weakly terminal set and consider the full subcategory $\mathcal{C}_0$ spanned by S. By Proposition 18.7 the inclusion $\mathcal{C}_0 \to \mathcal{C}$ is cofinal. Since $\mathcal{C}_0$ is small, the functor $\mathcal{C}_0 \to \mathcal{C}$ admits a colimit. From Corollary 16.9 we thus deduce that also the identity functor $\mathcal{C} \to \mathcal{C}$ admits a colimit. Such a colimit is a terminal object, by Lemma 15.13. □

Proof of Theorem 18.2. The fact that left adjoints preserve colimits was dealt with in Proposition 17.22. Let us therefore prove that F admits a right adjoint if it preserves colimits. By Proposition 18.4, it suffices to show that for every object d of $\mathcal{D}$, the slice $\mathcal{C}_{/d}$ has a terminal object.

We then observe that $\mathcal{C}_{/d}$ is again locally small and cocomplete. The cocompleteness follows from Proposition 17.23 because the functor $\mathcal{C} \to \mathcal{D}$ preserves colimits by assumption, and the functor $\mathcal{D}_{/d} \to \mathcal{D}$ preserves colimits by Proposition 15.29. To see that $\mathcal{C}_{/d}$ is again locally small, we calculate the mapping anima in terms of those in $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{D}_{/d}$. Those in $\mathcal{C}$ and $\mathcal{D}$ are essentially small by assumption, and those in $\mathcal{D}_{/d}$ are then also essentially small by Proposition 12.15. Thus the pullback is also essentially small.

Hence by Corollary 18.8 it suffices to establish the existence of a weakly terminal object, which we will deduce by means of Lemma 18.6. In other words, we have to show that $\mathcal{C}_{/d}$

contains an essentially small full subcategory which is colimit dense. We claim that $(\mathcal{C}_0)_{/d}$ is such a subcategory. It is essentially small since $\mathcal{D}$ is locally small, and colimit dense since F preserves colimits. $\square$

Appendix A. Exercises

A.1. Introduction.

Exercise 2. Let $h(\mathrm{CW})$ be the homotopy category of CW-complexes. Show that this category does not have all pushouts. More concretely, show that the diagram

$$* \longleftarrow S^1 \xrightarrow{\cdot 2} S^1$$

does not admit a pushout.

A.2. Section 1.

Exercise 3. Work out at least three of the following *simplicial identities*

(1) $d_i^* d_j^* = d_{j-1}^* d_i^*$ if $i < j$
(2) $d_i^* s_j^* = s_{j-1}^* d_i^*$ if $i < j$
(3) $d_i^* s_j^* = \mathrm{id}$ if $i = j, j+1$
(4) $d_i^* s_j^* = s_j^* d_{i-1}^*$ if $i > i+1$
(5) $s_i^* s_j^* = s_{j+1}^* s_i^*$ if $i \leq j$.

Here, for any simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, we denote the map $X(d_i)$ by d_i^*. Hint: Think about what this means for the maps d_i and s_j in Δ and prove the corresponding identities there.

Exercise 4. Show that every map in Δ can be uniquely factored as a composition of s_i's followed by a composition of d_j's. Thus a simplicial set is equivalently described by a sequence of sets X_n equipped with face and degeneracy maps satisfying the simplicial identities.

Exercise 5. Give examples of simplicial sets where the relation of Definition 1.10 of the lecture, leading to $\pi_0^{\Delta}(X)$, is not symmetric and not transitive.

Exercise 6. Show that every simplex $x \in X_n$ is of the form $\alpha^*(y)$ for a surjection $\alpha\colon [m] \to [n]$ and a non-degenerate n-simplex y, and show that the pair (α, y) is uniquely determined by x.

Exercise 7. Show that the category Set is bicomplete. Hint: General colimits are constructed as quotients of disjoint unions, and general limits are constructed as subsets of products.

Exercise 8. Let $F\colon I \to \mathcal{C}$ be a functor. Show that a colimit of F can equivalently be described as an initial cocone over F, and that a limit of F can be equivalently described as a terminal cone over F.

Exercise 9. Calculate the limit and colimit of a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$.

Exercise 10. Show that the datum of an adjunction is equivalent to the datum of a pair of functors (F, G) as above together with natural transformations $\varepsilon\colon FG \to \mathrm{id}$ and $\eta\colon GF \to \mathrm{id}$

satisfying the snake identities, that is, the obvious composites

$$F(X) \to F(GF(X)) \cong FG(FX) \to F(X)$$

and

$$G(X) \to GF(G(X)) \cong G(FG(X)) \to G(X)$$

are the identity of $F(X)$ and $G(X)$ respectively.

Exercise 11. Show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint if you can speficy objects Gy for all $y \in \mathcal{D}$ and maps $\varepsilon_y\colon FGy \to y$, which have the property that the induced map on hom-sets

$$\mathrm{Hom}_{\mathcal{C}}(x, Gy) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(Fx, FGy) \xrightarrow{\varepsilon_y} \mathrm{Hom}_{\mathcal{D}}(Fx, y)$$

is a bijection. There is an obvious dual notion which shows that F admits a left adjoint if one can specify objects Gy for all $y \in \mathcal{D}$ and maps $\eta_y\colon y \to FGy$ which make the induced map on hom-sets

$$\mathrm{Hom}_{\mathcal{D}}(Gy, x) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(FGy, Fx) \xrightarrow{\eta_y} \mathrm{Hom}_{\mathcal{C}}(y, Fx)$$

a bijection.

Exercise 12. Prove that if a simplicial set X has at most n-dimensional non-degenerate simplices, and Y has at most m-dimensional non-degenerate simplices, then their product $X \times Y$ has at most $(n+m)$-dimensional non-degenerate simplices.

Exercise 13. Show that for every simplicial set X, there is a canonical bijection $\pi_0^{\Delta}(X) \cong \pi_0(|X|)$.

Exercise 14. Show that there are inclusions $I^n \subseteq \Lambda^n_j$ provided $0 < j < n$ or $n \geq 3$, and $\Lambda^n_j \subseteq \partial\Delta^n \subseteq \Delta^n$ for all $n \geq 0$.

Exercise 15. Let I be a category with an initial object i and let J be a category with a terminal object t. Show that a limit of a functor $F\colon I \to \mathcal{C}$ is given by $F(i)$ (together with the canonical maps $F(i) \to F(x)$ for all $x \in I$). Similarly, show that a colimit of a functor $G\colon J \to \mathcal{C}$ is given by $G(t)$ (together with its maps $G(x) \to G(t)$ for all $x \in J$).

Exercise 16. Show that for every $n \geq 0$, there is a pushout

$$\begin{tikzcd} \coprod_{J_n} \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X) \arrow[d] \\ \coprod_{J_n} \Delta^n \arrow[r] & \mathrm{sk}_n(X) \end{tikzcd}$$

where J_n is the set of non-degenerate n-simplices. Furthermore $X \cong \mathrm{colim}_n\, \mathrm{sk}_n(X)$.

Exercise 17. Show that the following simplicial sets are not nerves of categories:

(1) $\partial\Delta^n$ for $n \geq 2$,
(2) Λ^n_j for $n \geq 2$ and $0 \leq j \leq n$,
(3) I^n for $n \geq 2$.

Exercise 18. Suppose X is a Kan complex. Show that for all $n \geq 0$, the simplicial set $\mathrm{cosk}_n(X)$ is again a Kan complex. Prove that the canonical map $X \to \mathrm{cosk}_n(X)$ induces a bijection

$$\pi_k^\Delta(X) \to \pi_k^\Delta(\mathrm{cosk}_n(X))$$

for $k < n$ and that $\pi_k^\Delta(\mathrm{cosk}_n(X)) = 0$ for $k \geq n$.

Exercise 19. Show that a natural transformation between functors $f, g \colon \mathcal{C} \to \mathcal{D}$ induces a homotopy between $\mathrm{N}(f), \mathrm{N}(g) \colon \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$. Use this to show that conjugation with an element determines a self map of BG which is homotopic to the identity. What does conjugation induce on $\pi_1(BG)$? Why does this not show that every group is abelian?

Exercise 20. Show that the nerve of a category $\mathcal{C}$ is 2-coskeletal, i.e. that the canonical map $\mathrm{N}(\mathcal{C}) \to \mathrm{cosk}_2(\mathrm{N}(\mathcal{C}))$ is an isomorphism of simplicial sets.

Exercise 21. Let X be a simplicial set and let $n \leq m$. Show that $\mathrm{sk}_n(\mathrm{sk}_m(X)) = \mathrm{sk}_n(X) = \mathrm{sk}_m(\mathrm{sk}_n(X))$. Deduce that $\mathrm{cosk}_n(\mathrm{cosk}_m(X)) \cong \mathrm{cosk}_m(\mathrm{cosk}_n(X))$. Is it also true that $\mathrm{sk}_n(\mathrm{cosk}_m(X)) \cong \mathrm{cosk}_m(\mathrm{sk}_n(X))$ (if not provide a counter example)? Is there a preferred map between these two simplicial sets?

Exercise 22. Let $\mathcal{C}$ be a category and X a simplicial set. Recall that X^{op} is the simplicial set with: $X_n^{\mathrm{op}} = X_n$ and $d_i^{\mathrm{op}} \colon X_n \to X_{n-1}$ is given by d_{n-i}, likewise that $s_i^{\mathrm{op}} = s_{n-i}$ as a map $X_n \to X_{n+1}$. Prove the following assertions:

(1) $\mathrm{N}(\mathcal{C}^{\mathrm{op}}) \cong \mathrm{N}(\mathcal{C})^{\mathrm{op}}$,
(2) $(\Delta^n)^{\mathrm{op}} \cong \Delta^n$,
(3) $(\Lambda_i^n)^{\mathrm{op}} \cong \Lambda_{n-i}^n$,
(4) $(\partial\Delta^n)^{\mathrm{op}} \cong \partial\Delta^n$.

Exercise 23. Let G be a group and let $\mathbb{B}G$ be the category with one object and G as endomorphisms of that object. Show that $\mathrm{N}(\mathbb{B}(G))$ has only one non-trivial homotopy group, namely $\pi_1^\Delta(\mathrm{N}(\mathbb{B}G))$ and that this group is canonically isomorphic to G.

A.3. **Section 2.**

Exercise 24. Let X be a composer and let $f \colon x \to y$ be a morphism in X. Show that f is a composition of id_x with f and of f with id_y.

Exercise 25. Consider the map $[0] \to [n]$ in Δ with image $\{0\}$. Show that this determines a map $0 \colon \Delta^0 \to \partial\Delta^n$. Calculate the simplicial homotopy sets $\pi_i^\Delta(\partial\Delta^n, 0)$ for $i \geq 1$ and $n \geq 2$. Deduce that $\partial\Delta^n$ is not a Kan complex.

Exercise 26. Show that the following simplicial sets are not ∞-categories:

(1) $\partial\Delta^n$ for $n \geq 2$,
(2) Λ_j^n for $n \geq 3$ and $0 \leq j \leq n$,
(3) I^n for $n \geq 2$.

Exercise 27. Determine the homotopy category of the following simplicial sets:

(1) $\partial\Delta^n$ for $n \geq 1$,

(2) Λ^n_j for $n \geq 2$ and $0 \leq j \leq n$, and
(3) I^n for $n \geq 0$.

Exercise 28. Let $f\colon X \to Y$ be a map of simplicial sets. Prove or give a counter example to the following statements.

(1) If f is a monomorphism, then $hX \to hY$ is fully faithful,
(2) If f is a degree-wise surjection, then $hX \to hY$ is *surjective and full* i.e. induces a surjection on objects and on hom-sets.
(3) If f induces a surjection on 0- and 1-simplices, then $hX \to hY$ is surjective and full.

Exercise 29. Prove or disprove the following statement: For any two simplicial sets X and Y, the canonical map $h(X \times Y) \to hX \times hY$ is an isomorphism of categories.

Exercise 30. A category $\mathcal{C}$ is called connected if $\pi_0^\Delta(\mathrm{N}(\mathcal{C}))$ consists only of one element. Show that a groupoid $\mathcal{G}$ is connected if and only if for every two objects $x, y \in \mathcal{G}$, the set $\mathrm{Hom}_\mathcal{G}(x, y)$ is non-empty. Show that a connected groupoid is equivalent to $\mathbb{B}G$ for a group G. Show however, that the category of connected groupoids is *not* equivalent to the category of groups.

Exercise 31. Let X be a topological space. Describe the category $h(\mathcal{S}(X))$. Show that the endomorphisms of each object form a group. Which group is it?

Exercise 32. Suppose X is a composer with the inner 3-horn extension property. Let $\sigma\colon \Delta^1 \times \Delta^1 \to X$ be a map such that

(1) $\sigma_{|\Delta^1 \times \{0\}} = f$,
(2) $\sigma_{|\Delta^1 \times \{1\}} = g$,
(3) $\sigma_{|\{0\} \times \Delta^1} = \mathrm{id}_x$, and
(4) $\sigma_{|\{0\} \times \Delta^1} = \mathrm{id}_y$,

for morphisms $f, g\colon x \to y$. Show that $f \sim g$ in the sense of Definition 2.4.

Exercise 33. Let X be a composer with the extension property for inner 3-horns. Show that for any two composable morphisms $f\colon x \to y$ and $g\colon y \to z$, the simplicial set $\mathrm{Comp}_X(f, g)$ is connected, i.e. that $\pi_0^\Delta(\mathrm{Comp}_X(f, g))$ consists only of one element.

Exercise 34. Let X be a simplicial set and consider the canonical map $X \to \mathrm{N}(hX)$.

(1) Show that this map factors through the canonical map $X \to \mathrm{cosk}_2(X)$.
(2) Show that the induced map $\mathrm{cosk}_2(X) \to \mathrm{N}(hX)$ is an isomorphism if X is isomorphic to the nerve of a category.
(3) Show that the map $\mathrm{cosk}_2(X) \to \mathrm{N}(hX)$ is in general not an isomorphism. Hint: Find an X which is 2-coskeletal, but not the nerve of a category.
(4) Prove or disprove the following statement: The map $\mathrm{cosk}_2(X) \to \mathrm{N}(hX)$ is an isomorphism if and only if X is isomorphic to the nerve of a category.

Exercise 35. Let $(V, \otimes, \mathbb{1})$ be a monoidal category. Then the functor $\mathrm{Hom}_V(\mathbb{1}, -)\colon V \to \mathrm{Set}$ is lax monoidal. Is it monoidal? Can you find a condition on $(V, \otimes, \mathbb{1})$ which ensures that it is?

Exercise 36. Let $\mathcal{C}$ be a category with finite products and finite coproducts. We say that $\mathcal{C}$ is pointed if the canonical map $\emptyset \to *$ from the inital to the terminal object is an isomorphism. Show that the identity canonically refines to a lax monoidal functor $(\mathcal{C}, \times, *) \to (\mathcal{C}, \amalg, \emptyset)$. When is this functor monoidal? Furthermore, show that any functor $F\colon \mathcal{C} \to \mathcal{D}$ refines canonically to a lax symmetric monoidal functor $(\mathcal{C}, \amalg, \emptyset) \to (\mathcal{D}, \amalg, \emptyset)$. When is it monoidal?

Exercise 37. The goal of this exercise is to show that any essentially surjective and fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ between ordinary categories is an equivalence.

(1) Show that F admits an adjoint G. Hint: Use Exercise 11.
(2) Show that G is itself fully faithful.
(3) Show that an adjoint pair (F, G) of fully faithful functors makes F an equivalence with G an inverse.

Exercise 38. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with right adjoint $G\colon \mathcal{D} \to \mathcal{C}$. Show that they are mutually inverse if F is fully faithful and G is conservative. Here, conservativity means that if $f\colon x \to y$ is a morphism in $\mathcal{D}$ and $G(f)$ is an isomorphism, then f is an isomorphism.

Exercise 39. Let $F\colon \mathcal{C} \to \mathcal{D}$ be left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$. Show that if G is lax monoidal, then F canonically refines to an oplax monoidal functor. Vice versa, show that if F is oplax monoidal, then G canonically refines to a lax monoidal functor.

Exercise 40. Show that the left adjoint of a monoidal adjunction is in fact monoidal. Recall that a monoidal adjunction consists of lax monoidal functors F and G, which are witnessed to be adjoint by a unit η and a counit ε where both η and ε are monoidal transformations.

Exercise 41. Suppose F is left adjoint to G, witnessed by a unit and counit (η, ε). Show that if F is monoidal, the induced lax monoidal structure on G of Exercise 6.1 makes $(F, G, \eta, \varepsilon)$ a monoidal adjunction.

Exercise 42. Show that the coherent nerve functor $\mathrm{N}\colon \mathrm{Cat}_\Delta \to \mathrm{sSet}$ commutes with coproducts. Show that $\mathfrak{C}$ is not right adjoint to N. Does N have a right adjoint at all? Hint: Does the ordinary nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ have a right adjoint? How are these two questions related?

Exercise 43. Show that if the coherent nerve $\mathrm{N}(\mathcal{C})$ of a simplicial category is isomorphic to the nerve of an ordinary category, then the underlying category $u\mathcal{C}$ is isomorphic to the homotopy category $\pi(\mathcal{C})$. Make explicit the coherent nerve of the following simplicial category $\mathbb{B}^{\mathrm{simp}}(G)$: There is only one object, and the simplicial set of endomorphisms of this object is given by $\mathrm{N}(G)$, where G is a group (of a monoid if you wish). Deduce from the explicit analysis that $\mathrm{N}(\mathbb{B}^{\mathrm{simp}}(G))$ is not isomorphic to the nerve of a category although $u\mathrm{N}(\mathbb{B}^{\mathrm{simp}}(G)) \cong \pi(\mathrm{N}(\mathbb{B}^{\mathrm{simp}}(G)))$.

Exercise 44. Prove or disprove the following statements:

(1) There exists a simplicial category $\mathcal{C}$, whose coherent nerve $\mathrm{N}(\mathcal{C})$ is not an ∞-category,
(2) There exists a simplicial category $\mathcal{C}$, whose coherent nerve $\mathrm{N}(\mathcal{C})$ is an ∞-category, but not a Kan complex.

Exercise 45. Suppose that $\mathcal{C}$ is a simplicial category all whose hom simplicial sets are Kan complexes. Work out what it means concretely that a morphism $f: x \to y$ in a simplicial category is an equivalence. Rephrase the condition of weakly fully faithful functors via this.

Exercise 46. Show that there exists monomorphisms $X \to Y$ where both X and Y are ∞-categories, but X is not a sub-∞-category in the sense of Definition 2.73.

Exercise 47. Let $Y \to X$ be an inclusion of topological spaces. When is the induced map $\mathcal{S}(Y) \to \mathcal{S}(X)$ a subcategory? When is it a full subcategory?

Exercise 48. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full sub-∞-category and let $\mathcal{D}$ be an ∞-category. Show that the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{C}_0)$ is the full sub-∞-category of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ on those functors $f: \mathcal{D} \to \mathcal{C}$ which factor through the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$.

Exercise 49. Show that a simplicial set X is an ∞-category if and only if X^{op} is an ∞-category and likewise that X is a Kan complex if and only if X^{op} is. Show that if X is an ∞-category then X is an ∞-groupoid if and only if X^{op} is.

Exercise 50. Let $\mathcal{C}$ be a simplicial category. Show that the coherent nerve $\mathrm{N}(\mathcal{C})$ is isomorphic to the nerve of an ordinary category if and only if $\mathcal{C}$ is in the image of the functor $c: \mathrm{Cat} \to \mathrm{Cat}_\Delta$.

Exercise 51. Show that the composite

$$\mathrm{Cat}_\Delta \xrightarrow{\mathrm{N}} \mathrm{sSet} \xrightarrow{h} \mathrm{Cat}$$

is isomorphic to the functor $\pi: \mathrm{Cat}_\Delta \to \mathrm{Cat}$.

A.4. **Section 3.**

Exercise 52. Show the following assertions.

(1) Let $\mathcal{C}$ be an ordinary category and X a simplicial set. Then X is an ∞-category if and only if every map $X \to \mathrm{N}(\mathcal{C})$ is an inner fibration.
(2) A map $f: X \to Y$ is an inner fibration if and only if $f^{\mathrm{op}}: X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is.
(3) A map $f: X \to Y$ is a left fibration if and only if the map $f^{\mathrm{op}}: X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a right fibration.

Exercise 53. Let $S \subseteq S'$ be sets of morphisms. Show that

(1) $\chi_R(S') \subseteq \chi_R(S)$,
(2) $S \subseteq \chi(S)$, and
(3) $\chi_R(S) = \chi_R(\chi(S))$.

Exercise 54. A category I is called *filtered* if every functor $K \to I$ from a finite category K extends over the inclusion $K \to K^{\triangleright}$. Show that a poset (viewed as a category) is filtered if and only if

(1) for every finite collection of objects $X_1, \dots, X_n$ of I, there exists an object X of I equipped with maps $X_k \to X$ for all $k = 1, \dots, n$.

(2) Any two morphisms $f, g\colon X \to Y$ can be equalized, i.e. there exists a morphism $h\colon Y \to Z$ such that $hf = hg$.

Exercise 55. Let I be a finite category.

(1) Show that I is filtered if it has a terminal object.
(2) Show that there are examples where I is filtered but does not have a terminal object.
(3) Show that I is a poset and filtered if and only if it has a terminal object.

In particular, notice that this shows that there are many filtered categories which are not posets.

Exercise 56. Show that every simplicial set A with only finitely many non-degenerate simplices is *compact*, i.e. that the canonical map

$$\operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathrm{sSet}}(A, X_i) \to \mathrm{Hom}_{\mathrm{sSet}}(A, \operatorname*{colim}_{i\in I} X_k)$$

is an isomorphism, provided I is a filtered category.

Exercise 57. We call a set S semi-saturated if it is closed under pushouts, retracts and countable compositions. Show that a semi-saturated set

(1) contains isomorphisms, if it contains the identity of an inital object $\emptyset$,
(2) is closed under finite coproducts, if it contains the identity of an initial object $\emptyset$,
(3) is closed under composition, i.e. if $f\colon A \to B$ and $g\colon B \to C$ are elements of S, then so is $gf\colon A \to C$, and
(4) is closed under countable coproducts if it is closed under finite coproducts, i.e. if $\{f_i\colon A_i \to B_i\}_{i\in I}$ is a countable family of elements if S, then the map $\coprod_{i\in I}\colon A_i \to B_i$ is an element of S as well.

Exercise 58. Show that a saturated set S in a category $\mathcal{C}$ contains all isomorphisms. Find an example of a category $\mathcal{C}$ and a semi-saturated set S of morphisms in $\mathcal{C}$ which is non-empty and does not contain all isomorphisms.

Exercise 59. Show the following assertions.

(1) The map $\emptyset \to \{*\}$ in Set generates the set of injections. What is $\chi_R(\emptyset \to *)$? Spell out the factorization obtained by the small object argument for a general map $f\colon M \to N$ of sets.
(2) The map $\{*, *\} \to \{*\}$ generates the class of surjections. What is $\chi_R(\{*, *\} \to *)$? Spell out the factorization obtained by the small object argument for a general map $f\colon M \to N$ of sets.

Exercise 60. Consider the set $S = \{\partial\Delta^n \to \Delta^n\}_{n\geq 0}$ given by the boundary inclusions. Show that $\chi(S)$ is given by all monomorphisms of simplicial sets.

Exercise 61. Show that J is not a compact simplicial set, i.e. that there are infinitely many non-degenerate simplices in J.

Exercise 62. Show that if a morphism $f\colon \Delta^1 \to \mathcal{C}$ in an ∞-category extends over the inclusion $\Delta^1 \to J$, then f is an equivalence.

Exercise 63. Fill in the missing steps of Lemma 3.30. More precisely show the following assertions:

(1) The horn inclusion $\Lambda^n_j \to \Delta^n$ for $0 \leq j < n$ is a retract of the pushout product map
$$\Delta^n \times \{0\} \amalg_{\Lambda^n_j \times \{0\}} \Lambda^n_j \times \Delta^1 \to \Delta^n \times \Delta^1.$$

(2) The pushout product map
$$\partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \to \Delta^n \times \Delta^1$$
is left anodyne.

Exercise 64. Show that a trivial fibration $f\colon X \to Y$ between Kan complexes induces an isomorphism in the category $\pi(\mathrm{Kan})$. Hint: Show that a trivial fibration between Kan complexes is a homotopy equivalence.

Exercise 65. Show that if $f\colon y \to z$ is an equivalence, then
$$\mathrm{map}_{\mathcal{C}}(x,y) \simeq \mathrm{map}_{\mathcal{C}}(x,y) \times \Delta^0 \xrightarrow{f} \mathrm{map}_{\mathcal{C}}(x,y) \times \mathrm{map}_{\mathcal{C}}(y,z) \to \mathrm{map}_{\mathcal{C}}(x,z)$$
is a homotopy equivalence.

Exercise 66. Show that that composition as defined in the lecture is associative up to homotopy, i.e. that composition in an ∞-category determines a category enriched in $\pi(\mathrm{Kan})$, the homotopy category of Kan complexes. Hint: Consider the diagram
$$\begin{tikzcd}
\Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{2,3\}} \arrow[r, "\star"] \arrow[d, "\star"] & \Delta^{\{0,1,2\}} \cup \Delta^{\{2,3\}} \arrow[d, "\star"] & \Delta^{\{0,2\}} \cup \Delta^{\{2,3\}} \arrow[l] \arrow[d, "\star"] \\
\Delta^{\{0,1\}} \cup \Delta^{\{1,2,3\}} \arrow[r, "\star"] & \Delta^3 & \Delta^{\{0,2,3\}} \arrow[l] \\
\Delta^{\{0,1\}} \cup \Delta^{\{1,3\}} \arrow[r, "\star"] \arrow[u] & \Delta^{\{0,1,3\}} \arrow[u] & \Delta^{\{0,3\}} \arrow[l] \arrow[u]
\end{tikzcd}$$
and show that all maps labelled with a $\star$ are inner anodyne.

Exercise 67. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Let $a\colon x \to x'$ and $b\colon y \to y'$ be morphisms in $\mathcal{C}$. Then there is a homotopy commutative diagram of Kan complexes
$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}}(x',y) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{D}}(fx',fy) \arrow[d] \\
\mathrm{map}_{\mathcal{C}}(x',y') \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{D}}(fx',fy') \arrow[d] \\
\mathrm{map}_{\mathcal{C}}(x,y') \arrow[r] & \mathrm{map}_{\mathcal{D}}(fx,fy')
\end{tikzcd}$$
induced by precomposition with a, respectively fa and postcomposition with b, respectively fb.

Deduce that there is a canonical functor F from Cat^1_∞ (the 1-category of ∞-categories) to $\mathrm{Cat}_{\pi(\mathrm{Kan})}$, the category of categories enriched in the homotopy category of Kan complexes, where $F(\mathcal{C})$ has objects the same as $\mathcal{C}$ and the hom object from x to y is given by the image of $\mathrm{map}_{\mathcal{C}}(x,y)$ in $\pi(\mathrm{Kan})$.

A.5. **Section 4.**

Exercise 68. Show that $[n] \star [m] = [n+m+1]$. Further show that $\mathcal{C} \star [0] = \mathcal{C}^{\triangleright}$ and $[0] \star \mathcal{C} = \mathcal{C}^{\triangleleft}$.

Exercise 69. For categories $\mathcal{C}$ and $\mathcal{D}$ we have that $\mathrm{N}(\mathcal{C}) \star \mathrm{N}(\mathcal{D}) \cong \mathrm{N}(\mathcal{C} \star \mathcal{D})$. In particular, there is a canonical isomorphism $\Delta^i \star \Delta^j \cong \Delta^{i+1+j}$.

Exercise 70. The functors $X \star -$ and $- \star X$ as functors $\mathrm{sSet} \to \mathrm{sSet}$ preserve pushouts. Find an example of a colimit that is not preserved by $X \star -$.

Exercise 71. The slice/join adjunction induces a bijection of lifting problems between diagrams of the kind

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X_{\varphi/} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X_{\varphi i/} \times_{Y_{f\varphi i/}} Y_{f\varphi/} \end{tikzcd}$$

and diagrams of the kind

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend left, "\varphi"] & A \star T \amalg_{A\star S} B \star S \arrow[r] \arrow[d] & X \arrow[d] \\ & B \star T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Exercise 72. Show that

$$\Lambda^n_j \star \Delta^m \cup \Delta^n \star \partial\Delta^m = \Lambda^{n+1+m}_j$$

and that

$$\partial\Delta^m \star \Delta^n \cup \Delta^m \star \Lambda^n_j = \Lambda^{n+1+m}_{m+1+j}$$

For this, determine explicitly the following sub simplicial sets of Δ^{n+1+m}:

(1) $\partial\Delta^n \star \Delta^m$,
(2) $\Lambda^n_j \star \Delta^m$,
(3) $\Delta^m \star \partial\Delta^n$,
(4) $\Delta^m \star \Lambda^n_j$.

Exercise 73. For an object x in an ∞-category $\mathcal{C}$, show that the canonical map $\mathcal{C}_{x/} \to \mathcal{C}$ is a left fibration and that $\mathcal{C}_{/x} \to \mathcal{C}$ is a right fibration.

Exercise 74. Show that for an object x of a general simplicial set X, the canonical map $X_{x/} \to X$ is not in general a left fibration.

Exercise 75. Show that an inner fibration $f\colon X \to Y$ is inner anodyne if and only if it is an isomorphism.

Exercise 76. Show that $\Delta^0 \to J$ is not inner anodyne.

Exercise 77. Show that the intersection of left and right anodyne maps strictly contains the inner anodyne maps.

Exercise 78. Show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is conservative if and only if the induced functor $hF\colon h\mathcal{C} \to h\mathcal{D}$ between the homotopy categories is conservative. Furthermore, show that the canonical functor $\mathcal{C} \to \mathrm{N}(h\mathcal{C})$ is conservative.

Exercise 79. Show that a functor $p\colon \mathcal{C} \to \mathcal{D}$ is conservative if and only if the following diagram is a pullback.

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{D}^{\simeq} \arrow[r] & \mathcal{D}. \end{tikzcd}$$

Exercise 80. An inner fibration $\mathcal{C} \to \mathcal{D}$ between ∞-categories is an isofibration if and only if the induced functor $\mathrm{N}(h\mathcal{C}) \to \mathrm{N}(h\mathcal{D})$ is an isofibration.

Exercise 81. A functor $\mathcal{C} \to \mathcal{D}$ between ∞-categories is an isofibration if and only if $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an isofibration.

Exercise 82. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Show that the inclusion $\mathcal{C}_0 \to \mathcal{C}$ is an isofibration if $\mathcal{C}_0$ is closed under equivalences in $\mathcal{C}$, i.e. that if $x \in \mathcal{C}_0$ and $y \in \mathcal{C}$ is equivalent to x, then y is also in $\mathcal{C}_0$.

Exercise 83. Show that if $f\colon x \to y$ is an equivalence, then the maps constructed in the last lecture $\mathcal{C}_{/x} \to \mathcal{C}_{/y}$ and $\mathcal{C}_{y/} \to \mathcal{C}_{x/}$ are Joyal equivalences.

Exercise 84. Show that there exists a functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories which is conservative, but does not satisfy the RLP wrt $\Delta^1 \to J$. This might be a hard one. Hint: Consider the map $J \to \mathcal{S}(|J|)$; as a map between Kan complexes it is clearly conservative. The idea now should be that given a 1-simplex in $\mathcal{S}(|J|)$ there should be more than extension to J. choose two distinct such extensions. They provide a commutative diagram

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & J \arrow[d] \\ J \arrow[r] & \mathcal{S}(|J|) \end{tikzcd}$$

If there exists a lift in this extension problem, then the map $J \to J$ restricts to the identity on Δ^1 and thus must be the identity of J. This would imply that for any map $\Delta^1 \to \mathcal{S}(|J|)$ there is a unique extension to J. This is simply not correct.

A.6. **Section 5.**

Exercise 85. Show that a left fibration $p\colon \mathcal{C} \to \mathcal{D}$ is a Kan fibration provided that $\mathcal{D}$ is an ∞-groupoid.

Exercise 86. Let $p\colon \mathcal{C} \to \mathcal{D}$ be an isofibration. Show that the set of all monomorphisms $K \to L$ such that the induced map

$$\mathcal{C}^L \to \mathcal{C}^K \times_{\mathcal{D}^K} \mathcal{D}^L$$

is again an isofibration is a saturated set.

A.7. **Section 7.**

Exercise 87. The class of essentially surjective and fully faithful functors satisfies the 3-for-2 property.

Exercise 88. Suppose that $f\colon \mathcal{C} \to \mathcal{D}$ is a Joyal equivalence. Then show that also the restricted map $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a Joyal equivalence.

Exercise 89. Show that two functors $f, g\colon \mathcal{C} \to \mathcal{D}$ are naturally equivalent if and only if f and g represent the same element of $\pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})$.

Exercise 90. Show that a Joyal equivalence $f\colon \mathcal{C} \to \mathcal{D}$ induces an equivalence of ordinary categories $h\mathcal{C} \to h\mathcal{D}$.

Exercise 91. Show that a functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a Joyal equivalence if and only if for every simplicial set K, the induced map

$$f_*\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{D})$$

is a Joyal equivalence.

Exercise 92. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Show that the induced map on nerves is inner anodyne if and only if F is an isomorphism.

A.8. **Section 8.**

Exercise 93. Let $\mathcal{C}$ be an ∞-category and S a set of morphisms of $\mathcal{C}$. Then S is called saturated if it coincides with the set $\bar{S}$ of all morphisms that are sent to equivalences under the functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$. Show that

(1) If $S \subseteq T$ and T is saturated, then $\bar{S} \subseteq T$ as well.
(2) Two sets of morphisms S and T of $\mathcal{C}$ give rise to Joyal equivalent localizations (compatible with the map from $\mathcal{C}$) if and only if $\bar{S} = \bar{T}$.

Exercise 94. Prove or disprove the following statements:

(1) For every ∞-category $\mathcal{C}$, a set S of morphisms of $\mathcal{C}$ and a set T of morphisms of $\mathcal{C}[S^{-1}]$, the functor $\mathcal{C} \to \mathcal{C}[S^{-1}][T^{-1}]$ is a localization of $\mathcal{C}$.
(2) For every ∞-category $\mathcal{C}$, a set S of morphisms of $\mathcal{C}$ and a set T of morphisms of $\mathcal{C}$, the functor $\mathcal{C} \to \mathcal{C}[S^{-1}][T^{-1}]$ is a localization of $\mathcal{C}$. Here, we view morphisms of $\mathcal{C}$ as morphisms of $\mathcal{C}[S^{-1}]$ via the canonical functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$.

Exercise 95. Let $\mathcal{C}$ be an ordinary category and S a set of morphisms. Show that $\mathcal{C} \to h\mathcal{C}[S^{-1}]$ is the initial functor $\mathcal{C} \to \mathcal{D}$ (up to natural isomorphism) between ordinary categories sending S to isomorphisms.

Exercise 96. Let $\mathcal{C}$ be an ordinary category and S a set of morphisms. Show that every morphism in $h\mathcal{C}[S^{-1}]$ can be represented by a zig zag of morphisms in $\mathcal{C}$, such that the maps pointing in the wrong direction are contained in S.

Exercise 97. Let $\mathcal{C}$ be an ordinary category. Consider a pushout of categories

$$\begin{array}{ccc} \coprod\limits_{f \in \mathrm{Mor}(\mathcal{C})} [1] & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \coprod\limits_{f \in \mathrm{Mor}(\mathcal{C})} J & \longrightarrow & \mathcal{D} \end{array}$$

where $\mathrm{Mor}(\mathcal{C})$ denotes the set of all morphisms of $\mathcal{C}$. Show that $\mathcal{D}$ is a groupoid.

Exercise 98. In this exercise you may use the fact that the unit map $K \to \mathcal{S}(|K|)$ is a homotopy equivalence for any Kan complex K. Recall that a map of simplicial sets is a weak equivalence if its geometric realization is a homotopy equivalence and let X be a simplicial set. Prove or disprove the following statements.

(1) the unit map $X \to \mathcal{S}(|X|)$ is a monomorphism,
(2) the unit map $X \to \mathcal{S}(|X|)$ is a weak equivalence,
(3) the unit map $X \to \mathcal{S}(|X|)$ is anodyne.

Exercise 99. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C} \to \mathcal{S}(|\mathcal{C}|)$ be the unit map of the adjunction $(\mathcal{S}, |-|)$. Show that this is a localization of $\mathcal{C}$ along all morphisms.

Exercise 100. Show that the factorization $\mathcal{C} \to P(f) \to \mathcal{D}$ for a functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is functorial, i.e. that for every solid commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & P(f) \arrow[r] \arrow[d, dashed] & \mathcal{D} \arrow[d] \\ \mathcal{C}' \arrow[r] & P(f') \arrow[r] & \mathcal{D}' \end{tikzcd}$$

a dashed arrow exists making both small squares commute.

A.9. **Section 9.**

Exercise 101. Consider a pushout diagram of simplicial sets

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

in which $X \to X'$ is a monomorphism and $X \to Y$ is a Joyal equivalence. Show that the map $X' \to Y'$ is also a Joyal equivalence.

Exercise 102. Let $X \to X'$ and $Y \to Y'$ be a Joyal equivalences between simplicial sets. Show that both maps $X \amalg Y \to X' \amalg Y'$ and $X \times Y \to X' \times Y'$ are Joyal equivalences.

Exercise 103. Show that

(1) A retract of a Joyal equivalence is a Joyal equivalence,
(2) The set of monomorphisms which are also Joyal equivalences is saturated

Prove or disprove that the set of Joyal equivalences saturated.

Exercise 104. Recall that a map $f\colon X \to Y$ is said to admit a pre-inverse if there exists maps $g\colon Y \to X$ and $\tau\colon \Delta^1 \to \mathrm{Hom}(X, X)$ and $\tau'\colon \Delta^1 \to \mathrm{Hom}(Y, Y)$ such that

(1) $\tau_\varepsilon = \mathrm{id}_X$ and $\tau_{1+\varepsilon} = gf$, where $\varepsilon \in \{0, 1\} \cong \mathbb{Z}/2$,
(2) $\tau'_\varepsilon = \mathrm{id}_Y$ and $\tau'_{1+\varepsilon} = fg$, where again $\varepsilon \in \{0, 1\} \cong \mathbb{Z}/2$,
(3) for all objects x of X, the morphism $\tau(x)\colon \Delta^1 \to X$ represents a degenerate edge of X, and for all objects y of Y, $\tau'(y)\colon \Delta^1 \to Y$ represents a degenerate edge of Y.

Show that a map $f\colon X \to Y$ which admits a pre-inverse is a Joyal equivalence.

Exercise 105. Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories which induces a surjection on 0-simplices and is a Joyal equivalence. Show that p is a trivial fibration.

Exercise 106. Show that for any two simplicial sets, $X \star Y$ is a retract of $X \diamond Y$.

Exercise 107. Show that the canonical map

$$|\underline{\mathrm{Hom}}(A, B)| \longrightarrow \mathrm{map}(|A|, |B|)$$

is a homotopy equivalence. You may use the fact that both the unit map $A \to \mathcal{S}(|A|)$ and the counit map $|\mathcal{S}(X)| \to X$ are weak equivalences.

The goal of the following exercises is to (almost) give a proof of the fact that anodyne maps are precisely those monomorphisms which are weak equivalences. Precisely, we will show that it is implied by the following statement. Let $p\colon X \to Y$ be a Kan fibration. Then there exists factorization of p as

$$X \xrightarrow{\alpha} Z \xrightarrow{\beta} Y$$

where β is a trivial fibration (i.e. has the RLP with respect to monomorphisms) and α is a minimal fibration. The following is what we need to know about minimal fibrations:

- a minimal fibration is a Kan fibration,
- a minimal fibration $\alpha\colon X \to Z$ is locally trivial, i.e. for every simplex $\Delta^n \to Z$, the pulled back fibration is isomorphic (over Δ^n) to a projection $\Delta^n \times B \to \Delta^n$.

Using this, exercises 127–131 will prove our main result:

Exercise 108. Show that a Kan fibration $p\colon X \to Y$ is a trivial fibration if and only if its fibres are contractible. **Hint:** For the interesting direction, consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd}$$

and for future reference we let let $y = b(n)$. Consider the map $g\colon \Delta^n \times \Delta^1 \to \Delta^n$ determined by $g(k, 0) = k$ and $g(k, 1) = n$. Argue that the map $\partial\Delta^n \times \{0\} \to \partial\Delta^n \times \Delta^1$ is anodyne, and use g to obtain a map

$$\Delta^1 \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}$$

sending 0 to the original square and 1 to the square

$$\begin{tikzcd}
\partial\Delta^n \arrow[rr] \arrow[d] & & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[urr, dashed] & \Delta^0 \arrow[r, "y"] & Y
\end{tikzcd}$$

where the map $\Delta^0 \to Y$ is the object y. Show that this lifting problem can be solved. Finally, show that this implies that also the original lifting problem can be solved.

Exercise 109. In this exercise we will use minimal fibrations to show that the geometric realization of a Kan fibration $p\colon X \to Y$ is a Serre fibration, i.e. has the RLP wrt the inclusions $D^n \times \{0\} \to D^n \times D^1$. Here, D^n is the n-dimensional topological cube.

(1) Show that the geometric realization of a minimal fibration is a Serre fibration whose fibre is given by the geometric realization of the fibre of the Kan fibration.
(2) Show that the geometric realization of a trivial fibration is a Serre fibration.

Hints: For (1) show that the realization of a locally trivial map is also locally trivial. Then show that a locally trivial map of spaces is a Serre fibration. For (2) show that there exists a monomorphism $X \to W$ with W a contractible Kan complex. Consider the maps $X \to W \times Y \to Y$. Show that the latter is a trivial fibration and deduce that p is a retract of $W \times Y \to Y$.

Exercise 110. Show that a Kan fibration $p\colon X \to Y$ which is in addition a weak equivalence has contractible fibres.

Exercise 111. Show that a monomorphism $i\colon A \to B$ is a weak equivalence if and only if it is anodyne.

Exercise 112. Show that a cocartesian fibration $p\colon X \to Y$ whose fibres are Joyal equivalent to Δ^0 is a trivial fibration.

Exercise 113. Let P be a poset and $\mathcal{C}$ an ∞-category. Suppose given a function $f\colon P \to \mathrm{ob}(\mathcal{C})$ having the following property: Whenever $x \leq y$ are elements of P, then the anima of maps $\mathrm{map}_{\mathcal{C}}(fx, fy)$ is contractible.

(1) Show that there exists a functor $F\colon P \to \mathcal{C}$ extending the given function f on objects,
(2) Show that any two such extensions are equivalent.

The “correct” version of (2) of course is the following: Show that there is a contractible anima parametrizing all possible choices of such extensions, i.e. that in the pullback diagram

$$\begin{tikzcd}
\mathrm{Ext}(f) \arrow[r] \arrow[d] & \mathrm{Fun}(P, \mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r] & \mathrm{Fun}(P_0, \mathcal{C})
\end{tikzcd}$$

the simplicial set $\mathrm{Ext}(f)$ is a contractible Kan complex. Can one replace P by an arbitrary 1-category and obtain the same results?

Exercise 114. Let $\mathcal{C}$ be an ∞-category and consider for each $i \geq 0$ a morphism $x_i \to x_{i+1}$ between objects in $\mathcal{C}$. Show that these maps assemble into a functor $\mathbb{N} \to \mathcal{C}$.

Exercise 115. Prove or disprove that an isofibration $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a trivial fibration if and only if its fibres are Joyal equivalent to Δ^0.

A.10. **Section 10.**

Exercise 116. Let $\mathcal{C}$ be an ∞-category and consider the inner fibration $p\colon \mathcal{C} \to \Delta^0$. Show that a morphism f in $\mathcal{C}$ is p-(co)cartesian if and only if f is an equivalence.

Exercise 117. Show that given a morphism between two squares of ∞-categories in which each comparison map is a Joyal equivalence, then the one square is homotopy cartesian if and only if the other is.

Exercise 118. Suppose given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in which the map p is a (co)cartesian fibration. Show that also the map p' is a (co)cartesian fibration.

A.11. **??.**

Exercise 119. Show that the functors $(-)^\flat, (-)^\sharp\colon \mathrm{sSet} \to \mathrm{sSet}_+$ and the functors $u, m\colon \mathrm{sSet}_+ \to \mathrm{sSet}$ are involved in various adjunctions. Here, u is the forgetful functor and m is the functor which sends a marked simplicial set (X, S) to the smallest sub simplicial set $X_0 \subseteq X$ containing S.

Exercise 120. Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration and K a marked simplicial set equipped with a map $f\colon K \to \mathcal{C}^\sharp$. Consider the sub-$\infty$-category of $\mathrm{Fun}_f^{\mathrm{mcc}}(K, \mathcal{E})$ on those 1-simplices whose corresponding map $K \times \Delta^1 \to \mathcal{E}$ is a map of marked simplicial sets $K \times (\Delta^1)^\sharp \to \mathcal{E}^\natural$, i.e. we consider only those transformations of functors whose components are pointwise p-cocartesian. Then this sub-∞-category is given by $\mathrm{Fun}_f^{\mathrm{mcc}}(K, \mathcal{E})^{\simeq}$.

Exercise 121. Show that if $K = \Delta^0$ and $f\colon \Delta^0 \to \mathcal{C}$ picks out an object z of $\mathcal{C}$, then $\mathrm{Fun}^{\mathrm{cc}}(K, \mathcal{E}) \cong \mathcal{E}$ and $\mathrm{Fun}_f^{\mathrm{cc}}(K, \mathcal{E}) \cong \mathcal{E}_z$.

Exercise 122. Let $X \to Y$ be a cocartesian fibration and $f\colon K \to Y^\sharp$ a map of marked simplicial sets. Show that $\mathrm{Map}_f^\flat(K, X^\natural)$ is an ∞-category and that $\mathrm{Map}_f^\sharp(K, X^\natural)$ is the largest sub ∞-groupoid inside $\mathrm{Map}_f^\flat(K, X^\natural)$.

Exercise 123. Show that the functor $(-)^\flat\colon \mathrm{sSet} \to \mathrm{sSet}^+$ sends Joyal equivalences to marked equivalences.

Exercise 124. Show that the category of marked simplicial sets is canonically enriched in simplicial sets.

Exercise 125. Show that the functor $LF\colon \mathrm{sSet} \to \mathrm{sSet}$ of Theorem 12.8 preserves colimits and monomorphisms.

Exercise 126. Show that the initial vertex maps assemble to a natural transformation $LF \Rightarrow \mathrm{id}$.

Exercise 127. Suppose given a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "f_0"] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & \dots \\ B_0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \dots \end{tikzcd}$$

in which the maps f_i are Joyal equivalences and all horizontal maps are monomorphisms. Then the induced map $f\colon A = \operatorname{colim} A_i \to \operatorname{colim} B_i = B$ is a Joyal equivalence.

Exercise 128. Show that for an ∞-groupoid X, there is a canonical Joyal equivalence $X^{\mathrm{op}} \simeq X$.

Exercise 129. Show that the functor $\mathrm{N}(\Delta^{\mathrm{op}}_{/\mathcal{C}}) \to \mathcal{C}$ is full in the sense that every morphism of $\mathcal{C}$ is the image of a morphism under this functor. Use this to show the composition $\mathcal{C} \to \mathrm{Cat}_\infty$ induces on objects and morphisms the constructions we have done earlier.

Exercise 130. Suppose given a cocartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$. Recall that we have constructed for every edge $f\colon \Delta^1 \to \mathcal{C}$ a functor $\mathcal{E}_x \times \Delta^1 \to \mathcal{E}$, whose restriction to $\mathcal{E}_x \times \{1\}$ is given by $f_!$. Show that this functor may equivalently be constructed as follows. Consider the diagram

$$\begin{tikzcd} & \mathcal{E}_x \times \Delta^{\{0\}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{E}_x \times \{1\} \arrow[r] & \mathcal{E}_x \times \Delta^1 \arrow[r, "f\circ\mathrm{pr}"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

and show that a dashed arrow exists having all the above properties.

Exercise 131. Given a cocartesian fibration $\mathcal{E} \to \mathcal{C} \times \Delta^1$, construct a functor $\mathcal{E}_0 \to \mathcal{E}_1$ which commutes with the projections to $\mathcal{C}$. Here, $\mathcal{E}_i$ is the pulled back cocartesian fibration along the inclusion $\mathcal{C} \times \{i\} \to \mathcal{C} \times \Delta^1$. Show that this functor is a morphism of cocartesian fibrations. Likewise, construct for a cocartesian fibration $\mathcal{E} \to \mathcal{C} \times \Delta^2$ a 2-simplex in the ∞-category $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$. If you are eager, do this for general n instead of 2.

Exercise 132. Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories. Suppose that for every map $f\colon \Delta^1 \to \mathcal{D}$, the induced map $\Delta^1 \times_{\mathcal{D}} \mathcal{C} \to \Delta^1$ is a Joyal equivalence. Show that p is a Joyal equivalence.

Exercise 133. Let $\mathcal{C}$ be an ∞-category and x and object of $\mathcal{C}$. Show that the object represented by $\Delta^0 \to \mathcal{C}_{/x}$ which is adjoint to the map $\Delta^0 \star \Delta^0 \to \mathcal{C}$ given by id_x is a terminal object.

Exercise 134. Let $\mathcal{C}$ be an ∞-category and x and initial object of $\mathcal{C}$. Show that if y is equivalent to x, then y is also initial in $\mathcal{C}$.

Exercise 135. Suppose that $\mathcal{C}_{\text{term}}$ is not empty. Show that any terminal object of $\mathrm{Fun}(K, \mathcal{C})$ takes values in $\mathcal{C}_{\text{term}}$.

Exercise 136. Show that there exists $\mathcal{C}$ and K such that $\mathrm{Fun}(K, \mathcal{C})$ has a terminal object but $\mathcal{C}$ does not.

Exercise 137. Show that there exists a simplicial set X such that the above described map $\mathrm{Tw}(X) \to X \times X^{\mathrm{op}}$ is not a right fibration.

Exercise 138. Prove or disprove that the diagram

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{D}) \arrow[d] \\ \mathcal{C} \times \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

is homotopy cartesian if and only if f is fully faithful.

Exercise 139. Let $p\colon K \to \mathcal{C}$ be a diagram. Suppose that there is a colimit cone $\tilde{p}$ with colimit x in $\mathcal{C}$. Let y be an object of $\mathcal{C}$ which is equivalent to x. Show that y is also the colimit of a colimit cone of p.

Exercise 140. Let $p\colon K \to \mathcal{C}$ be a diagram and let $q\colon I \to \mathcal{C}_{p/}$ be a further diagram. Let $\bar{q}\colon I \star K \to \mathcal{C}$ be the associated map. Then there is an isomorphism $(\mathcal{C}_{p/})_{q/} \cong \mathcal{C}_{\bar{q}/}$. Likewise, there is an isomorphism $(\mathcal{C}_{/p})_{q/} \cong (\mathcal{C}_{q'/})_{/p'}$ where q' is the restriction of $q\colon I \star K \to \mathcal{C}$ to I and p' is adjoint to $\bar{q}$.

Exercise 141. Let K be the coequalizer of two monomorphisms $f, g\colon A \to B$ and let $F\colon K \to \mathcal{C}$ be a functor. Suppose that the restrictions of F to B and A admit colimits and that $\mathcal{C}$ admits coequlizers. Show that then F admits a colimit.

Exercise 142. A map $f\colon K \to L$ is cofinal if and only if $f^{\mathrm{op}}\colon K^{\mathrm{op}} \to L^{\mathrm{op}}$ is coinitial.

Exercise 143. Show that left anodyne maps do not satisfy the left cancellation property among monomorphisms.

Exercise 144. Show that the set of right anodyne maps $i\colon A \to B$ whose pullback along any left fibration is again right anodyne is a saturated set and satisfies the right cancellation property.

Exercise 145. A right deformation retract is a monomorphism $i\colon A \to B$ such that there exists a retraction $p\colon B \to A$ and a simplicial homotopy $H\colon \Delta^1 \times B \to B$ with $H(0) = \mathrm{id}_B$, $H(1) = ip$ and whose restriction to $\Delta^1 \times A$ is constant the identity of A.

(1) Show that for every simplicial set, the map $\{1\} \times K \to \Delta^1 \times K$ is a right deformation retract.
(2) Show that a right deformation retract is a right anodyne map.
(3) Show that the pullback of a right deformation retract along a left fibration is again a right deformation retract.

Exercise 146. Give an example of a proper (or smooth) map which is not universally proper (or smooth).

Adjunctions.

Exercise 147. Let $p\colon \mathcal{E} \to \mathcal{D}$ be a cartesian fibration. Show that the canonical map $\mathcal{E}_d \to \mathcal{E}_{d/}$ admits a right adjoint. Deduce that for a cartesian fibration, the canonical map $\mathcal{E}_d \to \mathcal{E}_{d/}$ is a weak equivalence.

Exercise 148. Show that a fully faithful and essentially surjective functor is invertible.

Exercise 149. Let $F\colon K \to \mathrm{Cat}_\infty$ be a functor. Let $p\colon \mathcal{E} \to K$ be the associated cocartesian fibration. Show that the colimit of F is given by $\mathcal{E}[\mathrm{cc}^{-1}]$, where the set cc is the set of p-cocartesian edges. Likewise, show that the limit is given by $\mathrm{Fun}_K^{\mathrm{cc}}(K, \mathcal{E})$ is given by the category of cocartesian sections of p. Deduce the analogs for functors with values in An.

Exercise 150. Consider for a diagram $F\colon K \to \mathcal{C}$ and an object x of $\mathcal{C}$ the functor

$$\mathrm{map}_{\mathcal{C}}(F(-), x)\colon K \to \mathrm{An}^{\mathrm{op}}.$$

Show that its limit is given by $\mathrm{Map}_{\mathcal{C}}(F, x)$. Deduce that for a functor $f\colon \mathcal{C} \to \mathcal{D}$, with right adjoint g, there exists a canonical equivalence $\mathrm{Map}_{\mathcal{D}}(fF, z) \simeq \mathrm{Map}_{\mathcal{C}}(F, gz)$.

Exercise 151. Let $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{D} \to \mathcal{E}$ be composable functors which admit right adjoints g and g'. Show that then gg' is a right adjoint of $f'f$.

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