Little things to know about model categories

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Abstract

Notes in evolution!

1 Little facts about pushouts, liftings, and retracts

It is well known that pushouts compose horizontally and vertically. This can be rephrased as the following two commutation properties:

$p.o.(\circ)$ A pushout of a composition is a composition of pushouts.

$p.o.(p.o.)$ A pushout of a pushout is a pushout.

Property $p.o.(\circ)$ extends to transfinite compositions. We shall also use the following decomposition:

$+ = \circ(p.o.)$ A coproduct of morphisms is a (transfinite) composition of pushouts.

Indeed, $i_1 + i_2 = (i_1 + id) \circ (id + i_2)$.

Definition 1.1 *A lifting of a commutative diagram $g \circ h = k \circ f$ is a morphism l such that $l \circ f = h$ and $g \circ l = k$. We say that the pair (f, g) admits lifting if every commutative square of this kind (with f, g fixed) has a lifting. The convention is to draw the square with f, g as vertical East and West morphisms, respectively, and the lifting as pointing toward North-East. For any class of morphisms we write*

- $\rho(A) = \{g \mid \forall\, f \in A \;\; (f, g)$ *admits a lifting*$\}$.
- $\lambda(B) = \{f \mid \forall\, g \in B \;\; (f, g)$ *admits a lifting*$\}$.

We have the usual properties of orthogonals: λ and ρ are contravariant, $A \subseteq \lambda\rho(A)$ and $\rho(A) = \rho\lambda\rho(A)$, and dually for $\lambda(B)$. Also, one has clearly $A \subseteq \lambda(B)$ if and only if $B \subseteq \rho(A)$, since both inclusions just mean that any pair of morphisms, the first in A and the second in B, admits lifting. But of course, there is no relation between the converse inclusions.

Definition 1.2 *A retract of a morphism f is a morphism g together with morphisms u, v, u', v' such that*

$$v \circ u = id \quad f \circ u = u' \circ g \quad f \circ v = v' \circ f \quad v' \circ u' = id$$

A more economical way to say this is that (u, u') and (v, v') form a retraction pair between the *objects* f, g of the arrow category. By fixing a common codomain for f, g and by requiring $u' = v' = id$ we get a tighter notion of retraction, relative to a the comma category.

Example 1.3 *If $ps = id$, then p is a retract of sp. (Indeed, write ps as bottom horizontal morphism and $id \circ id$ as top horizontal one.)*

Lemma 1.4

1. *Every $\rho(A)$ or $\lambda(B)$ contains the identity morphisms (and more generally all the isos), is closed under composition, and under retracts.*
2. *Every $\rho(A)$ is closed under pullbacks.*
3. *Every $\lambda(B)$ is closed under pushouts and under transfinite compositions.*

PROOF. To show that $f, g \in \rho(A)$ implies $g \circ f \in \rho(A)$, one makes two successive liftings. To show the closure of $\lambda(B)$ under retracts, it suffices to put a lifting candidate square aside on the right of the retraction situation. One then derives a lifting for the rectangle obtained from the two rightmost squares, and one finally composes that lifting with the South-West horizontal morphism of the diagram to get the desired lifting. For $\rho(A)$, one proceeds likewise, after putting the candidate now on the left of the retraction rectangle.

The recipe for the stability with respect to pullbacks or pushouts is similar: one just has to put a lifting candidate square aside with the universal square, then derive a lifting for the rectangle, and the required lifting follows by universality. Finally, given a candidate commutative square for a transfinite composition f of morphisms in $\lambda(B)$, one may use any factorisation $f = h \circ g$ with $g \in \lambda(B)$ to get a lifting morphism from the target of g to the north-west corner C of the original square. In this way we can form a cocone to C, and from there the desired lifting follows by universality (remembering that the target of f is a colimit object). □

Note that, by $+ = \circ(p.o.)$ (cf. section 1), every $\lambda(B)$ is closed under (arbitrary) coproducts.

Note also that λ is "oriented towards colimits", while ρ is "oriented towards limits" (it is closed under transfinite "reverse composition" by duality). One may wonder whether there is an interesting general statement about $\lambda(B)$ of which both properties listed under (3) would be special cases.

Lemma 1.5

1. *If $id \circ f = g \circ h$ admits a lifting, then f is a retract of h.*
2. *If $g \circ h = f \circ id$ admits a lifting, then f is a retract of g.*

Proof. Note that the second situation is (up to the NW-SE symmetry) the dual of the first, hence we need only consider the first part of the statement, which follows obviously from displaying the data as two adjacent commuting squares, where k is the lifting. The third commutation in the assumptions serves to establish that the above horizontal composite is the identity. $\square$

2 Equivalent definitions of closed model categories

We adopt here the following definition, based on weak factorisation systems, which is quite economical (at least in words!), and which we shall therefore call "minimal".

Definition 2.1 *A weak factorisation system on a category $\mathbb{C}$ is given by two classes A, B of morphisms such that $A = \lambda(B)$ and $B = \rho(A)$ and such $\mathbb{C}_1 = B \circ A$.*

Definition 2.2 *A (closed) model category is a complete and cocomplete category endowed with three classes of morphisms $\mathbf{F}, \mathbf{C}, \mathbf{W}$ satisfying the following two properties:*

(WF) Both pairs $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ and $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ form weak factorisation systems. The members of $\mathbf{C}, \mathbf{F}, (\mathbf{C} \cap \mathbf{W}), (\mathbf{F} \cap \mathbf{W})$ are called cofibrations, fibrations, acyclic cofibrations, and acyclic fibrations, respectively.

(2/3) Given composable f, g, then as soon as two of the three morphisms in $\{f, g, g \circ f\}$ is in $\mathbf{W}$, so is the third one.

We show successively that this definition is equivalent to (an optimised version of) the one originally given by Quillen [6][section I.5] (and used in [4]), and with the definition given in most texbooks or tutorials on model categories [5]. The move from the minimal definition to Quillen's definition is just a trade-off between the two equalities $\mathbf{F} \cap \mathbf{W} = \rho(\mathbf{C})$ and $\mathbf{C} \cap \mathbf{W} = \lambda(\mathbf{F})$ on one hand and a single equality defining $\mathbf{W}$ in terms of $\mathbf{F}$ and $\mathbf{C}$.

Proposition 2.3 *Let $\mathbb{C}$ be a complete and cocomplete category with three distinguishes classes of morphisms $\mathbf{F}, \mathbf{C}, \mathbf{W}$ such that $\mathbf{W}$ satisfies the (2/3) axiom. Then this provides a model category structure if and only if the following properties are satisfied:*

1. $\lambda(\mathbf{F} \cap \mathbf{W}) \subseteq \mathbf{C}$, $\rho(\mathbf{C} \cap \mathbf{W}) \subseteq \mathbf{F}$, $\mathbf{W} = \rho(\mathbf{C}) \circ \lambda(\mathbf{F})$.
2. $\mathbb{C}_1 = \mathbf{F} \circ (\mathbf{C} \cap \mathbf{W}) = (\mathbf{F} \cap \mathbf{W}) \circ \mathbf{C}$.

Proof. Given a model category defined according to the minimal definition, all we have to check is that $\mathbf{W} = \rho(\mathbf{C}) \circ \lambda(\mathbf{F})$. Take $w \in \mathbf{W}$ and decompose it through any of the two weak factorisation systems: the (2/3) property gives us $\mathbf{W} = (\mathbf{F} \cap \mathbf{W}) \circ (\mathbf{C} \cap \mathbf{W}) = \rho(\mathbf{C}) \circ \lambda(\mathbf{F})$.

Conversely, we have to show $\rho(\mathbf{C}) = \mathbf{F} \cap \mathbf{W}$ and $\lambda(\mathbf{F}) = \mathbf{C} \cap \mathbf{W}$. Indeed:

- $\rho(\mathbf{C}) \subseteq \mathbf{W}$ is an immediate consequence of the equality defining $\mathbf{W}$.
- $\rho(\mathbf{C}) \subseteq \rho(\mathbf{C} \cap \mathbf{W}) \subseteq \mathbf{F}$.
- Let $f \in \mathbf{F} \cap \mathbf{W}$. Because $f \in \mathbf{W}$, we have $f \circ id \in \rho(\mathbf{C}) \circ \lambda(\mathbf{F})$, and because $f \in \mathbf{F}$ we can apply Lemma 1.5, hence f is a retract of a morphism in $\rho(\mathbf{C})$ and we conclude by Lemma 1.4.

The equality $\lambda(\mathbf{F}) = \mathbf{C} \cap \mathbf{W}$ is proved likewise. This concludes the proof (remember that $\mathbf{F} \cap \mathbf{W} \subseteq \rho(\mathbf{C})$ implies $\mathbf{C} \subseteq \lambda(\mathbf{F} \cap \mathbf{W})$). □

As a consequence of Proposition 2.3, we see that there is also a "(2/3)" property on $\{\mathbf{F}, \mathbf{C}, \mathbf{W}\}$: any choice of two of these three classes in a model category determines the third class.

For the sake of presenting the third, most common definition, we shall split axiom (WF) in three parts (WF1), (WF2), (WF3), which will be spelled out below. The trade-off is now between the four inclusions

(WF3) $\rho(\mathbf{C}) \subseteq \mathbf{F} \cap \mathbf{W}\ ,\ \lambda(\mathbf{F}) \subseteq \mathbf{C} \cap \mathbf{W}\ ,\ \lambda(\mathbf{F} \cap \mathbf{W}) \subseteq \mathbf{C}\ ,\ \rho(\mathbf{C} \cap \mathbf{W}) \subseteq \mathbf{F}$

on one hand and properties of closure by retracts on the other hand.

Proposition 2.4 *Let $\mathbb{C}$ be a complete and cocomplete category with three distinguishes classes of morphisms $\mathbf{F}, \mathbf{C}, \mathbf{W}$ and satisfying the (2/3) axiom. Then this provides a model category structure if and only if the following properties are satisfied:*

(ret) $\mathbf{F}, \mathbf{C}, \mathbf{W}$ *are closed under retracts.*

(WF1) $\mathbf{C} \subseteq \lambda(\mathbf{F} \cap \mathbf{W})$, $\mathbf{F} \subseteq \rho(\mathbf{C} \cap \mathbf{W})$.

(WF2) $\mathbb{C}_1 = \mathbf{F} \circ (\mathbf{C} \cap \mathbf{W}) = (\mathbf{F} \cap \mathbf{W}) \circ \mathbf{C}$.

Proof. Let $\mathbb{C}$ be a category satisfying the axioms listed in the statement. We have to prove that (WF3) holds. This is an easy consequence of

- the use of (WF2);
- the use of Lemma 1.5;
- the closure of each of the four classes $\mathbf{F} \cap \mathbf{W}$, $\mathbf{C} \cap \mathbf{W}$, $\mathbf{C}$, $\mathbf{F}$ under retracts.

For example, given a morphism f in $\rho(\mathbf{C})$, we perform the following steps: factorise it as $(\mathbf{F} \cap \mathbf{W}) \circ \mathbf{C}$, conclude from the lemma (case 1) that f is a retract of a morphism in $\mathbf{F} \cap \mathbf{W}$, and hence is itself in $\mathbf{F} \cap \mathbf{W}$ by the closure property.

Conversely, the closure of $\mathbf{F}$ and $\mathbf{C}$ under retracts follows immediately from the equalities defining $\mathbf{F}$ and $\mathbf{C}$ and from Lemma 1.4. The closure of $\mathbf{W}$ under retracts is trickier. We reproduce below an argument of Joyal, which we learned from Worytkiewicz [7]. Let $w \in \mathbf{W}$ and let f be a retract of w. We proceed in two steps.

1. We first treat the special case where $f \in \mathbf{F}$. We can reformulate our goal $f \in \mathbf{W}$ as $f \in \mathbf{F} \cap \mathbf{W} = \rho(\mathbf{C})$. We decompose w as $w_2 \circ w_1$, with $w_1 \in \lambda(\mathbf{F})$ and $w_2 \in \rho(\mathbf{C})$ (cf. Proposition 2.3), which allows us to view the right square of the retraction situation as a lifting situation (exploiting our additional assumption about f). The existence of the corresponding lifting allows us (factorising the North-East horizontal morphism) to exhibit f as a retract of w_2, and we conclude by Lemma 1.4 (stability under retract).

2. Now we consider the general case. We can factorise f as $f = p \circ i$ where $i \in \lambda(\mathbf{F})$ and $p \in \mathbf{F}$. Let u_0 and v_0 be the North-West and North-East horizontal morphisms (and hence $v_0 u_0 = id$), and let u_1 and v_1 be likewise the South morphisms. Take the pushout $ui = ju_0$ of u_0 and i. Let q, v be the universal morphisms induced by $w \circ u_0 = (u_1 p) \circ i$ and by $(iv_0) \circ u_0 = id \circ i$. We can now display the original retraction rectangle as a square of four squares, with the pushout diagram in the North-West corner and $w = q \circ j$ as middle vertical arrow. We make the following steps:

   (a) By Lemma 1.4 (stability under pushout), we have $j \in \lambda(\mathbf{F}) = \mathbf{C} \cap \mathbf{W}$.

   (b) By the (2/3) property, this implies $q \in \mathbf{W}$.

   (c) The lower rectangle of the diagram is a retraction situation: its top composed horizontal morphism is the identity by definition of v, and $p \circ v = v_1 \circ q$ by universality.

   (d) Thanks to the last two properties we are in the special situation examined above, so we get $p \in \mathbf{W}$.

   (e) Finally, we conclude $f \in \mathbf{W}$ by applying once more the (2/3) property.

□

As a final variation, we exhibit a variant of the minimal definition which fits better with the actual way in which the factorisations are proved. We modify axiom (WF) by leaving one of the two factorisation systems unchanged, while, in the other, one inequality is removed and in "compensation" the factorisation is strengthened.

Proposition 2.5 *Let $\mathbb{C}$ be a complete and cocomplete category with three distinguishes classes of morphisms $\mathbf{F}, \mathbf{C}, \mathbf{W}$, where $\mathbf{W}$ satisfies the (2/3) axiom. Then this provides a model category structure if and only if the following properties are satisfied:*

(WF)/2 $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ *forms a weak factorisation system.*

$(\overline{WF1})/2$ $\lambda(\mathbf{F} \cap \mathbf{W}) \subseteq \mathbf{C}$, $\rho(\mathbf{C}) \subseteq \mathbf{F} \cap \mathbf{W}$.

(WF2)/2 $\mathbb{C}_1 = \rho(\mathbf{C}) \circ \mathbf{C}$.

PROOF. A model category according to Definition 2.2 satisfies a fortiori $(\overline{\mathbf{WF1}})/2$, and also $\mathbb{C}_1 = (\mathbf{F} \cap \mathbf{W}) \circ \mathbf{C} = \rho(\mathbf{C}) \circ \mathbf{C}$.

Conversely, we show that the conditions of the statement imply the conditions in Proposition 2.3:

- Since $\rho(\mathbf{C}) \subseteq \mathbf{F} \cap \mathbf{W}$ and $\lambda(\mathbf{F}) \subseteq \mathbf{C} \cap \mathbf{W}$, we have $\rho(\mathbf{C}) \circ \lambda(\mathbf{F}) \subseteq \mathbf{W}$ by the (2/3) property.
- Let $f \in \mathbf{W}$. By (WF2)/2 we can write $f = jq$ with $j \in \rho(\mathbf{C})$ and $q \in \mathbf{C}$. Since $\rho(\mathbf{C}) \subseteq \mathbf{W}$ by $(\overline{\text{WF1}})/2$, we get $q \in \mathbf{W}$ by (2/3), and we conclude that $f \in \rho(\mathbf{C}) \circ \lambda(\mathbf{F})$ by $(WF)/2$.
- Finally, from (WF2)/2 and $(\overline{\text{WF1}})/2$, we have

$$\mathbb{C}_1 = \rho(\mathbf{C}) \circ \mathbf{C} \subseteq (\mathbf{F} \cap \mathbf{W}) \circ \mathbf{C} \subseteq \mathbb{C}_1$$

□

We turn now to the important case where the model category structure is generated from three classes $I, J, \mathbf{W}$, where $\mathbf{W}$ is as above, and where the members of I, J are called the *generating cofibrations* and the *generating acyclic cofibrations*, respectively.

Proposition 2.6 *Let $\mathbb{C}$ be a complete and cocomplete category with three distinguished classes of morphisms $I, J, \mathbf{W}$, where $\mathbf{W}$ satisfies the (2/3) axiom, and such that the following inequalities and factorisation properties hold:*

(IJ) $\rho(I) \subseteq \rho(J)$.

(IWJ) $\lambda\rho(I) \cap \mathbf{W} \subseteq \lambda\rho(J)$.

(IW) $\rho(I) \subseteq \mathbf{W}$.

(JW) $\lambda\rho(J) \subseteq \mathbf{W}$.

(WFIJ) $\mathbb{C}_1 = \rho(J) \circ \lambda\rho(J) = \rho(I) \circ \lambda\rho(I)$.

Then we have a model category structure, setting $\mathbf{F} = \rho(J)$ and $\mathbf{C} = \lambda\rho(I)$, and we have $I \subseteq \mathbf{C}$ and $J \subseteq \mathbf{C} \cap \mathbf{W}$.

In the above axioms, we can replace (IJ)+(JW) with

(JIW) $\lambda\rho(J) \subseteq \lambda\rho(I) \cap \mathbf{W}$.

Applying this change, our axioms reduce to

$$(IW) \quad \rho(I) \subseteq \mathbf{W} \qquad\qquad (IWJ) + (JIW) \quad \lambda\rho(I) \cap \mathbf{W} = \lambda\rho(J)$$

One can also replace (IWJ) with

(JWI) $\rho(J) \cap \mathbf{W} \subseteq \rho(I)$.

so that, applying the two changes, an alternative set of inequalities is

(IW)+(JIW)+(JWI)

PROOF. We check the conditions in Proposition 2.5.

(WF)/2 (a) $\lambda(\mathbf{F}) \subseteq \mathbf{C} \cap \mathbf{W}$ follows from (IJ) and (JW).

(b) $\rho(\mathbf{C} \cap \mathbf{W}) \subseteq \mathbf{F}$ is implied by $J \subseteq \lambda\rho(J) \subseteq \mathbf{C} \cap \mathbf{W}$, where the latter inequality follows from (IJ) and (JW).

The rest of $(WF)/2$ is then given by (IWJ) and (WFIJ) (which also gives us (WF2)/2).

($\overline{\text{WF1}}$)/2 (a) $\lambda(\mathbf{F} \cap \mathbf{W}) \subseteq \mathbf{C}$ follows from (IJ) and (IW).

(b) $\rho(\mathbf{C}) \subseteq \mathbf{F} \cap \mathbf{W}$ is the conjunction of (IJ) and (IW).

Finally, we have $I \subseteq \lambda\rho(I) = \mathbf{C}$ and $J \subseteq \lambda\rho(J) = \lambda(\mathbf{F}) = \mathbf{C} \cap \mathbf{W}$.

The equivalence beween (IJ)+(JW) and (JIW) follows from the observation that $\rho(I) \subseteq \rho(J)$ is equivalent to $\lambda\rho(J) \subseteq \lambda\rho(I)$.

To see that (JWI) holds, notice that it rephrases as $\mathbf{F} \cap \mathbf{W} \subseteq \rho(\mathbf{C})$ and is hence part of the model category structure. Conversely, if $\rho(J) \cap \mathbf{W} \subseteq \rho(I)$ holds, then from WFIJ),(JW), and (2/3) we get

$$\mathbf{W} \subseteq (\rho(J) \cap \mathbf{W}) \circ \lambda\rho(J) \subseteq \rho(I) \circ \lambda\rho(J)$$

from which it follows by Lemmas 1.5 and 1.4 that any morphism in $\lambda\rho(I) \cap \mathbf{W}$ is a retract of a morphism in $\lambda\rho(J)$ and hence is in $\lambda\rho(J)$. □

Note that in fact every model category arises in this way: given a model category $\mathbb{C}$, we can set $I = \mathbf{C}$ and $J = \mathbf{C} \cap \mathbf{W}$, and it is easily checked that the properties of the previous statement hold.

In the litterature, one often finds the (quite ugly) notations $A - inj$ and $A - cof$ for $\rho(A)$ and $\lambda\rho(A)$, respectively.

3 The small object argument

When $\mathbb{C}$ is locally presentable, and when I and J are *sets* of morphisms, then we get axiom (WFIJ) of Proposition 2.6 for free. All we have to know about a locally presentable category is that for any *set* K of objects there exists a limit ordinal κ associated with K such that any morphism $f : A \to \text{colim}\ _{\beta<\kappa} B^\beta$, with $A \in K$ factors through some B^β, i.e. $f = c^\beta \circ f'$ for some $f' : A \to B^\beta$, where c^β is the component of the colimiting cocone at B^β.

Proposition 3.1 *Let $\mathbb{C}$ be a locally presentable category, and let I be a set of morphisms of $\mathbb{C}$. Then the following factorisation property holds:*

$$\mathbb{C}_1 = \rho(I) \circ \lambda\rho(I)$$

Proof. Given a fixed $f : X \to Y$, we can collect all the commutative diagrams $f \circ u = v \circ i$ (f fixed, i ranging over I) into a single commutative square, taking the coproducts of all North-West corners and all South-West corners. One may then take the pushout of the top horizontal morphism and of the left vertical one, and get a factorisation $f = \rho_f \circ \lambda_f$, with $\lambda_f : X \to X_0 \in \lambda\rho(I)$ as a pushout of a coproduct of morphisms of I (cf. Proposition 1.4). Moreover, by construction, for every one of the commutative squares $f \circ u = v \circ i$, there exists w such that

$$v = \rho_f \circ w \quad \text{and} \quad \lambda_f \circ u = w \circ i$$

The idea is to repeat this construction on $\rho_f = f'$, i.e. $\rho_f = \rho_{f'} \circ \lambda_{f'}$, and hence $f = \rho_{f'} \circ (\lambda_{f'}\lambda_f)$, and to iterate these compositions on the right transfinitely up to the limit ordinal κ associated with the set of sources of all the morphisms of I. So, for every $\beta \leq \kappa$ we have $f = \rho^\beta \circ \lambda^\beta$, with (again by Proposition 1.4) $\lambda^\beta : X \to X^\beta \in \lambda\rho(I)$. As initial values we have $\lambda^0 = \lambda_f, \rho^0 = \rho_f, \lambda^1 = \lambda_{f'}\lambda_f, , \rho^1 = \rho_{f'}$. By construction, ρ^κ is the universal arrow from the universal cocone to the cocone with vertex Y formed by the ρ^β's, i.e. we have $\rho^\kappa \circ c^\beta = \rho^\beta$ for all $\beta < \kappa$.

Our proof will be completed if we show that $\rho_\kappa : X^\kappa \to Y$ is in $\rho(I)$. So, let $\rho^\kappa \circ u = v \circ i$ be a commutative square, with $i \in I$. By our choice of κ, we have that $u = c^\beta \circ u'$ factorises through X^β for some $\beta < \kappa$. Hence, using also the universality of ρ^κ, we have $\rho^\kappa \circ u = \rho^\beta \circ u'$, so that the square $\rho^\beta \circ u' = v \circ i$ is commutative. If follows, replacing X, X_0, u above by $X^\beta, X^{\beta+1}, u'$, that there exists w' such that

$$v = \rho^{\beta+1} \circ w' \quad \text{and} \quad \lambda^{\beta+1} \circ u' = w' \circ i$$

Then $c^{\beta+1} \circ w'$ is a lifting, which concludes the proof. □

The argument used in this proof, and the statement itself, are known as the "small object argument".

We have actually proved something stronger. Let *cell*(I) be the collection of transfinite compositions of pushouts of coproducts of elements of I, and let *ret*(A) be the collection of retractions of morphisms of A (in the tighter "comma category" sense mentioned after Definition 1.2).

Proposition 3.2 *Let $\mathbb{C}$ be a locally presentable category, and let I be a set of morphisms of $\mathbb{C}$. Then the following factorisation property holds:*

$$\mathbb{C}_1 = \rho(I) \circ \mathit{cell}(I)$$

Moreover, $\lambda\rho(I)$ is the closure of I under pushouts, transfinite compositions, and retracts, and, more specifically:

$$\lambda\rho(I) = \mathit{ret}(\mathit{cell}(I))$$

Proof. The morphism λ^κ in the proof of Proposition 3.1 is a transfinite composition of pushouts of coproducts, which by properties $+ = \circ(p.o.)$ and $p.o.(\circ)$ is a transfinite composition of pushouts of pushouts, and hence is in *cell*(I) by $p.o.(p.o.)$.

For the last part of the statement, we have $ret(cell(I)) \subseteq \lambda\rho(I)$ by Lemma 1.4, while the converse inclusion follows from Lemma 1.5 and from the first part of the statement applied to $f \in \lambda\rho(I)$. Finally, we have that the closure of I under the three listed operations contains $ret(cell(I))$ and is contained in $\lambda\rho(I)$, hence is equal to both. □

The morale is that if the category is locally presentable, then verifying the model category structure "boils down" to finding a class $\mathbf{W}$ satisfying (2/3) and sets I, J of morphisms satisfying the inequalities of Proposition 2.6. This situation is known as that of a *cofibrantly generated* model category. If moreover $\mathbf{W}$ is defined "in terms of bijections", then the verification of the (2/3) property is obvious.

We next give sufficient conditions for the existence of a cofibrantly generated model structure, due to Jeff Smith, as reported in [1][1].

Definition 3.3 *Let $\mathbb{C}$ be a category, let $\mathbf{W}$ be a class of morphisms, and let I, J be sets of morphisms, with $J \subseteq \mathbf{W}$. We say that J is a solution set for $\mathbf{W}$ at I if every commutative square with left vertical morphism in I and right vertical morphism in $\mathbf{W}$ factorizes as a juxtaposition of two commuting squares, with the middle vertical morphism in J. Given $\mathbf{W}, I$, we say that $\mathbf{W}$ satisfies the solution set condition at I if there exists a set $J \subseteq \mathbf{W}$ as just described.*

Lemma 3.4 *If J is a solution set for $\mathbf{W}$ at I, then $\rho(J) \cap \mathbf{W} \subseteq \rho(I)$.*

Proof. Take a commuting square with left vertical morphism in I and right vertical morphism $f \in \rho(J) \cap \mathbf{W}$. Because f is in $\mathbf{W}$, we get a middle vertical morphism in J. Because f is in $\rho(J)$, then there is lifting for the right square, which when prefixed with the South-West horizontal morphism becomes a lifting for the original square. □

Lemma 3.5 *Let $\mathbb{C}$ be a locally presentable category, let $\mathbf{W}$ be a class of morphisms and I be a set of morphisms such that (2/3) and (IW) hold. If $\mathbf{W}$ satisfies the solution set condition at I, then one can choose a solution set J such that $J \subseteq \lambda\rho(I) \cap \mathbf{W}$.*

Proof. Let $K \subseteq W$ be a solution set of $\mathbf{W}$ at I, and consider the set of all left squares witnessing the solution set condition (it is a set because $\mathbb{C}$ is locally small). For each such square with $i \in I$ and $k : X \to Y \in K$ as left and right vertical morphisms, respectively, take the pushout i' of i along the top horizontal morphism and decompose the universal morphism to Y as qp with $p \in \lambda\rho(I)$ and $q \in \rho(I)$, and set $j = pi'$. Let J be the set of all j's obtained in this way. Then we can split the original square involving i, k into two adjacent commutative squares with j as middle vertical morphism. And we have $J \subseteq \lambda\rho(I) \cap \mathbf{W}$:

- $J \subseteq \lambda\rho(I)$ holds since by construction both i' and p are in $\lambda\rho(I)$.

[1]Our presentation is more stream-lined than in op. cit., as it avoids reappealing to (a variant of) the small-object argument.

- Since $k = qj$, we get from our assumptions (IW) and (2/3) that $J \subseteq \mathbf{W}$. □

Proposition 3.6 *Let $\mathbb{C}$ be a locally presentable category, let $\mathbf{W}$ be a class of morphisms and I be a set of morphisms such that (2/3) and (IW) hold, and such that*

1. *$\lambda\rho(I) \cap \mathbf{W}$ is stable under retracts, transfinite compositions and pushouts;*
2. *$\mathbf{W}$ satisfies the solution set condition at I.*

Then there exists a set J such that $\mathbf{W}, I, J$ yield a structure of (cofibrantly generated) model category on $\mathbb{C}$.

Proof. All we have to do (cf. Proposition 2.6) is to check (JIW) and (JWI). The latter has been proved in Lemma 3.4. By Lemma 3.5, we can choose a solution set J included in $\lambda\rho(I) \cap \mathbf{W}$, from which (JIW) follows by the stability assumptions on $\lambda\rho(I) \cap \mathbf{W}$ and by Proposition 3.2. □

4 What the hell has this to do with homotopy?

Our goal in the end is to have a framework in which structures that are related through a homotopy equivalence are considered of the same type, i.e., in categorical language, are made isomorphic. The notion of homotopy equivalence is based on a notion of equivalence between maps called homotopy:

Definition 4.1 *Let $\sim$ be some equivalence relation defined on the homsets of some category $\mathbb{C}$, which we call the homotopy relation. Then we say that $f : A \to B$ is a homotopy equivalence if we have both $gf \sim id$ and $fg \sim id$.*

Now, how do we define the homotopy relations? Either through a cylinder object $cyl(A)$ or through a path object $path(A)$ over some object A. And a key property of that cylinder or path object over A is that it is related to A by ... homotopy equivalences. Let us illustrate this with topological spaces. The cylinder object of X is $X \times I$, with $I = [0, 1]$ the unit interval of $\mathbb{R}$. We show that the pairs $f_0 = x \mapsto (x, 0)$ and $g = (x, s) \mapsto x$ form a homotopy equivalence. Indeed, $gf = id$, and $H = ((x, s), t) \mapsto (x, st)$ is a homotopy from fg to id. In fact, all constant maps $f_t = x \mapsto (x, t)$ (t fixed) are homotopy inverses of the (fixed) projection map $(x, s) \mapsto x$. In other words, a saliant feature of $X \times I$ is that we have $g \circ [f_0, f_1] = [id, id]$ and that g is a homotopy equivalence.

In order to break this intuitive circularity, it is tempting to base an abstract definition of cylinder/path objects upon a preexisting notion of "weak equivalence", which itself if all goes well will turn out to be the same as homotopy equivalence. Hence the picture is

- a notion of homotopy relation defined using weak equivalences (defined next);
- a notion of homotopy equivalence based on that of homotopy relation (just given).

So far to explain why $\mathbf{W}$ in the definition of a model category. Why $\mathbf{F}, \mathbf{C}$ is more mysterious. First, note that they are not needed: there is a notion of homotopical category [2] whose axiomatisation is based on a single class $\mathbf{W}$ (see section **??**). Second, it just happens that the factorisation of morphisms plays a fundamental role in the elementary but non trivial manipulations leading to the main results of this section (holding for any model category $\mathbb{C}$), that connect the three notions:

1. weak equivalences and homotopy equivalences coincide on objects that are both fibrant and cofibrant (Proposition 4.10);
2. the localisation of the category by $\mathbf{W}$ is equivalent to the quotient by the homotopy relation of the full subcategory of $\mathbb{C}$ determined by these objects (Proposition 4.12).

This section is based on the very nice tutorial [3]. We assume throughout the section that a model category $\mathbb{C}$ is given.

Definition 4.2 *An object X is called fibrant if the morphism $X \to 1$ is a fibration. Equivalently, in view of the characterisation of fibrations, X is fibrant if for every acyclic cofibration $f : A \to B$ and every $g : A \to X$ there exists $l : B \to X$ such that $lf = g$. Dually, an object A is cofibrant if the morphism $0 \to A$ is a cofibration, which means that for every acyclic fibration $f : X \to Y$ and every $g : A \to Y$ there exists $l : A \to X$ such that $gl = f$.*

Definition 4.3 *Let A be an object. A cylinder object over A is a structure given by an object $cyl(A)$ and two morphisms $i : A + A \to cyl(A)$ and $w : cyl(A) \to A$ such that the following two conditions hold.*

1. $w \in \mathbf{W}$;
2. $wi = [id, id]$.

If in addition $i \in \mathbf{C}$ (resp. $i \in \mathbf{C}$ and $w \in \mathbf{F}$), then $cyl(A)$ is called good (resp. very good). Let $f, g : A \to X$ be two morphisms (X arbitrary). A left homotopy from f to g is a map $H : cyl(A) \to X$ (for some cylinder object over A) such that $Hi = [f, g]$. We say that f is left homotopic to g, notation $f \overset{l}{\sim} g$, if there is a cylinder object on A and a homotopy $H : cyl(A) \to X$ from f to g. We call $cyl(A)$ and H witnesses of $f \overset{l}{\sim} g$.

At first sight, the definition of cylinder object looks unreasonably liberal, since taking $i = [id, id]$ and $w = id$ (and hence $cyl(A) = A$) yields a cylinder object, which is particularly uninteresting since it can only witness $f \overset{l}{\sim} g$ when f, g coincide. But the notion is useful because the next lemma shows that if we have shown that $f \overset{l}{\sim} g$ through *some* cylinder object (not necessarily good, but surely not as stupid as the "degenerate" one), then we also have $f \overset{l}{\sim} g$ through a good cylinder object. The notion of very good cylinder object will serve in (the proof of) Lemma 4.6[2]. We make a few more observations:

Remark 4.4 *1. If $(cyl(A), i, w)$ is a cylinder object, then $i_1 = i \circ in_1$ and $i_2 = i \circ in_2$ (where in_1, in_2 are the canonical injections) are weak equivalences. If $f \overset{l}{\sim} g$, then f is a weak equivalence if and only g is a weak equivalence. If $cyl(A)$ is good and A is cofibrant, then i_1 and i_2 are acyclic cofibrations. (By (2/3) since $wi_1 = wi_2 = id$; the homotopy itself is then a weak equivalence; in_1 is a pushout of $0 \to A$ (along $0 \to A$).)*

2. *The homotopy relation is reflexive and symmetric. (Indeed, by definition, w is a homotopy from id to id; and if $(cyl(A), i, w)$ is a cylinder object, then so is $(cyl(A), i \circ [in_2, in_1], w)$.)*

3. *If $(cyl(A), i, w)$ is a cylinder object and if $i = w'i'$, with $w' : A' \to cyl(A) \in \mathbf{W}$, then (A', i', ww') is a cylinder object, and if $cyl(A)$ witnesses $f \overset{l}{\sim} g$, then so does A'. (If $Hi = [f, g]$, then set $H' = Hw'$.)*

4. *If $f_1 \overset{l}{\sim} f_2$, then $gf_1 \overset{l}{\sim} gf_2$. If moreover g is an acyclic fibration, then the converse implication holds. (Take gH as witness (H witness of $f_1 \overset{l}{\sim} f_2$) . For the partial converse, take a good cylinder object as witness (see Lemma 4.5) and apply the lifting property to i and g.)*

Lemma 4.5 *1. There exists a good cylinder object over any object. For any f, g, if $f \overset{l}{\sim} g$, then the witness can be taken to be a good cylinder object.*

2. *Under the additional assumption that the common target X of f, g is fibrant, then the witness can even be taken to be a very good cylinder object (on the common domain).*

Proof. The first part of the statement is an immediate consequence of (WF) applied to i and Remark 4.4[3]. As for the second part, starting from a witness that is already a good cylinder one applies (WF) to w to get $w = w'i'$ where both w' and i' are acyclic. Then the target of i' is a very good cylinder. Moreover, since X is fibrant, we can factorise H as $H'i'$, and H' is the "very good witness". □

We want the homotopy relation to be an equivalence relation, and we want it to be a congruence for composition. Remark 4.4 leaves us to prove transitivity and the other half of the congruence property: they hold under the additional assumption that the source is cofibrant and the target is fibrant, respectively.

Lemma 4.6 *1. If A is cofibrant, then $\overset{l}{\sim}$ is transitive on all homsets with source A.*

2. *If X is fibrant, then for all $f_1, f_2 : A \to X$, $f_1 \overset{l}{\sim} f_2$ implies $f_1 h \overset{l}{\sim} f_2 h$, for all h.*

Proof. (1) Consider $(cyl(A), [i_1, i_2], w)$ and H, and $(cyl(A)', [i'_1, i'_2], w')$ and H', witnessing $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$, respectively. We can take the two cylinders to be good, by Lemma 4.5. If $i_1^* \circ i_2 = i^*2 \circ i_1'$ is the pushout of i_2, i_1', then the following data witness $f \overset{l}{\sim} h$:

- $cyl(A)''$: South-East corner of the pushout diagram;
- w'': the universal morphism induced by the commutation $wi_2 = id = w'i_1'$;
- $i'' = [i_1^* i_1, i_2^* i_2']$;
- H'': the universal morphism induced by the commutation $Hi_2 = g = H'i_1$.

The only delicate point is to check that w'' is a weak equivalence. This is where we use the cofibrant assumption, which entails that i_1 is an acyclic cofibration (cf. Remark 4.4[1]). Hence i_1^* is an acyclic cofibration by stability of this class of morphisms under pushouts, and a fortiori is a weak equivalence, from which the claim on w'' follows by (2/3). Note that weak equivalences are not supposed to be closed under pushouts in general.

(2) Let $f_1, f_2 : A \to X$ and $h : A' \to A$. Using our assumption on X and Lemma 4.5 consider a *very good* $(cyl(A), i, w)$ and H witnessing $f_1 \overset{l}{\sim} f_2$. Let $(cyl(A'), i', w')$ be a good cylinder object over A'. It is enough to find a morphism $l = cyl(A') \to cyl(A)$ such that $li' = i(h + h)$, because then Hl will witness $f_1 h \overset{l}{\sim} f_2 h$. This l is obtained as the lifting in the square $w \circ i(h + h) = [h, h] = hw' \circ i'$ (whence the importance for w to be acyclic). □

Lemma 4.6 seems to cry for asking for objects to be both fibrant and cofibrant, in order to compose classes modulo homotopy. (We could leave out the cofibrant constraint and make a quotient category of morphisms modulo left homotopy out of the full subcategory of fibrant objects: the price to pay is to define the left homotopy equivalence as the transitive closure of $\overset{l}{\sim}$.)

Dually, one can define (good, very good) path objects, and right homotopy relations, and prove the dual properties, leading to a qotient category of cofibrant objects and morphisms modulo right homotopy.

We content ourselves with laying down the notion of path object over X and of right homotopy relation.

Definition 4.7 *A path object over X is a triple $(path(X), p, w)$ where $w : X \to path(X) \in \mathbf{W}$ and $pw = \langle id, id \rangle$. Let $f, g : A \to X$ be two morphisms (A arbitrary). A right homotopy from f to g is a map $H : A \to path(X)$ (for some path object over X) such that $pH = [f, g]$.*

The geometric intuition is to think of a left (resp. right) homotopy between f, g displayed as upper and lower border of a surface as filling the surface with "horizontal" (resp. "vertical") lines.

Where we really need objects to be *both* fibrant and cofibrant is when we want the two notions of homotopy to coincide.

Lemma 4.8 *Let $f, g : A \to X$. If A is cofibrant, then $f \stackrel{l}{\sim} g$ implies $f \stackrel{r}{\sim} g$. If X is fibrant, then $f \stackrel{r}{\sim} g$ implies $f \stackrel{l}{\sim} g$.*

PROOF. We prove the first part (the second is its dual). Take a good path object $(path(X), \langle p_1, p_2\rangle, q)$ over X, and $(cyl(A), [i_1, i_2], j)$ and H witnessing $f \stackrel{l}{\sim} g$, with $cyl(A)$ good. The idea is to "fill with vertical lines" progressively, covering a larger and larger vertical portion of the surface. Technically, we shall build a morphism $K : cyl(A) \to path(X)$ such that Ki_2 witnesses $f \stackrel{r}{\sim} g$, i.e., $\langle p_1, p_2\rangle \circ (Ki_2) = \langle f, g\rangle$. This morphism is the lifting in the commutative square

$$\langle p_1, p_2\rangle \circ (qf) = \langle fj, H\rangle \circ i_1$$

This lifting exists since our assumption on A yields that i_1 is an acyclic cofibration (cf. Remark 4.4). □

We therefore have obtained a category $\mathbb{C}_{\mathbf{FC}}/\sim$ whose objects are the fibrant cofibrant objects of $\mathbb{C}$ and whose morphisms are the homotopy classes of morphisms of $\mathbb{C}$.

We can now prove the two announced results, which hold under the assumption that $\mathbb{C}$ is a model category.

Lemma 4.9 *If $p : C \to X$ is an acyclic fibration and if X is cofibrant, then there exists $s : X \to C$ such that $ps = id_X$ and $sp \stackrel{l}{\sim} id_C$.*

PROOF. The existence of s such that $ps = id_X$ follows by definition of cofibrant object. On the other hand, since $psp = p$, we have a fortiori $psp \stackrel{l}{\sim} p$, from which, using again the fact that p is an acyclic fibration, we get $sp \stackrel{l}{\sim} id$, by Remark 4.4[4]. □

Proposition 4.10 *A morphism f between fibrant cofibrant objects is a weak equivalence if and only if it is a homotopy equivalence.*

PROOF. Let $f : A \to X$. We can write $f = pq$ where $p : C \to X$ is a fibration and $q : A \to C$ is an acyclic cofibration.

Suppose first that f is a weak equivalence. Then p is also a weak equivalence. By Lemma 4.9, we get s such that $ps = id$ and $sp \stackrel{l}{\sim} id$. Dually, using that A is fibrant and q is an acyclic cofibration, we obtain r such that $rq = id$ and $qr \stackrel{r}{\sim} id$. It follows, using Lemma 4.6[2] and its dual, and still that X is cofibrant and A is fibrant, that

$$(rs) \circ (pq) \stackrel{l}{\sim} id \quad \text{and} \quad (pq) \circ (rs) \stackrel{r}{\sim} id$$

Now we want the same homotopy relation! This is where we use our assumption that, say A is also cofibrant, which allows us to appeal to Lemma 4.8.

Conversely, suppose that f is a homotopy equivalence, and let g be a homotopy inverse. It is enough to show that p is a weak equivalence. Let $H : cyl(X) \to X$

be a witness of $fg \stackrel{l}{\sim} id$, with $(cyl(X), [i_1, i_2], w)$ good. Using Remark 4.4, i_1 is an acyclic cofibration since X is cofibrant. Hence there is a lifting H' for the square

$$p \circ (qg) = fg = H \circ i_1$$

Setting $s = H' i_2$ (and hence $qg \stackrel{l}{\sim} s$), we have:

- $ps = Hi_2 = id$;
- $sp \sim id$. This is established as follows. By (the dual of) Lemma 4.9 there exists r such that $rq = id$ and $qr \stackrel{r}{\sim} id$. We have:

  $$sp \sim qgp \sim qgpqr = qgfr \sim qr \sim id$$

  Note that by Lemma 4.6 and its dual, to justify $sp \stackrel{l}{\sim} qgp$ and $qgp \stackrel{r}{\sim} qgpqr$ we need C fibrant and cofibrant, respectively, which follows from X being fibrant and A being cofibrant.

We conclude by Remark 4.4[1] that sp is a weak equivalence, and then that p is a weak equivalence as a retract of sp (cf. Example 1.3). □

Our last result needs a bit of preparation. With every object X one can associate a fibrant object RX and an acyclic cofibration $i_X : X \to RX$ (factorise the canonical morphism $X \to 1$). Note that if X is cofibrant, then so is RX (since cofibrations compose). Dually, with every object we associate a cofibrant object QX and an acyclic fibration $p_X : QX \to X$, and we have that QX is fibrant if X is.

The next lemma shows that when these choices have been made, Q and R can be made functorial (up to homotopy in the target category).

Lemma 4.11 *Given $f : X \to Y$, there exists $\tilde{f} : QX \to QY$ such that $p_Y \tilde{f} = f p_X$. If two morphisms satisfy this property, then they are left homotopic. If f is a weak equivalence, then so is $\tilde{f}$. The same holds replacing Q by R and "left" by "right".*

Proof. The existence of $\tilde{f} : QX \to QY$ is obtained by unfolding the definition of cofibrant object at fp_X and p_Y. The uniqueness of $\tilde{f}$ up to left homotopy follows from Remark 4.4[4]. The preservation of weak equivalences is an immediate consequence of (2/3). □

Proposition 4.12 *The categories $Ho(\mathbb{C}) = \mathbb{C}[\mathbf{W}^{-1}]$ (the* homotopy category*) and $\mathbb{C}_{\mathbf{FC}}/\sim$ are equivalent.*

Proof. We make of a choice for R and Q on objects, choosing $QX = X$ and $p_X = id$ whenever X is already cofibrant, and likewise for R.

We extend RQ to a functor from $\mathbb{C}[\mathbf{W}^{-1}]$ to $\mathbb{C}_{\mathbf{FC}}/\sim$ in two steps. First, by Lemma 4.11, RQ extends from $\mathbb{C}$ to $\mathbb{C}_{\mathbf{FC}}/\sim$. Second, we observe, by Lemma 4.11 and Proposition 4.10 that RQ maps weak equivalences to isos, and hence factorises through $\mathbb{C}[\mathbf{W}^{-1}]$ by the defintion of a localised category. We therefore consider now RQ as a functor from $\mathbb{C}[\mathbf{W}^{-1}]$ to $\mathbb{C}_{\mathbf{FC}}/\sim$.

It is essentially surjective because when X is fibrant cofibrant, then so are QX and RQX. Hence the weak equivalences p_X and i_{QX} are homotopy equivalences by Proposition 4.10, and therefore X and RQX are isomorphic in $\mathbb{C}_{\mathbf{FC}}/\sim$. (In fact, by our convention at the beginning of the proof, we actually have that X is *equal* to RQX.)

In order to show that RQ is full and faithful, we exhibit a transformation from $(\mathbb{C}_{\mathbf{FC}}/\sim)[RQX, RQY]$ to $\mathbb{C}[\mathbf{W}^{-1}][X,Y]$:

$$f \mapsto \hat{f} \;=\; (p_Y i_{QY}^{-1}) \circ f \circ (i_{QX} p_X^{-1})$$

This is well defined as two homotopic maps of $\mathbb{C}$ are equal in $\mathbb{C}[\mathbf{W}^{-1}]$. Indeed, let $(cyl(A), [i_1, i_2], w)$ and H be witness of $f_1 \overset{l}{\sim} f_2$. Then $wi_1 = id$ and $wi_2 = id$ imply $i_1 = w^{-1} = i_2$ and hence $f_1 = Hi_1 = Hi_2 = f_2$. We prove that this transformation is inverse to RQ on morphisms:

- $RQ\hat{f} = f$: this follows from our choice of R, Q: clearly, if X is cofibrant, then we can take $\tilde{p_X} = id : QX \to QX$, etc..., so that $RQp_Y = id$, etc... yielding $RQ\hat{f} = id \circ f \circ id = f$.
- $\widehat{RQg} = g$ follows by filling with squares a rectangle whose South and North sides are occupied by g and RQg, and whose vertical morphisms are p_Z and i_{QZ} for all objects Z visited by the zigzag g. Then the conclusion follows from the commutation of all these squares. Along the way, one uses a few observations:
  - $Qh \circ q_{Z_1}^{-1} = q_{Z_2}^{-1} \circ h$ for all $h : Z_1 \to Z_2$.
  - Qw is a weak equivalence if w is (by (2/3)).
  - $(Qw)^{-1} \circ q_{Z_1}^{-1} = q_{Z_2}^{-1} \circ w^{-1}$.
  - $RQ(w^{-1}) = (RQw)^{-1}$. □

The following lemma gives us a useful bridge between the two sorts of quotient (by localisation, by homotopy).

Lemma 4.13 *Let A be a cofibrant object and X be a fibrant object. Then there is a (natural) bijection between $Ho(\mathbb{C})[A,X]$ and $\pi(A,X) = (\mathbb{C}_{\mathbf{FC}}/\sim)[A,X]$.*

Proof. We have the following sequence of bijections:

$$\begin{array}{llll} Ho(\mathbb{C})[A,X] & \cong & Ho(\mathbb{C})[RA,QX] & \text{(weak equivalences are isos in } Ho(\mathbb{C})) \\ & \cong & \pi(RA,QX) & \text{(cf. proof of Proposition 4.12)} \\ & \cong & \pi(A,QX) & (QX \text{ is fibrant and } i_A \text{ is an acyclic cofibration)} \\ & \cong & \pi(A,X) & (A \text{ is cofibrant and } p_X \text{ is an acyclic cofibration)} \end{array}$$

□

As a final result in this section, we prove that the class of weak equivalences satisfies a saturation property.

Proposition 4.14 *The morphisms of $\mathbb{C}$ that become isomorphisms in $\mathbb{C}[\mathbf{W}^{-1}]$ are exactly the weak equivalences.*

PROOF. A weak-equivalence w of $\mathbb{C}$ becomes invertible in $\mathbb{C}[\mathbf{W}^{-1}]$ by definition of $\mathbb{C}[\mathbf{W}^{-1}]$. Thanks to Proposition 4.12, the converse can be rephrased as: if RQf is a homotopy equivalence, then f is a weak equivalence, which thanks to Proposition 4.10 can be further rephrased as: if RQf is a weak equivalence, then f is a weak equivalence, which itself is an easy consequence of the construction of Qf and RQf, and of (2/3). □

(Non closed) model categories. The original notion of (non closed) model category, also due to Quillen [6][section I.1], is obtained by taking the axioms listed in Proposition 2.4 (Hovey's definition), replacing the axiom (ret) with the following list of properties:

- all isos are fibrations, cofibrations, and weak equivalences,
- fibrations and cofibrations are closed under composition,
- fibrations and acyclic fibrations are closed under pullbacks,
- cofibrations and acyclic cofibrations are closed under pushouts.

That this yields a weaker definition is an immediate consequence of (WF) and Lemma 1.4. It is not too difficult to produce an example proving that the notion is strictly weaker (taking less fibrations, or less cofibrations, see [6]). It seems more difficult to find an example of a non closed model category in which the closure of $\mathbf{W}$ by retraction fails. An even stronger counterexample would be one of a non closed model category in which Propositions 4.10 and 4.14 fail (see below).

By looking carefully at the arguments used in this section, the reader can check that all the statements of the section but Propositions 4.10 and 4.14 hold in non closed model categories. More precisely, the only place where the stronger closed category axioms are used is at the end of the proof that a homotopy equivalence between fibrant cofibrant objects is a weak equivalence when we appeal to the closure of $\mathbf{W}$ under retracts.

In [6], the notion of closed model category is indeed introduced to achieve the saturation property stated here as Proposition 4.14. Quillen also proved in the same seminal work (section I.5, Proposition 2) that the closure under retracts of $\mathbf{W}$ is a consequence of his definition.

Why the name "model categories"? In [6][introduction to chapter 1], Quillen explains that he chose the term "model category" by reference to the situation of a given (fixed) homotopy theory on one hand and the existence of several categories $\mathbb{C}_1$, $\mathbb{C}_2$,... having this (same) theory as their homotopy category (the most famous example being the category of topological spaces and the category of simplicial sets).

One may draw a comparison with logic, where several models may have the same theory, i.e. validate the same collection of formulas in some logical system.

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