$$\boxed{\begin{array}{ll} \mathcal{E}_n = Set/O_n & \mathcal{E}_{n+1} = \mathcal{E}_n/_{T_n 1} \\ T_n\left(\begin{array}{c} O_n \\ \downarrow \scriptstyle{id} \\ O_n \end{array}\right) = \begin{array}{c} O_{n+1} \\ \downarrow \\ O_n \end{array} & = (Set/O_n)/_{T_n 1} = Set/O_{n+1} \end{array}}$$

A l'origine, $\mathcal{E}_0 = Set$ and $T_0 = id$

A T_0-operad is a monoid $\in Set/_{T_0 1} = Set/_1 = Set$

We have $\mathcal{E}_1 = \mathcal{E}_0/_{T_0 1} = Set$ and T_1 = free-monoid monad

A T_1-operad is a plain operad. We have $T_1 1 = \mathbb{N}$

So, a T_1-operad is $\begin{array}{c} X \\ \downarrow \\ \mathbb{N} \end{array}$

Definition of T_n

$$T_n\begin{pmatrix} A \\ \downarrow \# \\ O_n\end{pmatrix} = \begin{matrix} A^{\oplus} \\ \downarrow \#^{\oplus} \\ O_n \end{matrix} \quad \text{where} \quad T_{n-1}\begin{pmatrix} O_{n-1} \\ \downarrow id \\ O_{n-1}\end{pmatrix} = \begin{matrix} O_n = O_{n-1}^{\oplus} \\ \downarrow t = id^{\oplus} \\ O_{n-1}\end{matrix}$$

and where, using that elements of O_n are themselves (O_{n-1}, O_{n-2})-trees $(S, p_-(S) : S^0 \to O_{n-1}, p_{S}' : S^1 \to O_{n-2})$

- $A^{\oplus} = \{(T, v_T : T^0 \to A, v'_T : T^1 \to O_{n-1}) \mid$ (A, O_{n-1})-trees
  $\forall x \in T^0,\ T^1_x = |\underbrace{\#(v_T(x))}_{\in O_n}|^0$ and
  $x <_u \Rightarrow p_u(\#(v_T(x))) = v'_T(u)$
  $<_u y \Rightarrow v'_T(u) = t(\#(v_T(y)))\}$

- $\#^{\oplus}(|\, v_{\in O_{n-1}}) = \underbrace{\eta_{n-1}(v)}_{\in O_n}$ (monad structure on T_{n-1})

  $$\#^{\oplus}\left(\begin{matrix}(T_u, v_u, v'_u) \\ \mid \\ \bullet\ a\end{matrix}\right) = \underbrace{\underbrace{|\#(a)|}_{\text{tree}}\{u \leftarrow \underbrace{\#^{\oplus}(T_u, v_u, v'_u)}_{(O_{n-1},\, O_{n-2})\text{-tree}}\}}_{\in O_n}$$

Definition of O_{n+1} and $\underline{t} : O_{n+1} \to O_n$

We define $\begin{array}{c} O_{n+1} \\ \downarrow \underline{t} \\ O_n \end{array}$ as $T_n\left(\begin{array}{c} O_n \\ \downarrow id \\ O_n \end{array}\right)$. Explicitly,

- $O_{n+1} = \{ (\underline{T}, s_-(T) : T^0 \to O_n, ss_-(T) : T^1 \to O_{n-1} \mid$
  $\forall x \in T^0, \ T^1_x = (s_x(T))^0$ and
  $x \prec_u \Rightarrow s_u(s_x(T)) = ss_u(T)$
  $\prec_u y \Rightarrow ss_u(T) = \underline{t}(s_y(T)) \}$

We use the notation $s_x(\omega) = s_x(T)$ for $\omega = (\underline{T}, s_-(T), ss_-(T))$ etc.

- $\underline{t}(\,|\,\underset{\in O_{n-1}}{v}\,) = \underbrace{\eta_{n-1}(v)}_{\in O_n}$ (monad structure on T_{n-1})

$$\underline{t}\left(\begin{array}{c} (\underline{T}_u, v_u, v'_u) \\ \diagdown \ \mid \ \diagup \\ \bullet \\ \omega \end{array}\right) = \underbrace{\underbrace{|\omega|}_{\text{free}} \{ u \leftarrow \underbrace{\underline{t}(T_u, v_u, v'_u)}_{[O_{n-1}, O_{n-2}]\text{-free}}}_{\in O_n}$$

In fact we can content ourselves defining O_{n+1} only:

$$T_n\left(\begin{array}{c} A \\ \downarrow \# \\ O_n \end{array}\right) = \begin{array}{c} A^{\oplus} \\ \downarrow \#^{\oplus} \\ O_n \end{array}$$ can be described as

$$A^{\oplus} = \Big\{ \underbrace{(\underline{T}, v_T, v'_T)}_{(A, O_{n-1})\text{-trees}} \;\Big|\; \underbrace{(\underline{T}, \# \circ v_T, v'_T)}_{(O_n, O_{n-1})\text{-trees}} \in O_{n+1} \Big\}$$

$$\#^{\oplus}(\underline{T}, v_T, v'_T) = \underline{t}\,(\underline{T}, \# \circ v_T, v'_T)$$

Monad structure on T_n

We need to define $1 \xrightarrow{\eta_1} T_n 1$ and $T_n T_n 1 \xrightarrow{\mu_1} T_n 1$ only, and then extend by cartesianity

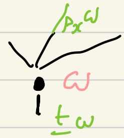
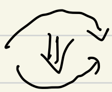
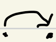
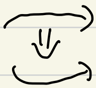
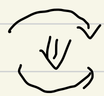
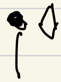
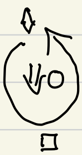
- $\eta_1 : O_n \to O_{n+1}$ $\quad \eta_1(\omega) =$  ← single-node tree

illustration : $\eta_1(\square) =$  

$\Diamond = \bullet$
$\square =$ 

$\eta_1\Big($  $\Big) =$  $\Longrightarrow$  $=$ 

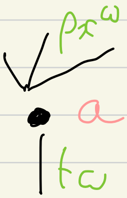
$\eta_1(\Diamond) = \square =$ 

$\underline{t}(|\Diamond) = \underline{t}$  $= \eta_1(\Diamond) = \square$

- $\mu_1 : T_n\left(\begin{array}{c} O_{n+1} \\ \downarrow \underline{t} \\ O_n \end{array}\right) \to \begin{array}{c} O_{n+2} \\ \downarrow \underline{t} \\ O_n \end{array}$

(O_n, O_{n-1})-tree
$\|$
$\in O_{n+1}$

$\mu_1(\underline{T}, v_T, v'_T) = \underline{T}\{x \leftarrow \overbrace{v_T(x)}\}$ ← substitution in node

$\eta\left(\begin{array}{c} A \\ \downarrow \# \\ O_n \end{array}\right)(a) =$ 

$(\omega = \#(a))$

$\mu_A : T_n\left[\begin{array}{c} A^{\oplus} \\ \downarrow \#^{\oplus} \\ O_n \end{array}\right] \to \begin{array}{c} A^{\oplus} \\ \downarrow \#^{\oplus} \\ O_n \end{array}$

Same formula as for μ_1!
$v_T(x)$ is now a (A, O_{n-1})-tree

Heuristics for the definition of $\mathcal{E}_n \overset{T_n}{\circlearrowleft}$: analysis of $\begin{matrix} A \\ \downarrow\# \\ O_n \end{matrix} \otimes \begin{matrix} A \\ \downarrow\# \\ O_n \end{matrix}$ in $\mathcal{E}_n$

$$(a, \underline{T}, w_T : T^0 \to A)$$

$$\begin{tikzcd}
& B \arrow[dl] \arrow[dr] & \\
T_{n-1}\begin{pmatrix} A \\ \downarrow\# \\ O_1 \\ \downarrow \underline{t} \\ O_{n-1} \end{pmatrix} \arrow[dr, "T_{n-1}!"] & & A \arrow[dl, "\#"] \\
& O_n &
\end{tikzcd}$$

$(\underline{T}, w_T) \mapsto (\underline{T}, \underline{t} \circ \# \circ w_T)$

$a \mapsto \#(a) = (\underline{T}', p_-(T'))$

$$O_n \to O_{n-1} = T_{n-1}\begin{pmatrix} O_{n-1} \\ \downarrow id \\ O_{n-1} \end{pmatrix}$$

(note that for an endofunctor F on a slice cat C/X we have $!_F = F\left(\downarrow id\right)$) It follows that $\underline{T}' = \underline{T}$ and $\forall x \in T^0$

$$p_x(\#(a)) = p_x(T') = \underline{t}(\#(w_T(x)))$$

Therefore, an element of the pullback B is a pair $(\underline{S}, v_S : S^0 \to A)$, where

$$S^0 = \{\bullet\} \cup \{\bullet_x \mid x \in (\#(v_S(\bullet)))^0\}$$

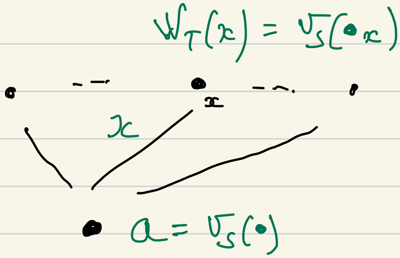
$$S^1_n = (\#(v_S(n)))^0 \quad (\forall n \in S^0)$$

$$o(\bullet) = !_\emptyset \qquad o(\bullet_x) = x$$

2-level tree!

with the constraint that for $\bullet \prec_x \bullet_x$

we have $p_x(\#(v_S(\bullet))) = \underline{t}(\#(v_S(\bullet_x)))$



$$p_x(\#(a)) = p_x(T) = \underline{t}(\#(w_T(x)))$$

Then the definition of T_n generalises this to arbitrary trees $\underline{S}$!

Maeva, let us examine $\begin{array}{c} B \\ \downarrow \\ O_n \end{array}$

It is defined as

$(a, I, w_T : T^0 \to A)$

(I, w_T)

$(I, \# \circ w_T)$

$$\begin{tikzcd}
 & & B \arrow[dl, "w"'] \arrow[dr] & \\
 & T_{n-1}\left(\begin{array}{c} A \\ \downarrow \\ O_{n-1} \end{array}\right) \arrow[dl] \arrow[dr] & & A \arrow[d, "\#"] \\
T_{n-1}\left(T_{n-1}\left(\begin{array}{c} O_{n-1} \\ \downarrow id \\ O_{n-1} \end{array}\right)\right) \arrow[d, "\mu"] & & T_{n-1}\left(\begin{array}{c} O_{n-1} \\ \downarrow id \\ O_{n-2} \end{array}\right) & T_{n-1}\left(\begin{array}{c} O_{n-1} \\ \downarrow id \\ O_{n-2} \end{array}\right) = \begin{array}{c} O_n \\ \downarrow \\ O_{n-1} \end{array} \\
T_{n-1}\left(\begin{array}{c} O_{n-1} \\ \downarrow id \\ O_{n-2} \end{array}\right) & & &
\end{tikzcd}$$

$$I\{x \leftarrow \#(w_T(x))\} = \#^{\oplus}(S, \sigma_S) \; !$$

of last slide

The structure we needed on O_n to be able to define T_n can be organised as a polynomial functor :

$$P(O_n) \qquad \begin{tikzcd} & O_n^\bullet \arrow[dl, "s"'] \arrow[r, "p"] & O_n \arrow[d, "t"] \\ O_{n-1} & & O_{n-1} \end{tikzcd}$$

where $O_n^\bullet = \{(T, \rho_-(T), x) \mid (T, \rho_-(T)) \in O_n,\ x \in T^0\}$

$$p(T, \rho_-(T), x) = (T, \rho_-(T))$$

$$s(T, \rho_- T, x) = \rho_x(T)$$

TO DO

- show that $P(O_n)$ is a polynomial monad
- show that $P(\underbrace{O_{n+1}}_{= O_n^{+}}) = P(O_n)^+$

the + construction of Kock-Joyal ...