A poset-like approach to positive opetopes

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October 5, 2023

Abstract

We introduce in this paper a new formalisation of positive opetopes where faces are organised in a poset. Then we show that our definition is equivalent to that of positives opetopes as given by Marek ZAWADOWSKI in [9].

Introduction

We begin by motivating the notion of opetope that is lying at the heart of this article. When manipulating categories, we find ourselves considering two types of forms: zero dimensional elements, which are the objects (the 0-cells):

$$\bullet$$

And one dimensional ones: the arrows (the 1-cells).

$$\bullet \longrightarrow \bullet$$

Category theory is mainly about composing arrows, so the category-theorist often draws diagrams as below:

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd}$$

and says that it commutes if $g \circ f = h$. It can be understood as a kind of surface whose boundary is given by the arrows f, g and h, which ensures that there is a way to go from the upper path to the lower one. Since in a weak setting we do not like equalities, it would be better to replace this $g \circ f = h$ by some kind of algebraic data

$$\alpha : g \circ f \Longrightarrow h$$

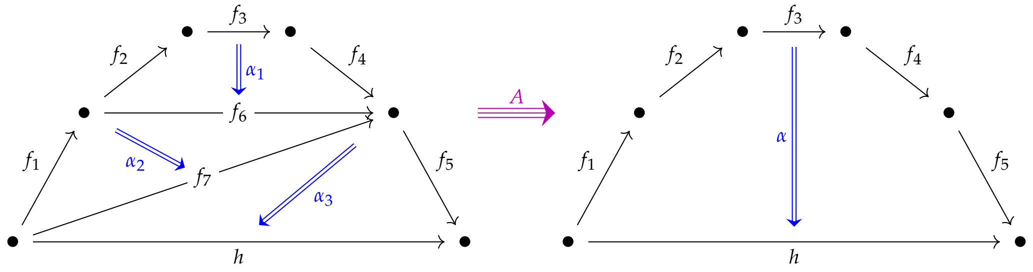
Hence we give a name to our (oriented) surface (or 2-cell), α, and depict it as below:

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, Rightarrow, "\alpha"'] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \bullet \end{tikzcd}$$

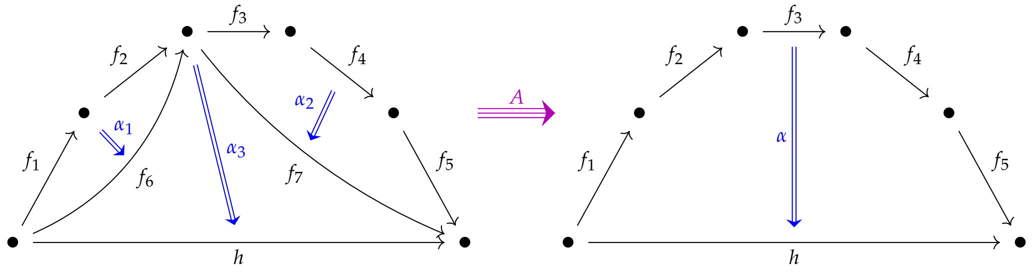
More generally, one is interested in n-ary compositions of arrows, so we may also consider diagrams such as

$$\begin{tikzcd} & & \bullet \arrow[r, "f_3"'] & \bullet \arrow[dr, "f_4"] & & \\ & \bullet \arrow[ur, "f_2"'] & & & \bullet \arrow[dr, "f_5"] & \\ \bullet \arrow[ur, "f_1"'] \arrow[rrrrr, "h"] & & {} & & & \bullet \arrow[from=1-3, to=3-3, Rightarrow, "\alpha"'] \end{tikzcd}$$

Since our surfaces are now algebraic data, we will soon be interested in comparing them, too. Once again, since asking if they are equal or not is more a strict-minded approach to category theory, we wish to introduce named volumes (3-cells), which may assert that there is a way to go from a composite of surfaces to another one. An illustration is given below:



And as another example:



Typically, in those drawings, (which are in fact 3-dimensional *opetopes*) the rightmost 2-cell α will be called the *target* of the 3-cell A. The 2-cells α_1, α_2 and α_3 will be called its *sources*. Notice that each 1-cell (our basic arrows $\bullet \to \bullet$) has exacly one source, and one target. But in higher dimensions, 2-cells and 3-cells may have many sources, while still having exactly one target. We may also say that 0-dimensional cells, (*i.e.* points $\bullet$), have no sources, and no target. Pictures as above take the shape of a polytope when increasing dimension, and they represent algebraic operations, whence the term introduced by BAEZ and DOLAN in [2]: *opetopes*, for "ope(ration-poly)topes".

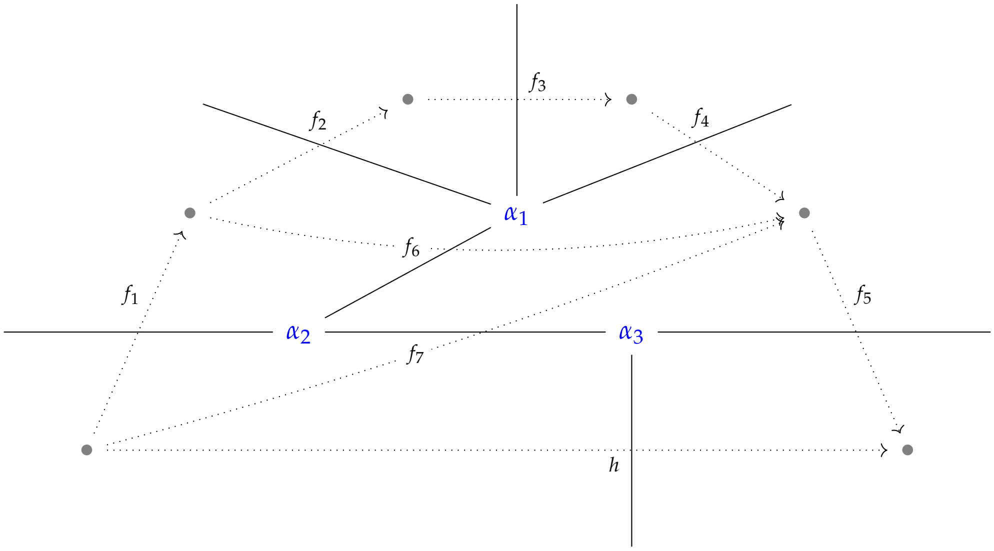
We have several ways to encode such shapes combinatorially.

- An approach, taken by Marek ZAWADOWSKI in [9], is to name all cells as above (including 0-cells), and store them in a poset, where every element has a dimension (its geometric dimension), and a relation $z \leq x$ will mean that z is a subface of x in the opetope. Then we need to identify axioms to ensure that those posets fit our intuition of opetopes as above. This is exposed in Section 2.

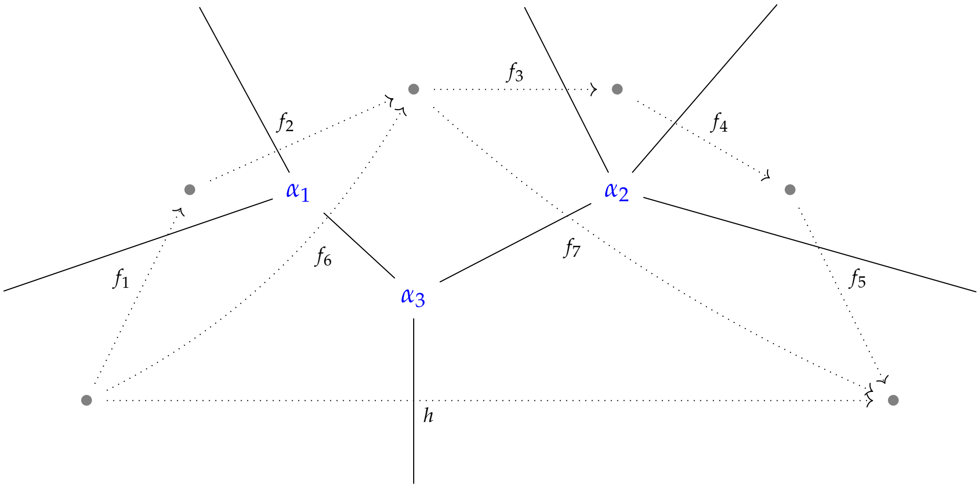
  Using the ideas of Amar HADZIHASANOVIC [3] and Pierre-Louis CURIEN, we were able to identify a second formalism using a similar principle, presented in Section 1.

  In the sections 3 and 4, we will show that the two formalisms are equivalent.

- We may also notice that there are trees hidden in opetopes, which may be retrieved by POINCARÉ duality:

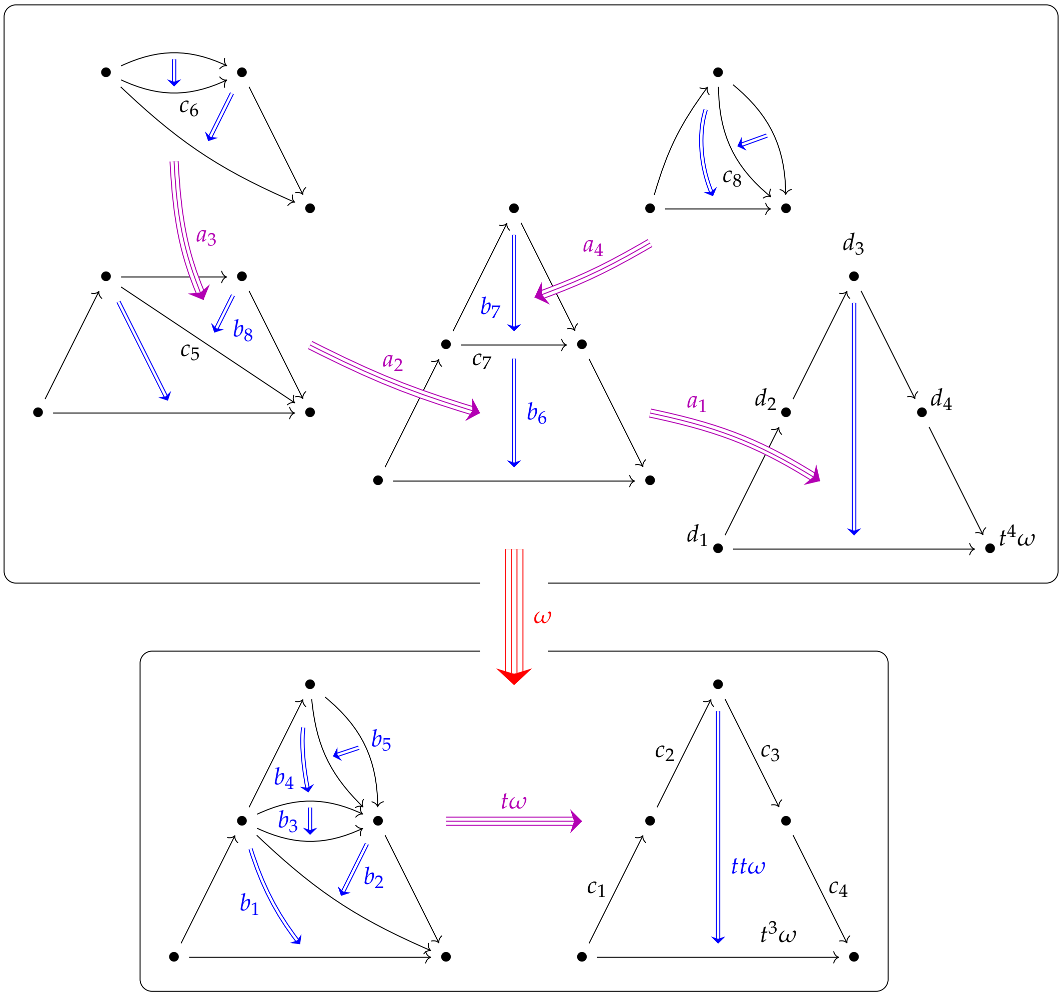


And for the second example:



Here, cells α_1, α_2 and α_3 are lower dimensional opetopes, and should have an associated tree too. Hence we may represent an opetope with a bunch of trees in several dimensions, interconnected by gluing relations. This is the approach taken by Joachim KOCK, André JOYAL, Michael BATANIN and Jean-François MASCARI in [5], or by Cédric HO THANH, Pierre-Louis CURIEN and Samuel MIMRAM in [4] And also, from a different perspective originating from Pierre-Louis CURIEN, with the formalisation of *epiphytes*, which will be presented in a future paper.

In order to provide the reader with a better intuition of opetopes in higher dimension, an illustration of a 4-dimensional opetope is given below:



For the reader introduced to the ideas of higher category theory, we specify that opetopes are a form of cell, like globes, or simplexes. They fit into the broader context of opetopic cardinals, which plays a role similar to that of pasting diagrams (see [6], or [1] where they are called *globular sums*) in the globular setting. In particular, opetopic cardinals are arranged in a strict ω-category, freely generated by opetopes of all dimensions. The study of opetopes is (for instance) motivated by the following result (see Corollary 13.5 in [9], which is proved by using the aforementioned opetopic cardinals):

Theorem : Zawadowski

There is an equivalence of categories between $\widehat{\mathbf{pOpe}}$ and $\mathbf{pPoly}$.

where $\mathbf{pOpe}$ denotes the category of positive opetopes, $\widehat{\mathbf{pOpe}}$ the associated presheaf category, and $\mathbf{pPoly}$ the category of positive-to-one polygraphs. That is, positive opetopes may be used to present a strict ω-category.

1 Dendritic face complexes

The poset structure that will be introduced below makes implicit use of a notion of rooted tree, which we start by defining below.

Definition 1.1 : Rooted tree

A *rooted tree* T consist of:

- A *finite* set of nodes $T^\bullet$.
- For each node $a \in T^\bullet$, a *finite* set $A(a)$, called the arity of a.
- A (necessarily finite) set of *triplets*, denoted $a \mathrel{-\!\!<}_b a'$ for some $a, a' \in T^\bullet$ and $b \in A(a)$. Moreover we ask that for each $a \in T^\bullet$ and $b \in A(a)$, there is at most one triplet $a \mathrel{-\!\!<}_b a'$. If there is none, the pair (a, b) is said to be a *leaf* of T, and we let

$$T^{|} := \{(a, b) \text{ leaf of } T\}$$

We moreover ask for a distinguished element $\rho(T) \in T^\bullet$, called the *root* of T, satisfying the following property: for each node $a \in T^\bullet$, there is a unique *(descending) path* in T

$$a = a_0 \mathrel{>\!\!-}_{b_1} a_1 \mathrel{>\!\!-}_{b_1} \cdots \mathrel{>\!\!-}_{b_p} a_p = \rho(T)$$

from a to the root of T.

1.2
Notice that if it exists, the root is uniquely determined.

1.3
Below is a representation of the rooted tree T having

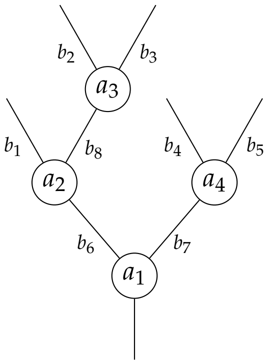
- as nodes $T^\bullet := \{a_1, a_2, a_3, a_4\}$
- as arities

$$A(a_1) := \{b_6, b_7\} \qquad A(a_2) := \{b_1, b_8\} \qquad A(a_3) := \{b_2, b_3\} \qquad A(a_4) := \{b_4, b_5\}$$

- as triplets

$$a_1 \mathrel{-\!\!<}_{b_6} a_2 \qquad a_1 \mathrel{-\!\!<}_{b_7} a_4 \qquad a_2 \mathrel{-\!\!<}_{b_8} a_3$$

- as root a_1.



1.4. We now present the notion of dendritic face complexes (DFC), defined in order to encode opetopes. Formally, they are partially ordered sets, where relations between elements are labelled by a sign ($+$ or $-$) in the Hasse diagram of which. They shall also satisfy some properties that will be given below. The elements of the poset, sometimes called *faces*, stand for the faces of the opetope. All faces have a dimension in the poset, which reflects the intuitive dimension in the geometrical sense. A relation $y \prec x$ should be thought as "y is a codimension-1 subface of x". Since in an opetope, faces come with an orientation, the sign tells if y is an input or an output of x.

For exemple, if the picture belows is part of an opetope, we should have

$$b_1 \prec^- a \qquad b_2 \prec^- a \qquad b_3 \prec^- a \qquad b_4 \prec^+ a$$

$$\begin{tikzcd}
& \bullet \arrow[rr, "b_2"] & {} \arrow[d, Rightarrow, "a"] & \bullet \arrow[dr, "b_3"] & \\
\bullet \arrow[ur, "b_1"] \arrow[rrrr, "b_4"'] & & {} & & \bullet
\end{tikzcd}$$

In order to define formally DFCs, we first need the notion of *positive-to-one poset* (POP).

Definition 1.5 : Positive-to-one poset

A positive-to-one poset consists of:

- A finite set of elements P.
- A gradation $\mathsf{dim} : P \to \mathbb{N}$.
- Two binary relations $\prec^-$ and $\prec^+$ on P, and we let $x \prec y$ iff $x \prec^- y$ or $x \prec^+ y$.

With the following properties:

- $\forall x, y \in P, \quad y \prec x \to \mathsf{dim}(x) = \mathsf{dim}(y) + 1$.
- $\forall x, y \in P, \quad \neg\,(y \prec^- x \wedge y \prec^+ x)$.
- $\forall x \in P, \quad \mathsf{dim}(x) \geq 1 \to (\exists! y, y \prec^+ x) \wedge (\exists y, y \prec^- x)$.

In particular: $\prec$, $\prec^-$ and $\prec^+$ are asymmetric, and the reflexive transitive closure of $\prec$ equips P with a structure of partially ordered set, such that dim is an increasing map.
Following the conventions of [9], for $x \in P$, we denote

$$\delta(x) := \{y \in P \mid y \prec^- x\}$$

and when $\mathsf{dim}(x) \geq 1$,

$$\gamma(x) := \{y \in P \mid y \prec^+ x\}$$

because of the third property, $\gamma(x)$ is always a singleton, hence we sometimes identify $\gamma(x)$ with its unique element, which we call the *target* of x. For $k \in \mathbb{N}$, we also denote

$$P_k := \mathsf{dim}^{-1}(\{k\}), \ P_{\geq k} := \bigcup_{i \geq k} P_i, \ P_{>k} := \bigcup_{i > k} P_i$$

and we let $\mathsf{dim}(P) := \max\{\mathsf{dim}(x)\}_{x \in P}$ be the *dimension* of P.

1.6. From now on, we will use a kind of HASSE diagrams to represent POPs. The convention will be as follows:

$$\begin{tikzcd}
x \arrow[d, no head] & x \arrow[d, no head, "+" description] & x \arrow[d, no head, "\alpha" description] \\
y & y & y
\end{tikzcd}$$

$$y \prec^- x \qquad\qquad y \prec^+ x \qquad\qquad y \prec^\alpha x$$

Definition 1.7 : Dendritic face complex

A dendritic face complex is a positive-to-one poset C, satisfying the following extra axioms:

- (*greatest element*)
  There is a greatest element in C, for the partial order induced by $\prec$.
- (*oriented thinness*)
  For $z \prec y \prec x$ in P, there is a unique $y' \neq y$ in P such that $z \prec y' \prec x$. Hence there is a lozenge as in Figure 1.1 below. Moreover, we ask for the *sign rule* $\alpha\beta = -\alpha'\beta'$ to be satisfied. When finding such a y' we say that we *complete the half lozenge* $z \prec y \prec x$.
- (*acyclicity*)
  For $x \in P_1$, $\delta(x)$ is a singleton.
  Let $x \in P_{\geq 1}$, then $\delta(x) \neq \emptyset$ and there is no cycle as in Figure 1.2 below.

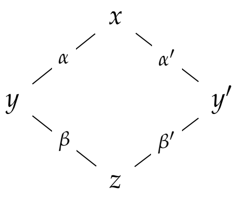


Figure 1.1: Lozenge

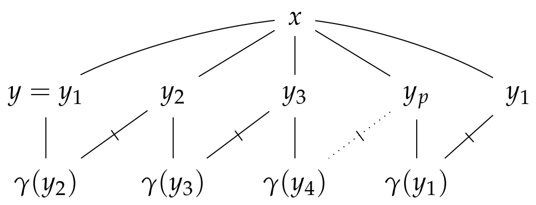


Figure 1.2: Cycle

Proposition 1.8 : Tree structure

Let P be a DFC and $x \in P_{\geq 1}$. There is a rooted tree structure on $\delta(x)$, given by the following data:

- The set of nodes is $\delta(x)$.
- For each $y \in \delta(x)$, $A(y) := \delta(y)$.
- There is a triplet $y \mathrel{-\!\!\prec}_z y'$ iff there is a lozenge as in Figure 1.3 below.
- When $\mathsf{dim}(x) \geq 2$, the root is given, as in [9], by the unique $\rho(x)$ completing the lozenge as in Figure 1.4 below.

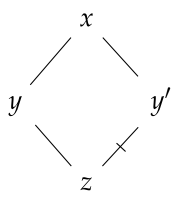


Figure 1.3: Triplet

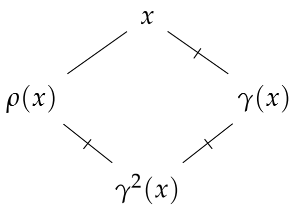
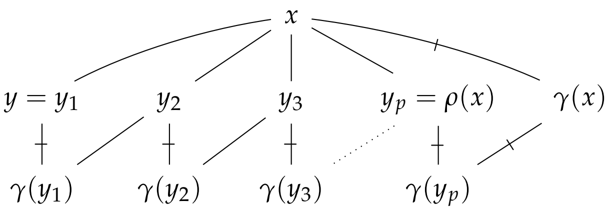


Figure 1.4: Root

Proof. If $\mathsf{dim}(x) = 1$, then $\delta(x)$ is a singleton, hence there is a structure of tree as stated.
Suppose that $\mathsf{dim}(x) > 1$. By oriented thinness, we may define the root of $\delta(x)$ as above.
If there is a triplet $y \mathrel{-\!\!\prec}_z y'$ in $\delta(x)$, then y' is uniquely determined by the uniqueness of lozenge completion.
Let $y \in \delta(x)$, then by successively completing lozenges (from left to right in the following diagram) we

obtain a configuration as below:



That is, we keep completing lozenges while the completion of $\gamma(y_i) \prec^+ y_i \prec^- x$ yields $\gamma(y_i) \prec^- y_{i+1} \prec^- x$. Because of acyclicity, there is some $p \geq 1$ such that the completion of $\gamma(y_p) \prec^+ y_p \prec^- x$ must yield the rightmost lozenge as above. Hence, this gives a descending path from y to the root of $\delta(x)$. Because of uniqueness of lozenge completion, this is the unique such path. ☐

Definition 1.9 : morphism of DFC

Let C and D be two dendritic faces structures. Then a morphism $f : C \to D$ of DFCs corresponds to the data of such a map f between the underlying sets of C and D, such that:

- f preserves the dimension.
- When $y \prec^+ x$ in C, $f(y) \prec^+ f(x)$ in D.
- When $y \prec^- x$ in C, $f(y) \prec^- f(x)$ in D.
- $f_x := f|_{\delta(x)} : \delta(x) \to \delta(f(x))$ is a bijection.

1.10
It follows from the definition that every morphism $f : C \to D$ induces an isomorphism onto its image $\mathsf{cl}(f(\top))$ where $\top$ denotes the greatest element of C, and cl the downward closure.

Definition 1.11

Let C be an POP of dimension n. For $k \in [\![0, n]\!]$, we introduce the two following sets:

$$\Gamma_k := \gamma(C_{k+1}) \qquad \text{and its complement} \qquad \Lambda_k := C_k \setminus \gamma(C_{k+1})$$

2 Positive opetopes

In this section, we describe positive opetopes, as they are defined by ZAWADOWSKI in [9]. They are another formalism dedicated to encoding opetopes, in the wider context of opetopic cardinals. We need first a notion of *positive hypergraphs*.

Definition 2.1 : Positive hypergraph

A positive hypergraph consists of:

- A family of finite sets $(S_k)_{k\in\mathbb{N}}$ of *faces* such that for k large enough, $S_k = \emptyset$.
- For all $k \in N$, a function $\gamma_k : S_{k+1} \to S_k$ and a total relation $\delta_k : S_{k+1} \to S_k$. We moreover ask for δ_0 to be functional.

We introduce the notations $S_{\geq k}$ for $\bigcup_{i\geq k} S_i$ and $S_{>k}$ for $\bigcup_{i>k} S_i$. Following [9], we will omit indices for δ_i's and γ_i's, and denote for any $x \in S$, $\gamma^{(k)}(x)$ the iterate $\gamma^q(x)$ such that $\gamma^q(x) \in S_k$.
We also let, for $x \in S$, $\iota(x) := \delta\delta(x) \cap \gamma\delta(x)$.

Definition 2.2 : Morphism of positive hypergraphs

Let $S = (S_k)_{k\in\mathbb{N}}$ and $T = (T_k)_{k\in\mathbb{N}}$ be two positive hypergraphs, then a *morphism* between S and T corresponds to a family of maps $(f_k : S_k \to T_k)_{k\in\mathbb{N}}$ such that:

- $\forall x \in S_{\geq 1}$, the restriction $f_x := f|_{\delta(x)} : \delta(x) \to \delta(f(x))$ is a bijection.
- $\forall x \in S_{\geq 1}$, $\gamma(f(x)) = f(\gamma(x))$.

2.3. The positive-to-one posets are the analogue of positive hypergraphs in the formalism of dendritic face structures. More precisely: let **POP** be the category of positive-to-one posets, and **pHg** the category of positive hypergraphs. Then we may associate to a POP $(P = \coprod_{k\in\mathbb{N}} P_k, \prec^+, \prec^-)$ a positive hypergaph $F(P)$ such that:

- For all $k \in \mathbb{N}$, the set of k-dimensional faces of $F(P)$ is defined as P_k.
- For all $k > 0$, we define the function γ_k as $\gamma| : P_{k+1} \to P_k$.
- For all $k > 0$, we define the total relation δ_k as $\delta| : P_{k+1} \to \mathcal{P}(P_k)$.

Conversely, to any positive hypergraph $(S = (S_k)_{k\in\mathbb{N}}, \gamma, \delta)$, we may associate a positive-to-one poset $G(S)$ defined as follows:

- The set of elements is $\coprod_{k\in\mathbb{N}} S_k$.
- For all k, the gradation dim sends any $x \in S_k$ on k.
- $\prec^-$ is defined by $y \prec^- x$ iff $y \in \delta(x)$, and $\prec^+$ by $y \prec^+ x$ iff $y = \gamma(x)$.

We may also associate to any morphism $f : P \to Q$ of POPs a morphism of positive hypergaphs $F(f) : F(P) \to F(Q)$ defined by $F(f)_k := f| : P_k \to Q_k$. Conversely, to any morphism of positive hypergraphs $g : S \to T$ we may associate a morphism of POPs $G(g) : G(S) \to G(T)$ defined by $G(g)|_{S_k}^{T_k} := g_k$. A straightforward check yields the following result:

Theorem 2.4

The functors $\mathbf{POP} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{pHg}$ form an equivalence of categories.

2.5. From now on, we will make implicit use of this correspondance. For instance we may use the notation $\prec^-$, $\prec^+$ and $\prec$ for positive hypergraphs.

Definition 2.6 : $\lhd^{S_k,+}$, $\lhd^{S_k,-}$, $<^{S_k,+}$, $<^{S_k,-}$

We define the following relations:

- $<^{S_0,-}$ is the empty relation.
  For $k > 0$, $<^{S_k,-}$ is the transitive closure of the relation $\lhd^{S_k,-}$ on S_k, such that $x \lhd^{S_k,-} x'$ iff $\gamma(x) \in \delta(x')$. We write $x \perp^- x'$ iff either $x <^- x'$ or $x' <^- x$, and we write $x \leq^- x'$ iff either $x = x'$ or $x <^- x'$.
  A *lower path* is a sequence $x_0 \succ^+ y_0 \prec^- \cdots \prec^- x_{p-1} \succ^+ y_p \prec^- x_p$.
- $<^{S_k,+}$ is the transitive closure of the relation $\lhd^{S_k,+}$ on S_k, such that $x \lhd^{S_k,+} x'$ iff there is $w \in S_{k+1}$, such that $x \in \delta(w)$ and $\gamma(w) = x'$. We write $x \perp^+ x'$ iff either $x <^+ x'$ or $x' <^+ x$, and we write $x \leq^+ x'$ iff either $x = x'$ or $x <^+ x'$.
  An *upper path* is a sequence $y_0 \prec^- x_1 \succ^+ y_1 \prec^- \cdots \prec^- x_p \succ^+ y_p$.

Definition 2.7 : Opetopic cardinal, positive opetope

An opetopic cardinal corresponds to the data of a positive hypergraph S satisfying the following axioms:

- (*Globularity*)
  For $x \in S_{\geq 2}$,
  $$\gamma\gamma(x) = \gamma\delta(x) \setminus \delta\delta(x) \qquad\qquad \delta\gamma(x) = \delta\delta(x) \setminus \gamma\delta(x).$$
- (*Strictness*)
  For $k \in \mathbb{N}$, $<^{S_k,+}$ is a strict order and $<^{S_0,+}$ is linear.
- (*Disjointness*)
  For $k > 0$, $\perp^{S_k,-} \cap \perp^{S_k,+} = \emptyset$.
- (*Pencil linearity*)
  For any $k > 0$ and $y \in S_{k-1}$, the sets below are linearly ordered by $<^{S_k,+}$.
  $$\{x \in S_k \mid y = \gamma(x)\} \qquad \text{and} \qquad \{x \in S_k \mid y \in \delta(x)\}.$$

An opetopic cardinal S is called *principal* if for all $k \leq \mathsf{dim}(S)$, $|S_k \setminus \delta(S_{k+1})| = 1$.
A *positive opetope* is a principal opetopic cardinal.

3 From dendritic face complexes to ZAWADOWSKI's positive opetopes

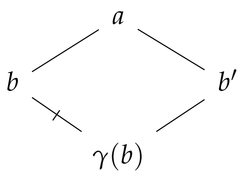
3.1. Let $(C = \coprod_{k\in\mathbb{N}} C_k, \prec^+, \prec^-)$ be a dendritic face complex. Recall from Theorem 2.4 that C may be given a structure of positive hypergraph. Denote by n the dimension of its greatest element ω. Our goal is to prove that C is indeed a positive opetope, in the sense of ZAWADOWSKI. We prove the required properties in the following order: globularity, strictness and principality, pencil linearity and then disjointness. In what follows, $<^+$, $<^-$, $\lhd^+$ and $\lhd^-$ refer to the corresponding notations introduced in Definition 2.6.

Proposition 3.2 : globularity

The positive hypergraph C satisfies the property of globularity. That is:

$$\gamma\gamma = \gamma\delta - \delta\delta \qquad \text{and} \qquad \delta\gamma = \delta\delta - \gamma\delta$$

Proof. We first prove that $\gamma\gamma = \gamma\delta - \delta\delta$:
Let $k \geq 2$ and $a \in C_k$, and let $b \in \delta(a)$. Then we have $\gamma(b) \prec^+ b \prec^- a$, and hence two possible lozenge completions:



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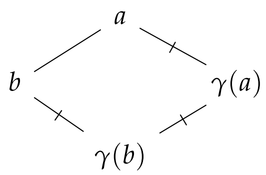
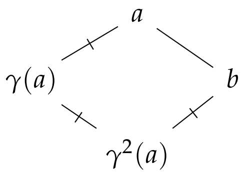
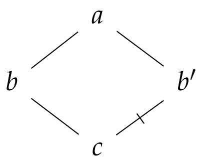


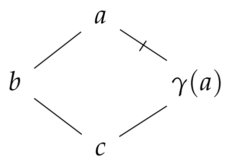
In the first case, $\gamma(b) \in \delta\delta(a)$. In the second case, $\gamma(b) = \gamma^2(a)$. Hence $\gamma\delta(a) - \delta\delta(a) \subseteq \gamma\gamma(a)$ For the converse inclusion, we observe that there is only one possibility of lozenge completion:



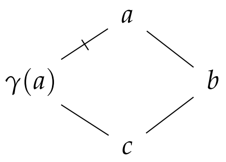
Hence, $\gamma^2(a) \in \gamma\delta(a)$ and because of the sign rule, it is impossible to have $\gamma^2(a) \in \delta\delta(a)$. So by double inclusion, we have shown that $\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a)$.
We now prove the second equation: $\delta\gamma = \delta\delta - \gamma\delta$. Let $k \geq 2$, $a \in C_k$, and $c \prec^- b \prec^- a$. Then we have two possible lozenge completions:



or



In the first case, $c \in \gamma\delta(a)$ and in the second one, $c \in \delta\gamma(a)$. Hence $\delta\delta(a) - \gamma\delta(a) \subseteq \delta\gamma(a)$. On the other hand, if $c \prec^- \gamma(a) \prec^+ a$, then the only possible shape for lozenge completion is



Hence $c \in \delta\delta(a)$, and because of the sign rule, $c \notin \gamma\delta(a)$.
Whence the second equality: $\delta\gamma(a) = \delta\delta(a) - \gamma\delta(a)$. □

Proposition 3.3 : strictness, first part

C satisfies the first half of the axiom of strictness. That is: $<^+$ is a strict partial order.

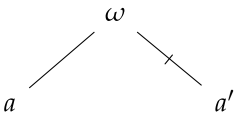
Proof. We shall prove the following three properties:

$$\begin{array}{rll}
(k \in [\![0, n]\!]) & \mathcal{P}_k : & <^{C_k,+} \text{ is a strict order} \\
(k \in [\![0, n-1]\!]) & \mathcal{Q}_k : & \forall e \in \Lambda_k, \exists! d \in \Lambda_{k+1} \text{ s.t. } e \prec^- d \\
(k \in [\![0, n-1]\!]) & \mathcal{R}_k : & \forall e \in \Lambda_k, e \prec^- \gamma^{(k+1)}\omega
\end{array}$$

by induction on the codimension $n - k$.

- $\boxed{k = n}$
  Since ω is a greatest element, $C_n = \{\omega\}$. Hence $\mathcal{P}_n$ is clear.

- $\boxed{k = n - 1}$.
  Because ω is a greatest element, when $a \lhd^{C_{n-1},+} a'$, the only possible situation is the following:

  Hence a cycle $a_0 \lhd^{C_n,+} a_1 \lhd^{C_n,+} \cdots \lhd^{C_n,+} a_0$ must have the form $a_0 \lhd^{C_n,+} a_0$ with $n = 0$ and $a_0 = \gamma(\omega)$. But $\gamma(\omega) \not\prec^- \omega$. So there is no such cycle. Whence $\mathcal{P}_{n-1}$.

  Because $\omega \in \Lambda_n$, and because it is a greatest element, $\mathcal{Q}_{n-1}$ and $\mathcal{R}_{n-1}$ are clear.

- $\boxed{\text{Induction}}$ We suppose now $\mathcal{P}_{k+1}$, $\mathcal{Q}_{k+1}$ and $\mathcal{R}_{k+1}$. First, we prove $\mathcal{Q}_k$.
  - *Uniqueness*: Let $e \in C_k$ and suppose $e \prec^- d$ and $e \prec^- d'$, with $d, d' \in \Lambda_{k+1}$. By using $\mathcal{R}_{k+1}$ we may construct a lozenge as in Figure 3.1 below:

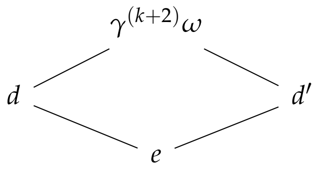
    

    Figure 3.1: Degenerated lozenge

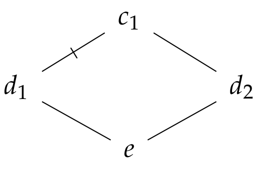
    

    Figure 3.2: Lozenge

    and because of the sign rule, it must be the case that $d = d'$.
  - *Existence*. Let $e \in \Lambda_k$. Since e must be a codimension-1 subface of some d, we may choose such a $d =: d_1$. And because e is not a target, we must have $e \prec^- d_1$. If d is in Λ_{k+1}, then we are done. Else, let c_1 be such that $d_1 = \gamma(c_1)$. Then we may complete $e \prec^- d_1 \prec^+ c_1$ into a lozenge as in Figure 3.2 above. If $d_2 \in \Lambda_{k+1}$, we have finished. Else, we continue with d_2, taking c_2 such that $d_2 \prec^+ c_2$ *etc.*. While iterating this construction, we cannot produce a loop as in Figure 3.3 below:

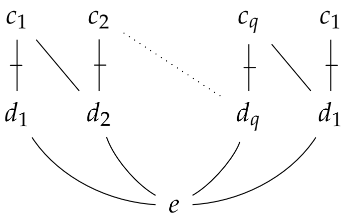
    

    Figure 3.3: Impossible loop

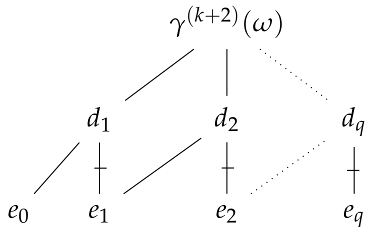
    

    Figure 3.4: Path

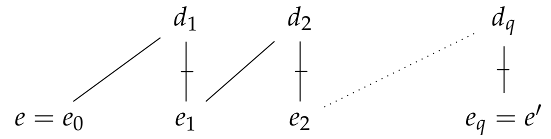
    because it would contradict $\mathcal{P}_{k+1}$. Hence, each time $d_i \notin \Lambda_{k+1}$, the next element d_{i+1} is a new one, and this construction must finish because the poset is finite. So when the construction ends, it produces an element $e \prec^- d_q \in \Lambda_{k+1}$. Whence the existence.

We then prove $\mathcal{R}_k$. Let $e \in \Lambda_{k+1}$, then using $\mathcal{Q}_k, \mathcal{R}_{k+1}$, we may find some $d \in \Lambda_{k+1}$ such that $e \prec^- d \prec^- \gamma^{(k+2)}\omega$. Since e is not a source, the only possible lozenge completion for this triple yield $e \prec^- \gamma^{(k+1)}\omega \prec^+ \gamma^{(k+2)}\omega$. Whence $\mathcal{R}_k$.

Now we prove $\mathcal{P}_k$. The strategy is to replace a related pair $e \lhd^{C_{k'},+} e'$ by a path $e = e_0 \lhd^{C_{k'},+} e_1 \lhd^{C_{k'},+} \cdots \lhd^{C_{k'},+} e_q = e'$ as in Figure 3.4 above: Indeed, if we are able to do this, every cyclic path $e_0 \lhd^{C_{k'},+} e_1 \lhd^{C_{k'},+} \cdots \lhd^{C_{k'},+} e_q = e_0$ will induce a longer path $e_0 = e'_0 \lhd^{C_{k'},+} e'_1 \lhd^{C_{k'},+} \cdots \lhd^{C_{k'},+} e'_m = e_0$ as above, which cannot exist because it would imply the existence of a cycle in the tree structure of $\delta(\gamma^{(k+2)}\omega)$. First, notice that if we know each d_i is in Λ_{k+1} then we know that $d_i \prec^- \gamma^{(k+2)}\omega$ (by $\mathcal{R}_{k+1}$). So we will be able to conclude if we prove 3.4 below.

Lemma 3.4

if $e \lhd^{C_{k'},+} e'$, then we have a sequence as follows, with $d_1, d_2, \cdots, d_q \in \Lambda_{k+1}$.



Proof. If $e \prec^- d \succ^+ e'$ with $d \in \Lambda_{k+1}$, then the result is proven. So we may suppose that d is the target of some element: $d = \gamma(c)$, as in Figure 3.5 below.

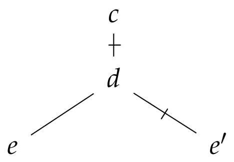


Figure 3.5

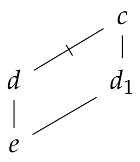


Figure 3.6

First, complete the left half lozenge with some d_1 as in Figure 3.6 above.

Then we consider its target $\gamma(d_1)$, and complete $\gamma(d_1) \prec^+ d_1 \prec^- c$ as a lozenge. We have two possibilities. Either the completion is of the type 1 as in Figure 3.7 below, or it of the type 2 as in Figure 3.8 below:

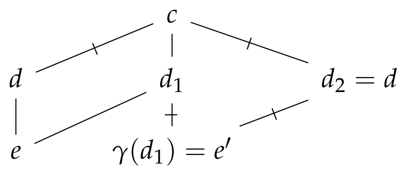


Figure 3.7: Type 1

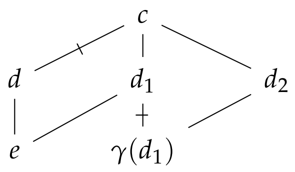


Figure 3.8: Type 2

In the second case, we keep completing lozenges on the right until ending with a diagram of the shape of Figure 3.9 below:

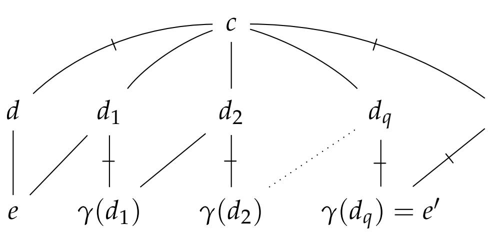


Figure 3.9: Path

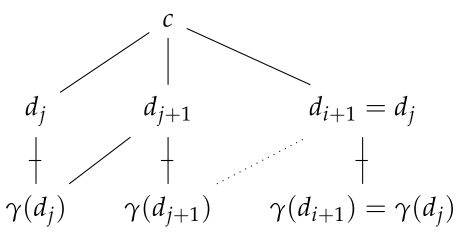
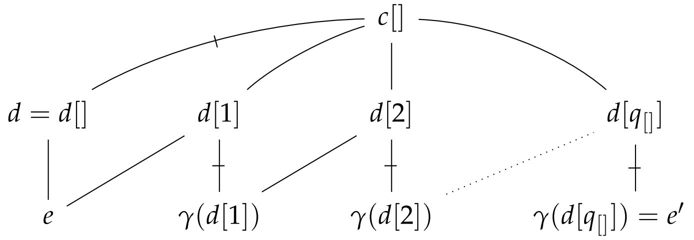


Figure 3.10: Cycle

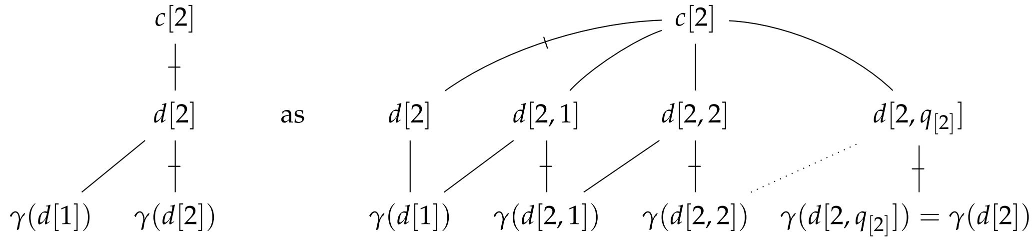
First, we will never encounter a situation where $d_{i+1} = d_j$ for a $j \leq i$ as in Figure 3.10 above, because it would imply having a cycle

$$d_j \prec_{\gamma(d_j)} d_{j+1} \prec_{\gamma(d_{j+1})} \cdots \prec_{\gamma(d_i)} d_{i+1} = d_j$$

hence a cycle in the tree structure of $\delta(c)$. And this construction must finish as above because of the finitness of the poset C. Now, after renaming the elements, we end up with this situation.



If each $d[i]$ is in Λ_{k+1} then we are done. In the other case, for example if $d[2] \prec^+ c[2]$, then we can once again *unfold*



Now we may iterate this unfolding process. This will produce a tree shaped collection of d's and c's, with relations

$$d[a_1, \cdots, a_p] \prec^+ c[a_1, \cdots, a_p] \qquad \text{and} \qquad c[a_1, \cdots, a_p] \succ^- d[a_1, \cdots, a_p, a]$$

for each $a \in \left[\!\left[1, q_{[a_1,\cdots,a_p]} \right]\!\right]$. Each branch

$$d[] \prec^+ c[] \succ^- d[a_1] \prec^+ c[a_1] \succ^- d[a_1, a_2] \prec^+ \cdots \succ^- d[a_1, \cdots, a_p] \prec^+ \cdots$$

must be finite, otherwise it would contradict $\mathcal{P}_{k+1}$. *i.e.* at some point, all $d[a_1, \cdots, a_p]$ are in Λ_{k+1}. We thus obtain a path of the desired form, which completes the proof of Lemma 3.4. ☐

Notice that through the proof above, we have seen the following result (*c.f.* $\mathcal{R}_k$):

Lemma 3.5

$\forall k \in [\![0, n-1]\!], \quad \Lambda_k \subseteq \delta(\gamma^{(k+1)}\omega).$

Lemma 3.6

Suppose $d \in C_k$ is not a source (*i.e.* there is no $c \in C_{k-1}$ such that $d \prec^- c$). Then $d = \gamma^{(k)}\omega$.

Proof. We will proceed by induction on the codimension of d.

- $\boxed{k = n}$
  In this case, $d = \omega$ is the only n-dimensional cell.

- $\boxed{\text{Heredity}}$
  Suppose that we know the result for all cells of dimension $k+1$, and let $d \in C_k$ not being a source. Because ω is a greatest element, we know that there is a chain from d to ω. Hence, there is $c_0 \in C_{k+1}$ such that $d \prec^+ c_0$, *i.e.* $d = \gamma(c)$. If c_0 is not a source, then we are done by induction hypothesis. In the other case, there is $b_0 \succ^- c$, thus there is a lozenge completion as in Figure 3.11 below: with $d \prec^+ c_1$, because d is not a source. If c_1 is not a source, then we are done by induction hypothesis. Else, there is $b_1 \in C_{k+1}$, such that $b_1 \succ^- c_1$. Hence we may keep completing lozenges as in Figure 3.12 above, until coming accross some c_p which is not a source. This proccess must end because of strictness (first part). By induction hypothesis, c_p is an iterated target, and so is d. □

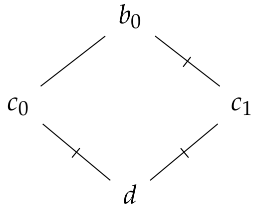


Figure 3.11: Lozenge completion

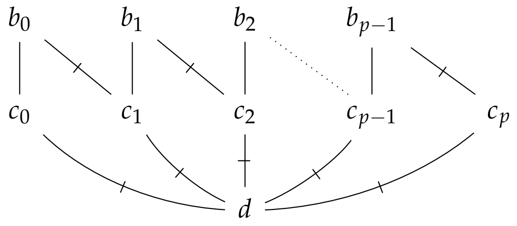


Figure 3.12: Path

Lemma 3.7

If d is a source, then $\exists! c \in \Lambda_{k+1}$ s.t. $d \prec^- c$.

Proof. Let c_0 be such that $d \prec^- c_0$. Suppose $c_0 \in \Lambda_{k+1}$, then we are done taking $c = c_0$. Else as in the proof of Lemma 3.6 we may complete lozenges from left to right (refer to Figure 3.13 below), until coming accross some $c \in \Lambda_{k+1}$. Notice that it must finish because $<^+$ is a strict order. The uniqueness follows also from the same argument as in the proof of $\mathcal{Q}_k$. □

Lemma 3.8

If $d \in C_k$ is a target, then it is the target of a unique $c \in \Lambda_{k+1}$.

Proof. The proof of existence goes by the same kind of construction as above, by filling lozenges as below from left to right until coming accross some $c \in \Lambda_{k+1}$ (refer to Figure 3.14 below). The uniqueness comes from the same argument as for the uniqueness in $\mathcal{Q}_k$. □

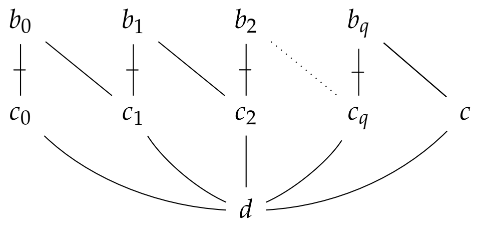


Figure 3.13

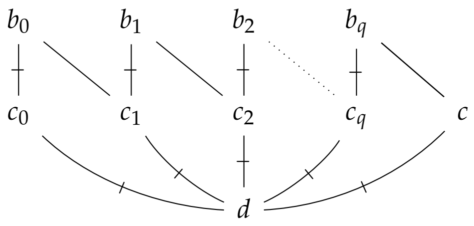


Figure 3.14

We mention that Lemma 3.5 has a converse, although we will not use it later on.

Lemma 3.9

If $d \prec^- \gamma^{(k+1)}\omega$, then $d \in \Lambda_k$.

Proof. Suppose d is a target, then because of Lemma 3.8, there is $c \in \Lambda_{k+1}$ s.t. $d = \gamma(c)$. But because c is in Λ_{k+1}, we have seen in the proof of strictness that $c \prec^- \gamma^{(k+2)}\omega$. Hence we are in the situation of Figure 3.15 below, which is prohibited by the sign rule, whence $d \in \Lambda_{k+1}$. ☐

Lemma 3.10

$\gamma^{(k)}\omega$ is not a source.

Proof. Suppose that it is, then by Lemma 3.7 it is the source of some $c \in \Lambda_k$. Hence we have the lozenge of Figure 3.16 below, which is prohibited by the sign rule. ☐

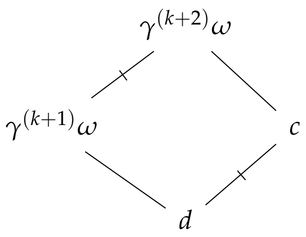


Figure 3.15: Forbidden lozenge

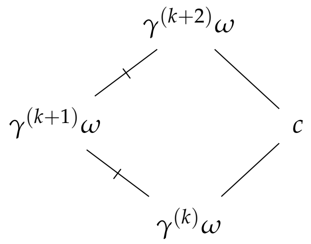


Figure 3.16: Forbidden lozenge

We also have the following corollary of Lemma 3.6, Lemma 3.7 and Lemma 3.8:

Lemma 3.11

If $d \in C_k$ for some $k < n$, it is the source of a unique $c \in \Lambda_{k+1}$ or the target of a unique $c \in \Lambda_{k+1}$.

Finally, we may state the following theorem:

Theorem 3.12 : Sources partition

$$(0 \leq k \leq n) \quad C_k \setminus \{\gamma^{(k)}\omega\} = \coprod_{c \in \Lambda_{k+1}} \delta(c)$$

Proof.

- $C_k \setminus \{\gamma^{(k)}\omega\} \subseteq \coprod_{c \in \Lambda_{k+1}} \delta(c)$ is given by Lemma 3.6 and Lemma 3.7.
- $C_k \setminus \{\gamma^{(k)}\omega\} \supseteq \coprod_{c \in \Lambda_{k+1}} \delta(c)$ is given by Lemma 3.10.

$\square$

Proposition 3.13 : principality

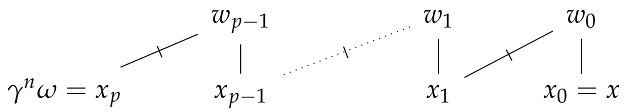
The positive hypergraph C is principal.

Proof. It follows directly from Theorem 3.12. $\square$

Proposition 3.14 : strictness ($<^{C_0,+}$ is total)

C satisfies the second half of the strictness property. That is: $<^{C_0,+}$ is a linear total order.

Proof. Let $x \in C_0$. From Theorem 3.12, we know that either $x = \gamma^n\omega$, or x is the source of a unique $w \in \Lambda_1$. By iterating this case disjunction, we may produce a unique path as below:



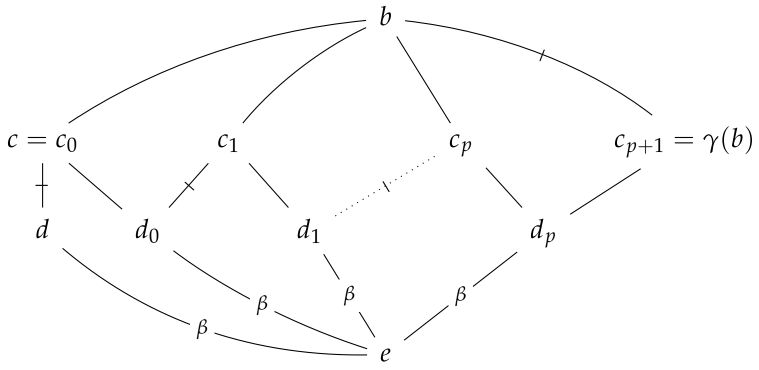
where the w_i's are in Λ_1. Because of the functionality of δ_0 and Lemma 3.8, the path from left to right starting from $\gamma^n\omega$ is also unique (and this is independant of x). Hence by extending this path as far to the right as possible (which ends because $<_{c_0,+}$ is a strict order), all 0-dimensional elements must appear somewhere along the path. Which proves that $<^{C_0,+}$ is total. $\square$

Lemma 3.15

For every configuration $e \prec^\beta d \prec^+ c \prec^- b$, there is a (unique) chain

$$c = c_0 \succ^- d_0 \prec^+ c_1 \succ^- d_1 \prec^+ \cdots \prec^+ c_p \succ^- d_p \prec^- c_{p+1} = \gamma(b)$$

with $p \geq 0$, as below:



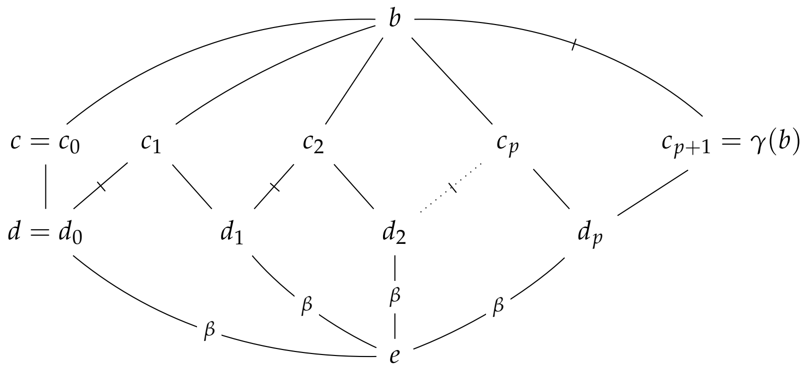
Proof. By lozenge completion, we may find a unique d_0 completing the lozenge (c, d, e, d_0). Because of the sign rule, we have $e \prec^\beta d_0 \prec^- c =: c_0$. Then we may find a unique c_1 completing the lozenge (b, c_0, d_0, c_1). If $d_0 \prec^- c_1 \prec^+ b$, then we are done taking $p = 1$. Else, we continue this process starting from d_0. It must finish because of strictness. $\square$

Lemma 3.16

For every configuration $e \prec^\beta d \prec^- c \prec^- b$, there is a (unique) chain

$$c = c_0 \succ^- d = d_0 \prec^+ c_1 \succ^- d_1 \prec^+ \cdots \prec^+ c_p \succ^- d_p \prec^- c_{p+1} = \gamma(b)$$

with $p \geq 0$ as below:



Proof. We follow the same argument as for Lemma 3.15. By lozenge completion, we may find a unique c_1 completing the lozenge (b, c_0, d_0, c_1). If $d_0 \prec^- c_1 \prec^+ b$, then we are done taking $p = 0$. Else, we may find a unique d_1 completing the lozenge (c_1, d_0, e, d_1). Because of the sign rule, $e \prec^\beta d_1 \prec^- c_1$. Then we continue this process starting from c_1. It must finish because of strictness. ☐

The two previous lemmas yield the following:

Lemma 3.17

For every hexagon as in Figure 3.17, there is – up to potentially exchanging c and c' – a (maybe trivial) lower path from c to c' as in Figure 3.18 below:

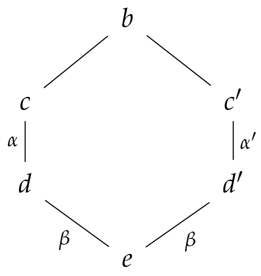


Figure 3.17: Hexagon

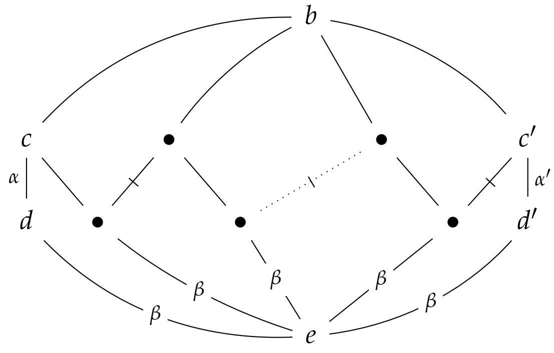


Figure 3.18: Filled hexagon

Proof. Suppose that we have such a hexagon, then using either Lemma 3.15 or Lemma 3.16, we may produce two paths as described in the those lemmas, starting from the left side and the right side of the hexagon, respectively. But those two paths must finish at the same source of $\gamma(b)$. Indeed, because of uniqueness of lozenge completion, there is at most one d'' with $e \prec^\beta d'' \prec^- \gamma(b)$. Hence, both paths are ascending paths in $\delta(b)$, reaching the same leaf. More precisely, in the tree structure of $\delta(b)$, they both are of the form:

$$c_0 \mathrel{-\!\!\prec}_{d_0} c_1 \mathrel{-\!\!\prec}_{d_1} \cdots \mathrel{-\!\!\prec}_{d_p} c_p \mathrel{-\!\!\prec}_{d_{p+1}=d''}$$

Because of the tree structure, one of those paths must be an extension of the other one, which yields the result. ☐

Definition 3.18 : zig-zag

A *zig-zag* from c to c' is the data of a sequence as follows:

$$c = c_0 \succ^{\alpha_0} d_0 \prec^{-\alpha_0} c_1 \succ^{\alpha_1} d_1 \prec^{-\alpha_1} c_2 \succ^{\alpha_2} \cdots \succ^{\alpha_{p-1}} d_{p-1} \prec^{-\alpha_{p-1}} c_p = c'$$

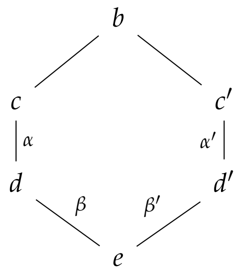
Such a zig-zag is said to be *simple* whenever $\forall i,\ d_i \neq d_{i+1}$.
It is said to be *non-trivial* if $p > 0$. If $\forall i,\ c_i \prec^- b$, the zig-zag will be called a *$\delta(b)$-zig-zag*.
Notice that because of the tree-structure on $\delta(b)$, if $c,\ c' \prec^- b$ there is a unique simple $\delta(b)$-zig-zag between c and c'. In term of rooted trees, a simple zig-zag is a sequence of the following form:

$$c_0 \succ_{d_0} c_1 \succ_{d_1} \cdots \succ_{d_{r-1}} c_r \prec_{d_r} \cdots \prec_{d_{p-1}} c_{p-1} \prec_{d_p} c_p$$

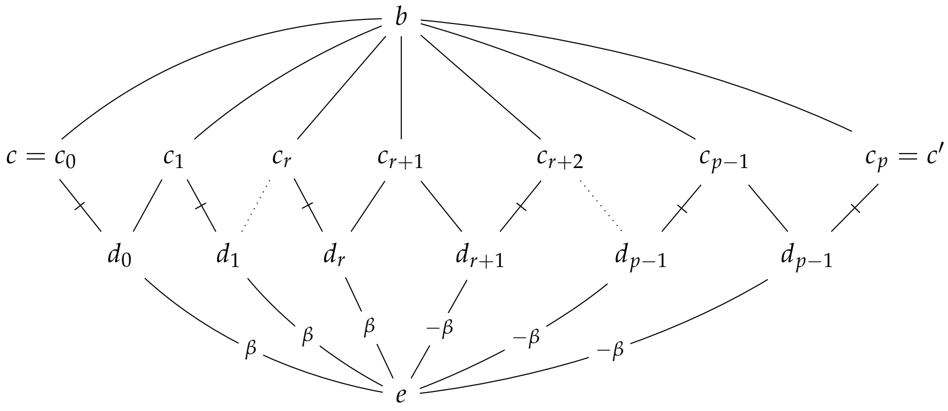
where no two triplets are the same.

Proposition 3.19 : Hexagon property

For every hexagon



Either $c = c'$, or (potentially by exchanging the role of c and c') there is a non-trivial simple $\delta(b)$-zig-zag as below:



Proof. Lemma 3.17 shows the case $\beta = \beta'$. We should now consider the case where we have a hexagon as above, with $\beta' = -\beta$. Starting from the left part of the hexagon $e \prec^\beta d \prec^\alpha c \prec^- b$, we may either construct a sequence as in Figure 3.19, or as in Figure 3.20 below (with $p \geq 0$):

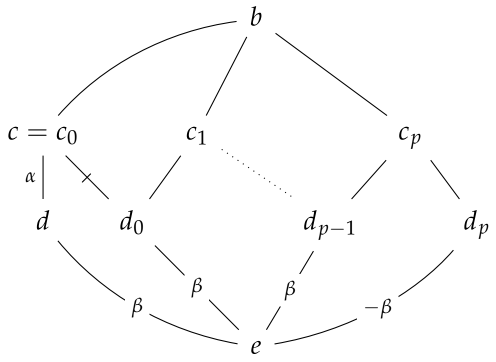


Figure 3.19: First case

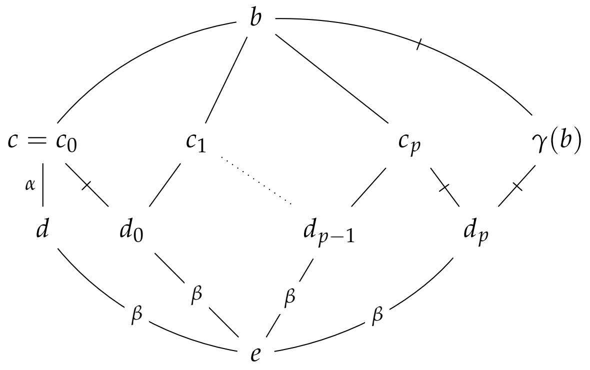


Figure 3.20: Second case

To see this we fill lozenges from left to right as follows: First, we let $d_0 := d$ if $\alpha = +$, and in the other case, we find d_0 as the unique one completing the lozenge (c_0, d, e, d_0). If $e \prec^{-\beta} d_0 \prec^{-} c_0$, then we are in the first case, with $p = 0$. Else we find the unique c_1 filling the lozenge (b, c_0, d_0, c_1). If $d_0 \prec^{+} c_1 \prec^{+} b$, then we are in the second case with $p = 0$. Else, we continue the same process, filling the lozenge (c_1, d_0, e, d_1) *etc.* This process must finish because of strictness.

Now, if we end up in the first case, we may use Lemma 3.17 to find a path between c_p and c', and conclude. If we end up in the second case, we repeat the same argument with the right parenthesis $e \prec^{-\beta} d' \prec^{\alpha'} c' \prec^{-} b$. We thus find a path, either with the shape of Figure 3.21, or with the shape of Figure 3.22 below (with $q \geq 0$):

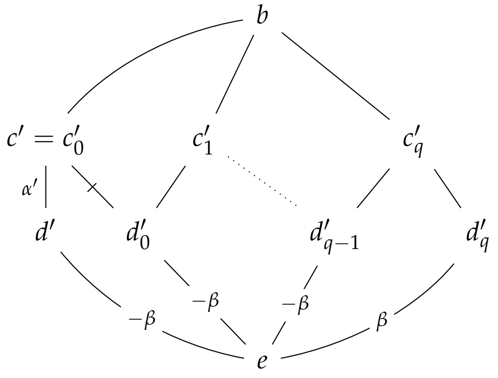


Figure 3.21: First case

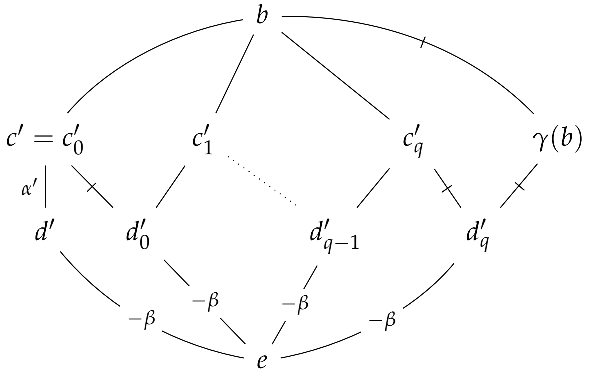
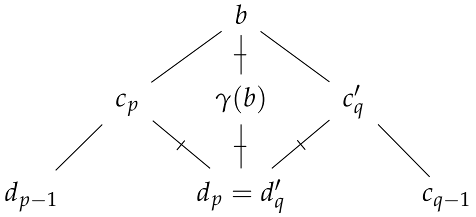
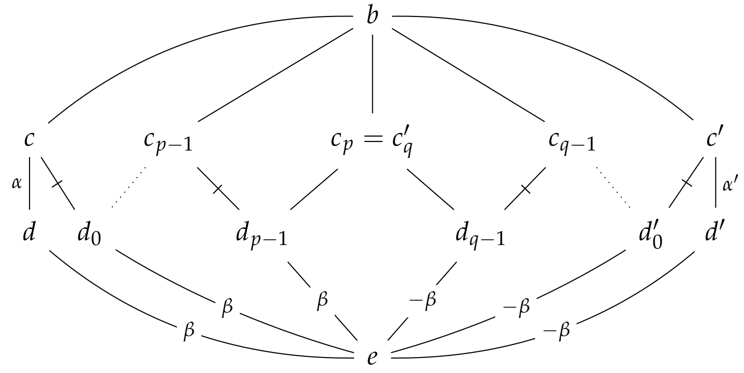


Figure 3.22: Second case

In the first case, we may conclude using Lemma 3.17. In the second case, we end up with the following configuration:



So, by uniqueness of lozenge completion, $c_p = c'_q$ and we have the following zig-zag between c and c':

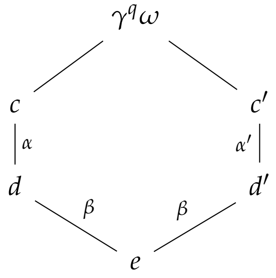


$\square$

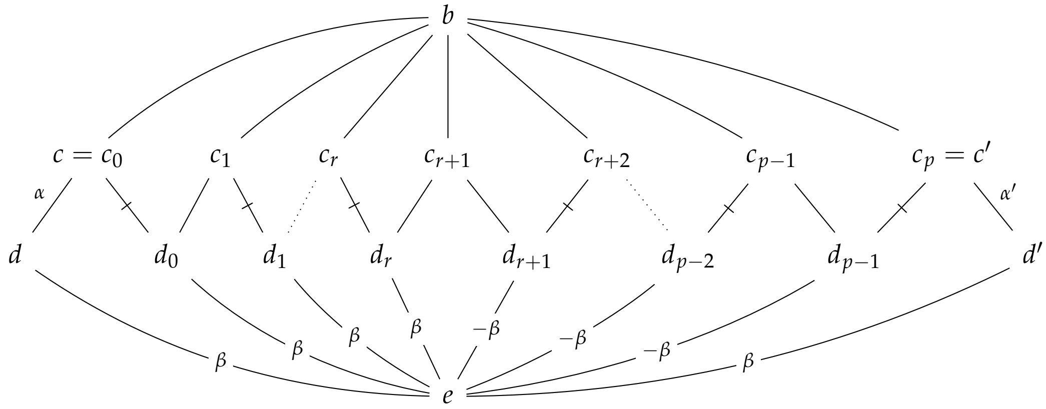
Proposition 3.20 : pencil linearity

C satisfies the axiom of pencil linearity. That is:
$\forall k > 0$, $\forall e \in C_{k-1}$, $\forall \beta \in \{+, -\}$, $\quad \{d \in C_k \mid e \prec^{\beta} d\}$ is linearly ordered by $<^{+}$.

Proof. Suppose $d \succ^{\beta} e \prec^{\beta} d'$, with $d \neq d'$. We know that every element is always the source or the target of some element in Λ. Because elements in Λ are sources of an iterated target of ω, we may find a hexagone as follows:

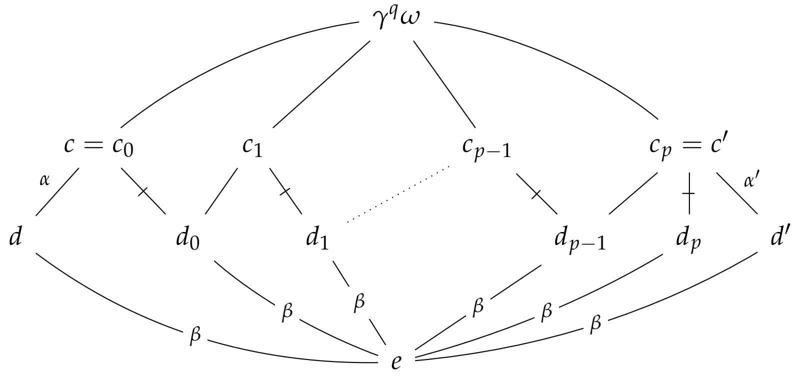


If $c = c'$, then by the sign rule $\alpha = -\alpha'$, hence d and d' are $<^{+}$-comparable and we are done. In what follows, we suppose $c \neq c'$. So – up to potentially exchanging c and c' – there is a simple non-trivial zig-zag with $d_0 = \gamma(c_0)$ as follows:



In this situation, distinguishing on the sign of α', we see on the rightmost lozenge that necessarily $r = p - 1$.

Hence the zig-zag is a path:

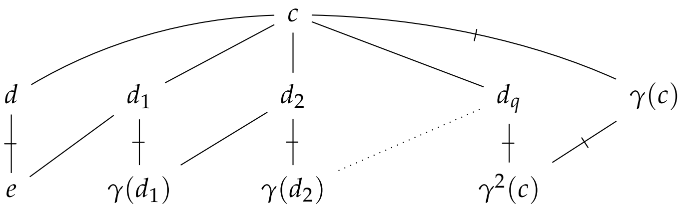


Either $\alpha = +$ and $d = d_0$, or $\alpha = -$ and $d \triangleleft^{C,+} d_0$. Also, either $\alpha' = +$ and $d = d_p$, or $\alpha' = -$, and because of the uniqueness of lozenge completion, $d' = d_{p-1}$. Summing up all cases, we always have an upper path from c to c' (in the tree structure of $\gamma^q\omega$, and of the shape depicted above). Hence it proves that all faces having a given e as a source are linearly ordered for $<^+$, and that all faces having e as a target also are linearly ordered by $<^+$. ☐

3.21. We shall finally focus on the remaining axiom of positive opetopes: disjointness. We will need some lemmas, and we will split the proof into two parts.

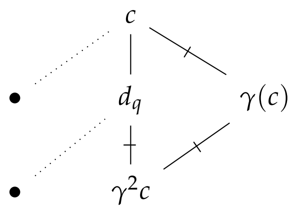
Lemma 3.22

If $d \prec^- c$ with $d \in C_{\geq 1}$, then there is a (unique) path as follows.



In fact this is the unique path from d to $\rho(\delta(c))$ in the tree structure on $\delta(c)$.

Proof. This is seen by completing lozenges from left to right until coming accross one with the shape

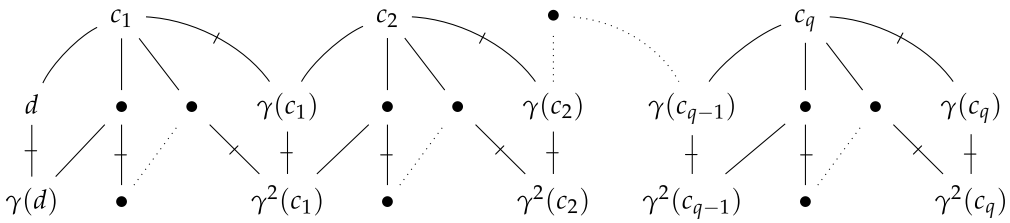


which ends the path. The process must finish because there is no infinite branch in the tree structure of $\delta(c)$. Moreover the path obtained this way is unique because all lozenge completions are. ☐

This lemma yields the following one, by concatenating paths.

Lemma 3.23

If there is an upper path: $d = d_0 \prec^- c_1 \succ^+ \gamma(c_1) \prec^- \cdots \prec^- c_q \succ^+ \gamma(c_q) = d'$, then there is a path:



hence yielding an upper path from $\gamma(d)$ to $\gamma(d')$

Now using Lemma **??**, we are able to prove the first half of disjointness:

Proposition 3.24

If $k > 0$, and d, $d' \in C_k$ then it cannot be the case that $d <^+ d'$ and $d' <^- d$.

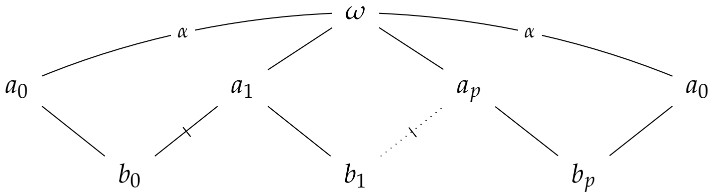
Proof. If $d <^+ d'$ and $d' <^- d$, then we have $\gamma(d) \leq^+ \gamma(d')$ by Lemma **??**, and $\gamma(d') <^+ \gamma(d)$ by the second hypothesis (because $d' <^- d \Rightarrow \gamma(d') <^+ \gamma(d)$). Thus, we have a cycle $\gamma(d) <^+ \gamma(d)$, which is absurd by strictness. ☐

3.25. We will now focus on the second half by handling the case where $d <^+ d'$, and $d <^- d'$. We begin with another lemma.

Lemma 3.26

There is no (non-trivial) upper path between two sources of a common cell.

Proof. First, if the common cell is ω, one of the sources should also be the target of ω, which is absurd. If the common cell a_0 is of codimension 1, assuming the upper path $b_p \prec^- a_p \lhd^- a_{p-1} \lhd^- \cdots \lhd^- a_1 \succ^+ b_0$ takes place in Λ, we have a diagram as follows:



and we cannot choose α consistently.

From now on, we will assume that the dimension of the common cell is $n - k$ with $k \geq 2$.

Notice that we have the following corollary of the hexagon property: If there is a hexagon as in Figure 3.23 below,

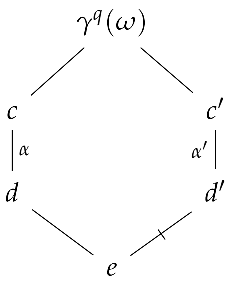


Figure 3.23: Hexagon

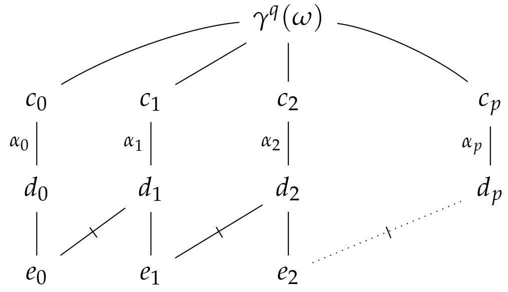


Figure 3.24: Succesive hexagons

then there is a simple zig-zag from c to c' in the tree structure of $\gamma^q\omega$, where every lower element has e as a codimension-1 subface. We can generalize this assertion a bit: When there is a shape as in Figure 3.24 above, there is a zig-zag from c_0 to c_p in the tree-structure of $\gamma^q\omega$ where lower elements successively have e_1, then e_2 ... then e_n as a codimension-1 subface. It is seen by considering such zig-zags between each c_i and c_{i+1}. More precisely, assuming that for all i, $\alpha_i = -$, the – *a priori* non simple – zig-zag has the shape depicted in Figure 3.25. Then, by shortening this zig-zag whenever possible, one obtains a simple zig-zag between c_0 and c_p made of consecutive triplets

$$c_{\varphi(i),\sigma(i)} \succ^{\alpha_i} d_{\varphi(i),\sigma(i)} \prec^{-\alpha_i} c_{\varphi(i),\sigma(i)+1}$$

for $0 \leq i < l$ with $\forall i \leq l-2,\ c_{\varphi(i),\sigma(i)+1} = c_{\varphi(i+1),\sigma(i+1)}$.
Moreover, φ is non-decreasing, $(\alpha_i)_i$ is non-increasing, and if $\varphi(i) = \varphi(i+1)$ then $\sigma(i) < \sigma(i+1)$. Notice that for every i, $e_{\varphi(i)} \prec^{-\alpha_i} d_{\varphi(i),\sigma(i)}$.

Now suppose that we have a path as in Figure 3.26 below, such that e_p and e_0 are both sources of a same element: e_0, $e_p \prec^- d_0 \in C_{n-k}$. We may moreover suppose that all d_i's for $i > 0$ are in Λ.

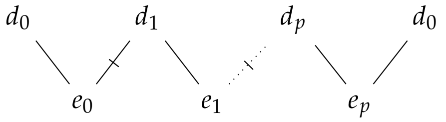


Figure 3.26: Hypothetical path

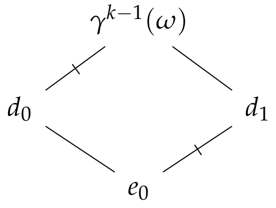


Figure 3.27: Impossible lozenge

If $d_0 = \gamma^k\omega$, then there is a lozenge as in Figure 3.27 above, which is not possible by the sign rule. Hence d_0 is not $\gamma^k(\omega)$, and is (by Lemma 3.6 and Lemma 3.7) the source of some $c_0 \in \Lambda$.

So we may extend this path with hexagons as in Figure 3.28 below.

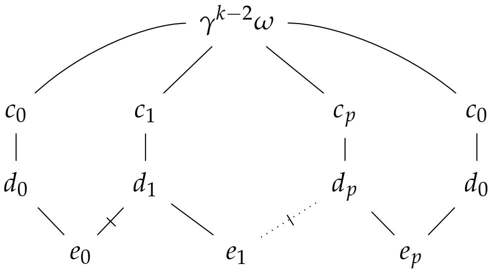


Figure 3.28: Path with hexagons

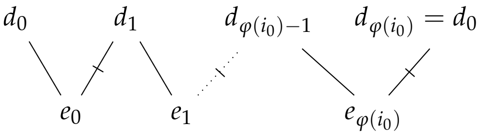


Figure 3.29: Impossible path

where for every $i \geq 0$, c_i is chosen as the unique element in Λ_{n-k+1} such that $d_i \prec^- c_i$.

Using the hexagon property in the rightmost hexagon yields a path in the tree structure of $\delta(\gamma^{k-2}\omega)$ (the same as in the proof of pencil linearity), which must be from c_p to c_0 because d_p is not a target. On the other hand, there is a simple zig-zag from c_p to c_0 obtained by the previous construction. But because of the uniqueness of simple zig-zag between two nodes of a tree, the path from c_p to c_0 and the simple zig-zag from c_0 to c_p constituted of triplets

$$c_{\varphi(i),\sigma(i)} \succ^{\alpha_i} d_{\varphi(i),\sigma(i)} \prec^{-\alpha_i} c_{\varphi(i),\sigma(i)+1}$$

for $0 \leq i < l$ must be the symmetric of each other. Hence it only contains such triplets with $\alpha_i = -$, and $e_i \prec^+ d_{\varphi(i),\sigma(i)}$. In particular the last triplet $\cdot \succ^+ d_0 \prec^- c_0$ of the path from c_p to c_0 corresponds to one of the triplets above. Suppose it is the i_0'th, then we have $e_{\varphi(i_0)} \prec^+ d_{\varphi(i_0),\sigma(i_0)} = d_0$. Hence we have a path as in Figure 3.29 above; this is absurd by strictness, and ends the proof of the lemma. □

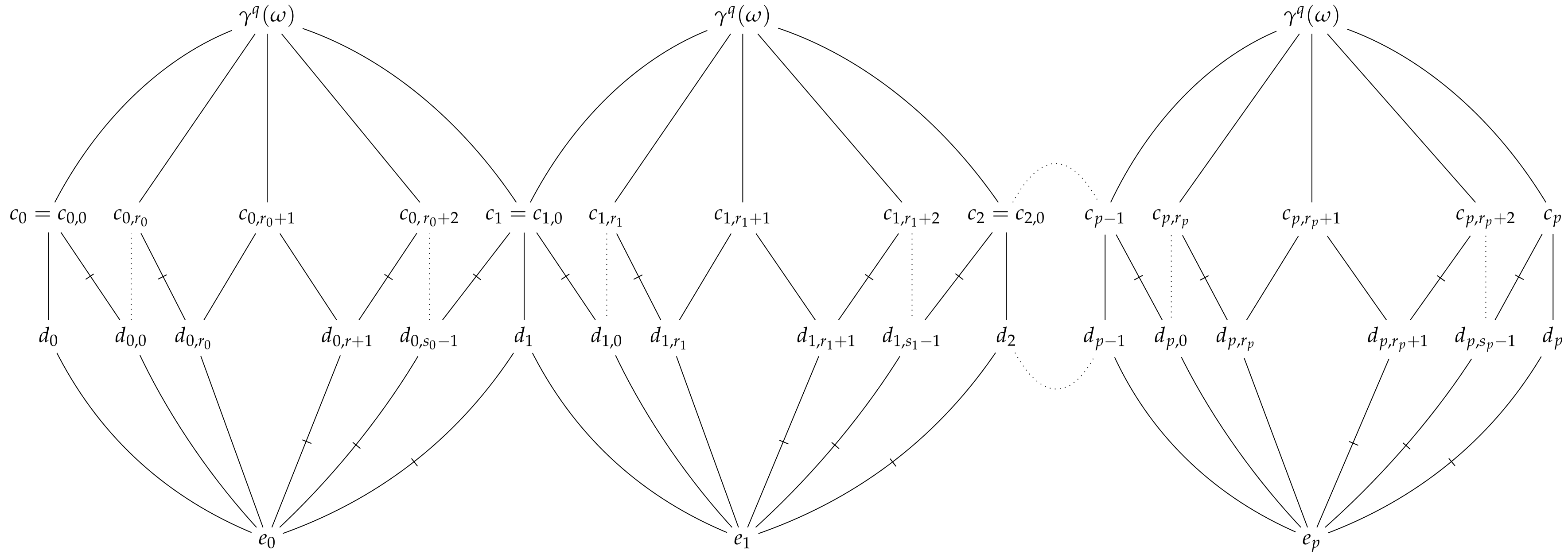
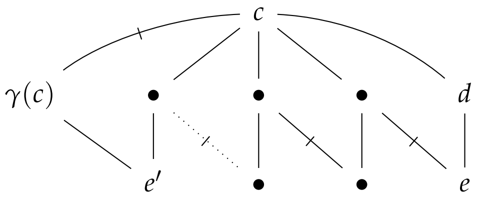


Figure 3.25: Long zig-zag

Lemma 3.27

If $e \prec^- d \prec^- c$, then there is a (unique) upper path from a source e' of $\gamma(c)$ to e as follows:



Proof. We keep completing half lozenges from right to left as above, until coming across the leftmost pattern. $\square$

Proposition 3.28

If $k > 0$, and $d, d' \in C_k$ then it cannot be the case that $d <^+ d'$ and $d <^- d'$.

Proof. Suppose that we have $d <^+ d'$ and $d <^- d'$. Then by concatenating paths obtained as in Lemma 3.27, $d <^+ d'$ yields a non-trivial upper path from e to $\gamma(d)$ where $e \prec^- d'$ is a source of d'. On the other hand, $d <^- d'$, hence $\gamma(d) \leq^+ e'$ for some e' a source of d'. Now by concatenating the path from e to $\gamma(d)$ and the path from $\gamma(d)$ to e', we obtain a non-trivial upper path from $e \prec^- \gamma(d')$ to $e' \prec^- \gamma(d')$, which is impossible by Lemma 3.26. $\square$

Proposition 3.29 : disjointness

C satisfies the axiom of disjointness. That is: two elements d and d' cannot be comparable for both relations $<^+$ and $<^-$.

Proof. This is a consequence of Proposition 3.24 and Proposition 3.28. $\square$

Theorem 3.30

The dendritic face complex C is a positive opetope.

Proof. This is a consequence of Propositions 3.2, 3.3, 3.14, 3.24, 3.28, 3.20 and 3.13. $\square$

4 From ZAWADOWSKI's positive opetopes to dendritic face complexes

4.1. In this section, we consider an opetopic cardinal $S = ((S_k)_{k\in\mathbb{N}}, \gamma, \delta)$. We aim to prove that S gives rise to a dendritic face complex. (At a certain point, we will require S to be principal in order to conclude.) Recall from Theorem 2.4 that S may be given the structure of a *POP*, we will then focus on proving the axioms of DFCs, in the following order: oriented thinness, acyclicity and the existence of a greatest element.

Proposition 4.2 : Oriented Thinness

Let S be an opetopic cardinal, S seen as a positive-to-one poset satisfies the property of oriented thinness.

Proof. We consider a chain $c \prec b \prec a$ in S. And we distinguish on the signs appearing in this relation.

- $\boxed{c \prec^{\beta} b \prec^{+} c}$
  Because of globularity, $\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a)$ and $\delta\gamma(a) = \delta\delta(a) - \gamma\delta(a)$. Hence there is some b' as in Figure 4.1 below. Suppose that there is another b'' such that $c \prec b'' \prec a$.
  Then still because of globularity , either $c \prec^{-\beta} b'' \prec^{+} a$ or $c \prec^{\beta} b'' \prec^{-} a$.
  In the first case, $b'' = b$ by target uniqueness, hence $c \prec^{-} b$ and $c \prec^{+} b$ which is impossible. Hence only the second case may occur.
  In the second case, the point (2.) of the proposition 5.1 in [9] shows that necessarily $b'' = b'$, whence the uniqueness.

- $\boxed{c \prec^{+} b \prec^{-} c}$ Lemma 4.1 in [9] gives $\gamma\delta(a) = \gamma\gamma(a) \sqcup \iota(a)$. Hence either $c \in \gamma\gamma(a)$ or $c \in \delta\delta(a)$ (those two cases are exclusive), and there is some b' as in Figure 4.2 below. Suppose that there is another b'' such that $c \prec b'' \prec a$.
  Then because of globularity $c \notin \delta\gamma(a)$, hence there is some $\alpha'' \in \{+, -\}$ such that $c \prec^{\alpha''} b'' \prec^{\alpha''} a$. And because the union $\gamma\delta(a) = \gamma\gamma(a) \sqcup \iota(a)$ is disjoint, $\alpha'' = \alpha'$. If $\alpha' = -$, we may conclude again by the point (2.) of the proposition 5.1 in [9] that $b' = b''$. And if $\alpha' = +$, then by uniqueness of the target, $b' = \gamma(a) = b''$.

- $\boxed{c \prec^{-} b \prec^{-} c}$ Lemma 4.1 in [9] gives $\delta\delta(a) = \delta\gamma(a) \sqcup \iota(a)$. Hence either $c \in \delta\gamma(a)$ or $c \in \gamma\delta(a)$ (those two cases are exclusive), and there is some b' as in Figure 4.3 below. Suppose that there is another b'' such that $c \prec b'' \prec a$.
  Then because of globularity $c \notin \gamma\gamma(a)$, hence there is some $\alpha'' \in \{+, -\}$ such that $c \prec^{-\alpha''} b'' \prec^{\alpha''} a$. And because the union $\delta\delta(a) = \delta\gamma(a) \sqcup \iota(a)$ is disjoint, $\alpha'' = \alpha'$. If $\alpha' = -$, we may conclude again by the point (2.) of the proposition 5.1 in [9] that $b' = b''$. And if $\alpha' = +$, then by uniqueness of the target, $b' = \gamma(a) = b''$. ☐

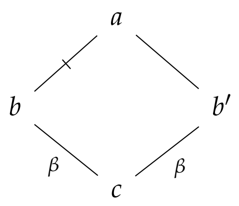


Figure 4.1: Lozenge 1

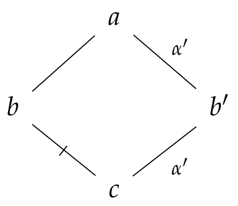


Figure 4.2: Lozenge 2

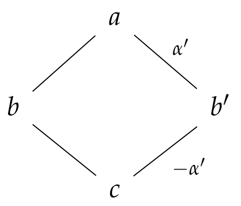


Figure 4.3: Lozenge 3

Proposition 4.3 : Acyclicity

Let S be an opetopic cardinal, S satisfies the axiom of acyclicity.

Proof. If $x \in S_1$, then $\delta(x)$ is a singleton because δ_0 is functional. If $x \in S_{\geq 1}$, then $\delta(x) \neq \varnothing$ because for all k, δ_k is total. There is no cycle as in Figure 1.2 because of strictness. ☐

Proposition 4.4 : Greatest element

If S is supposed to be principal, then $\hat{S}$ admits a greatest element ω.

Proof. Because of principality, $|\{S_n \setminus \delta(S_{n+1})\}| = 1$. But S_{n+1} is empty, hence S_n is a singleton. The fact that its only element is indeed a greatest element is given by the point (1.) of Lemma 7.1 in [9]. ☐

Theorem 4.5

Let S be a positive opetope, S is a dendritic face complex.

Proof. This is a consequence of Proposition 4.2, Proposition 4.3 and Proposition 4.4. ☐

4.6. Note that Theorem 4.5 is the converse of Theorem 3.30. Because of our previous results, we may also state a converse to Proposition 4.4:

Theorem 4.7

If an opetopic cardinal S admits a greatest element, then it is principal. That is, it is a positive opetope.

Proof. Because of Proposition 4.2 and Proposition 4.3, we know that S may be seen as a DFC. We may then use the Proposition 3.13 to conclude. ☐

Conclusion and related works

We have shown in this paper a way to associate a (ZAWADOWSKI's) positive opetope to any dendritic face complex, and vice versa. Noting that these constructions extend to morphisms, we obtain two functors F and G as depicted below.

$$\begin{tikzcd}
\mathbf{DFC} \arrow[r, bend left, "F"] & \mathbf{pOpe} \arrow[l, bend left, "G"]
\end{tikzcd}$$

where **DFC** and **pOpe** denote respectively the categories of dendritic face complexes and positive opetopes. Since $G \circ F$ and $F \circ G$ leave the structure unchanged as proved in Theorem 2.4, they form an equivalence of categories.

This result should be extended in a future paper, dealing with the equivalence with epiphytes (which will be defined then) and zoom complexes (see [5]).

There is also a definition of opetopes, namely *dendrotopes*, due to Thorsten PALM (see [8] for an introduction and [7] for a more complete description), which in many aspects is close to that of DFCs. The author is convinced that there should be a functor from **DFC** to the category of PALM's dendrotopes, although the details remain to be checked.

Acknowledgements

Thanks to Amar HADZIHASANOVIĆ for discussions about this article, and for ideas which helped develop the formalism of DFCs. I also want to thank Marek ZAWADOWSKI for his inspiring and wide-ranging work on opetopes and their formalization as posets. Finally, I would like to express a special thank you to the supervisor of my master's thesis, Pierre-Louis CURIEN, without whom this article would never have been.

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