A recursive tree-shaped definition for positive opetopes.

Louise LECLERC Pierre-Louis CURIEN

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1 Epiphytes

In this section, we describe epiphytes. They consist in trees where nodes are labelled by trees, whose nodes are labelled by trees... *etc*. They are a structure dedicated to encoding the combinatorics of opetopes. We need first a notion of rooted tree, which is defined below:

Definition 1.1 : Rooted tree

A *rooted tree* T consist of:

- A *finite* set of nodes $T^\bullet$.
- For each node $a \in T^\bullet$, a *finite* set $A(a)$, called the arity of a.
- A (necessarily finite) set of *triplets*, denoted $a \prec_b a'$ for some $a, a' \in T^\bullet$ and $b \in A(a)$. Moreover we ask that for each $a \in T^\bullet$ and $b \in A(a)$, there is at most one triplet $a \prec_b a'$. If there is at least one, (a, b) is called an *inner edge* of T, otherwise it is said to be a *leaf* of T. We let

$$T^{|} := \{(a, b) \text{ leaf of } T\} \qquad \text{and} \qquad T^{⫯} := \{(a, b) \text{ inner edge of } T\}.$$

We moreover ask for a distinguished element $\rho(T) \in T^\bullet$, called the *root* of T, satisfying the following property: for each node $a \in T^\bullet$, there is a unique *(descending) path* in T

$$a = a_0 \succ_{b_1} a_1 \succ_{b_1} \cdots \succ_{b_p} a_p = \rho(T)$$

from a to the root of T.

Remark 1.2
Notice that if it exists, the root is uniquely determined.

Exemple 1.3
Below is a representation of the rooted tree T having

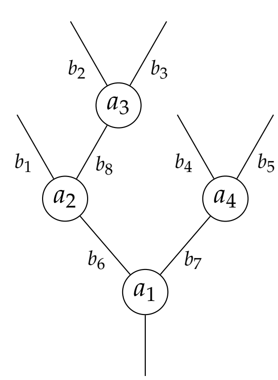
- as nodes $T^\bullet := \{a_1, a_2, a_3, a_4\}$
- as arities

$$A(a_1) := \{b_6, b_7\} \qquad A(a_2) := \{b_1, b_8\} \qquad A(a_3) := \{b_2, b_3\} \qquad A(a_4) := \{b_4, b_5\}$$

- as triplets

$$a_1 \prec_{b_6} a_2 \qquad a_1 \prec_{b_7} a_4 \qquad a_2 \prec_{b_8} a_3$$

- as root a_1.



Definition 1.4 : neat rooted tree

Let T be a rooted tree, T will be called *neat* iff the second projection

$$\begin{array}{rccl} \mathsf{pr}_2 : & T^{|} & \to & \bigcup_{a \in T^{\bullet}} T(a) \\ & (a, b) & \mapsto & b \end{array}$$

is injective. We then identify the leaf (a, b) with $b \in A(a)$, and let $\eta(b) := a$ (or $\eta_T(b) := a$ if needed). For a neat rooted tree T, the set $T^{|}$ will be replaced by its second projection.

1.5. We now describe the main definition of this section. Since the notion of epiphyte is closely entangled with that of their target with which they call each other, we blend those definitions in the following three mutually recursives ones. In the following, all the trees considered will be neat.

Definition 1.6 : Epiphyte

We define inductively *epiphytes* ω and their dimension $\mathsf{dim}(\omega)$, as follows:

- There is only one epiphyte of dimension 0, which is denoted by $\blacklozenge$. We let $\blacklozenge^{\bullet} := \emptyset$.
- Suppose that we have defined epiphytes of dimension $k \leq n$ for some $n \in \mathbb{N}$, together with their targets. Then a $(n+1)$-epiphyte ω consists in the following data:
  - A structure of neat rooted tree, which we also denote ω.
  - For each $a \in \omega^{\bullet}$, a n-epiphyte $s_a\omega$ with $(s_a\omega)^{\bullet} = A(a)$, called the *source* at a.

  Such that we have, for each triplet $a \prec_b a'$ of ω, the equality of epiphytes $s_b s_a \omega = t s_{a'} \omega$.

Remark 1.7
Notice that since $\blacklozenge^{\bullet} = \emptyset$, a 1-epiphyte is always of the form $\blacksquare_a$ with $\blacksquare_a^{\bullet} = \{a\}$, $A(a) = \emptyset$ and no triplets.

Definition 1.8 : λ, κ

Let ω be an epiphyte of dimension ≥ 1, and $a \in \omega^{\bullet}$, $b \in A(a)$.

- For $c \in A(b)$ (in $s_a\omega$) or equivalently $c \in (s_b s_a \omega)^{\bullet}$, we define $\lambda_c(a, b) \in \omega^{|}$ by increasing induction on the height of a in the tree structure of ω.
  - If $(a, b) \in \omega^{|}$, then $\lambda_c(a, b) := b$.
  - If there is a triplet $a \prec_b a'$ in ω, then using the equality $s_b s_a \omega = t s_{a'} \omega$, we have $c \in (s_{a'}\omega)^{|}$. Hence we may define $\lambda_c(a, b) := \lambda_c(a', \eta(c))$.
- We define $\kappa(a, b) \in \omega^{|}$ by increasing induction on the height of a in ω.
  - If $(a, b) \in \omega^{|}$, then $\kappa(a, b) := b$.
  - If there is a triplet $a \prec_b a'$ in ω, then $\kappa(a, b) := \kappa(a', \rho(s_a\omega))$.

Definition 1.9 : target

Let ω be a $(n+1)$-epiphyte with $n \geq 0$, then its *target* $t\omega$, a n-epiphyte, is defined as follows.

- $(t\omega)^{\bullet} := \omega^{|}$.
- For each $b \in \omega^{|}$, we let $A(b) := (s_b s_{\eta(b)} \omega)^{\bullet}$ and $s_b t\omega := s_b s_{\eta(b)} \omega$.
- For every $a \in \omega^{\bullet}$ and every triplet $b \prec_c b'$ in $s_a\omega$, there is a triplet $\lambda_c(a, b) \prec_c \kappa(a, b')$ in $t\omega$, and the root is $\kappa(\rho(\omega), \rho(s_{\rho(\omega)}\omega))$.

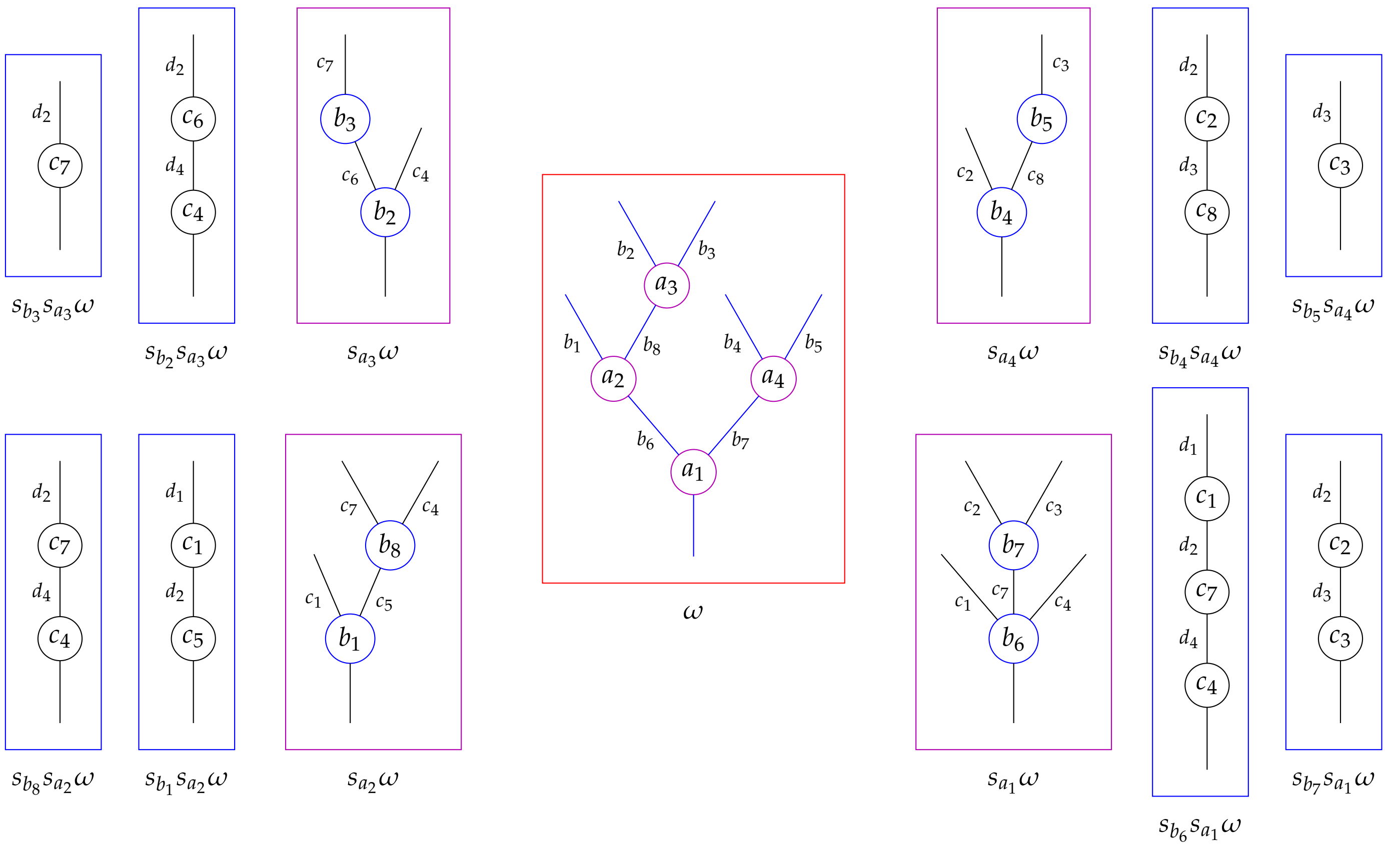
1.10. Correctness of Definition 1.9 (which implies that of Definition 1.6) will be the aim of Theorem 2.9.

Remark 1.11
Suppose that there is a path

$$\succ_{b_0} a_0 \succ_{b_1} \cdots \succ_{b_p=b} a_p = a$$

in an epiphyte ω with $\mathsf{dim}(\omega) \geq 1$, and there is $c \in (s_b s_a \omega)^{\bullet}$ with $\lambda_c(a, b) = b_0, \eta(b_0) = a_0$.
Then we have

$$s_c s_{\lambda_c(a,b)} t\omega = s_c s_{b_0} s_{a_0} \omega = s_c s_{b_1} s_{a_1} \omega = \cdots = s_c s_{b_p} s_{a_p} \omega = s_c s_b s_a \omega$$



Exemple 1.12
We have depicted above a 4-epiphyte (that is, of dim. 4), which formalise the picture of the introduction. For brevity, the 1 and 0-dimensional sources are left implicit.

Exemple 1.13
In the case of the $4-$epiphyte above the target is the $3-$epiphyte $t\omega$ having

- As nodes the leaves b_1, b_2, b_3, b_4 and b_5 of ω.
- As arities

$$A(b_1) = \{c_1, c_5\} \qquad A(b_2) = \{c_4, c_6\} \qquad A(b_3) = \{c_7\}$$

$$A(b_4) = \{c_2, c_8\} \qquad A(b_5) = \{c_3\}$$

  and the corresponding sources as in the definition above.

- As triplets

$$b_3 \prec_{c_7} b_4 \text{ induced by } b_6 \prec_{c_7} b_7 \text{ in } s_{a_1}\omega$$

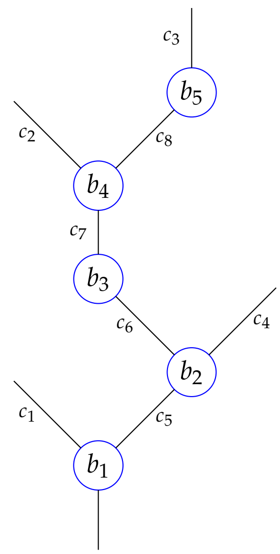
$$b_1 \prec_{c_5} b_2 \text{ induced by } b_1 \prec_{c_5} b_8 \text{ in } s_{a_2}\omega$$

$$b_2 \prec_{c_6} b_3 \text{ induced by } b_2 \prec_{c_6} b_3 \text{ in } s_{a_3}\omega$$

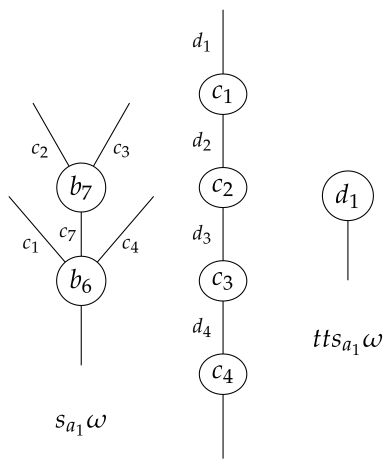
$$b_4 \prec_{c_8} b_5 \text{ induced by } b_4 \prec_{c_8} b_5 \text{ in } s_{a_4}\omega$$

- As root b_1.

And below is a graphical representation of the tree associated to $t\omega$.



We may also compute $ts_{a_1}\omega$ and $tts_{a_1}\omega$ as below:



$tts_{a_1}\omega$

$s_{a_1}\omega$

$ts_{a_1}\omega$

Lemma 1.14

Let ω be a n-epiphyte with $n \geq 1$, and $b \in (s_a\omega)^\bullet$, $b_0 \in \omega^{|}$ such that there is a path

$$\rightarrowtail_{b_0} \eta(b_0) \twoheadrightarrow_b a \qquad (b_0 = b \text{ when } \eta(b_0) = a)$$

in ω. Then there is a path

$$b_0 \rightarrowtail b_1 \rightarrowtail \cdots \rightarrowtail b_p = \kappa(a, b)$$

in $t\omega$, such that

$$\forall i, \eta(b_i) \twoheadrightarrow_b a$$

Proof. We proceed by decreasing induction on the height of a in the tree structure of ω.
When b is a leaf, $\kappa(a, b) = b = b_0$, hence there is a trivial path between b_0 and $\kappa(a, b)$.
Suppose now that b is not a leaf, and let a' be such that $a \leftarrowtail_b a'$. We suppose that the result is known for every $b' \in A(a')$. Let b' be such that $\eta(b_0) \twoheadrightarrow_{b'} a' \rightarrowtail_b a$. We again proceed by induction: on the increasing height of b' in the tree structure of $s_{a'}\omega$.

- Suppose that $b' = \rho(s_a\omega)$.
  Then $\kappa(a', b') = \kappa(a, b)$. Because of the induction hypothesis on b', we may find a path

$$b_0 \succ\!\!- b_1 \succ\!\!- \cdots \succ\!\!- b_p = \kappa(a', b') = \kappa(a, b)$$

  in $t\omega$, such that for for all i, we have $\eta(b_i) \gg\!\!-_{b'} a' \succ\!\!-_b a$. Whence the result.

- Suppose that there is a triplet $b'' \prec_c b'$ in $s_{a'}\omega$.
  Then there is a triplet $\kappa(a', b') \succ\!\!-_c \lambda_c(a', b'')$ in $t\omega$. Hence, using the first induction hypothesis on $b' \in A(a')$, we may find a path

$$b_0 \succ\!\!- b_1 \succ\!\!- \cdots \succ\!\!- b_p = \kappa(a', b') \succ\!\!-_c \lambda_c(a', b'')$$

  such that

$$\forall i, \eta(b_i) \gg\!\!-_{b'} a' \succ\!\!-_b a$$

  Using the second induction hypothesis, we may find a second path

$$\lambda_c(a', b'') = b'_0 \succ\!\!- b'_1 \succ\!\!- \cdots \succ\!\!- b'_q = \kappa(a, b)$$

  such that

$$\forall i, \eta(b'_i) \gg\!\!-_b a$$

  Concatenating those two paths yields the result.

□

2 Correctness

2.1. The aim of this section is to prove the following:

$$\begin{array}{lll} (n \geq 0) & \mathcal{P}_n : & n\text{-epiphytes and their target are well defined} \\ (n \geq 2) & \mathcal{Q}_n : & \text{any } n\text{-epiphyte } \omega \text{ satisfies } ts_{\rho(\omega)}\omega = tt\omega \end{array}$$

by strong induction on $n \geq 0$. In the case $n = 0$ and $n = 1$, $\mathcal{P}_n$ is clear. From now on, we let $n \geq 2$, and suppose $\mathcal{P}_k$, $\mathcal{Q}_k$ for $k < n$ such that those are defined.

Lemma 2.2

The target of a n-epiphyte is a neat rooted tree.

Proof. Let ω be a $n-$epiphyte.

At most one triplet $b \prec_c -$:

Let b, b' be two leaves of ω and let $a := \eta(b)$, $a' := \eta(b')$, suppose that $c \in (s_b t\omega)^\bullet = (s_b s_a \omega)^\bullet$, and that there is a triplet $b \prec_c b'$. Let

$$a = a_0 \succ\!\!-_{b_1} a_1 \succ\!\!-_{b_2} \cdots \succ\!\!-_{b_p} a_p = \rho(\omega) \qquad \Big(\text{resp. } a' = a_q \succ\!\!-_{b'_1} a'_1 \succ\!\!-_{b'_2} \cdots \succ\!\!-_{b'_q} a'_q = \rho(\omega)\Big)$$

be the descending path from a (resp. a') to the root in ω. By definition of triplets in $t\omega$, there are two integers l, m with $l \leq p$ and $m \leq q$ such that

- $a_l = a_m$.
- $\forall i < l$, $c \in (s_{a_i}\omega)^|$ with $b_i = \eta_{s_{a_i}\omega}(c)$.
- $\forall i < m$, $b'_i = \rho(s_{a'_i}\omega)$.
- there is a triplet $b_l \prec_c b'_m$ in $s_{a_l}\omega$.

This forces

- $l = \min\{i \mid c \notin (s_{a_i}\omega)^|\}$.

– $m = \min\{i \mid b'_i \neq \rho(s_{a'_i}\omega)\}$.

Thus b_l, and hence b'_m are uniquely determined, and b' is uniquely determined as being $\kappa(a_l, b'_m)$.

Unique path to the root :

The fact that for every $b \in (t\omega)^\bullet$ there is a path to the root is given by Lemma 1.14. As to see the unicity of such a path, it suffices to show that for every $b \in (t\omega)^\bullet$, there is at most one triplet of the form $b' \prec_c b$ in $t\omega$.
Because of the definition of triplets in $t\omega$, there is such a triplet iff there is a path

$$\succ_{b=b_0} a = a_0 \succ_{b_1} \cdots \succ_{b_p} a_p$$

with, $\forall i < p$, $b_i = \rho(s_{a_i})$, and a path

$$\succ_{b'=b'_0} a' = a'_0 \succ_{b'_1} \cdots \succ_{b'_q} a'_q = a_p$$

such that $\forall i < p$, $c \in (s_{a'_i}\omega)^|$, with a triplet $b'_q \prec_c b_p$ in $s_{a_p}\omega$. Hence, a_p is entirely determined as the first node on the path from a to the root such that $b_p \neq \rho(s_{a_p}\omega)$. Then, c and b'_q are also characterised by the triplet $b'_q \prec_c b_p$ in $s_{a_p}\omega$. It determines b' as $\lambda_c(a_p, b'_q)$.

Neatness :

Let $c \in (t\omega)^|$, where $c \in (s_b s_a \omega)^\bullet$, we need to check that b can be recovered from c. In fact, we will show that $b = \eta_{t\omega}(c) = \lambda_c\Big(\rho(\omega), \eta_{s_{\rho(\omega)}\omega}(c)\Big)$. Let

$$\succ_{b=b_0} a = a_0 \succ_{b_1} \cdots \succ_{b_p=b'} a_p = \rho(\omega)$$

be the descending path from b to the root in ω. We show by induction on $i \leq p$ the following property

$$(0 \leq i \leq p) \qquad \mathcal{P}_i \;:\; c \in (s_{a_i}\omega)^| \text{ and } \eta_{s_{a_i}\omega}(c) = b_i$$

- *Initialisation* $(i = 0)$: Suppose that there is a triplet $b \prec_c b'$ in $s_a\omega$, then by definition of $t\omega$, there is a triplet $\lambda_c(a, b) \prec_c \kappa(a, b')$ in $t\omega$. Since c is a leaf in $t\omega$ this is impossible, whence $c \in (s_a\omega)^|$. $\eta_{s_a\omega}(c) = b$ is by assumption.
- *Heredity* : Suppose the result known for some $i < p$. Using the equality $(s_{b_i}s_{a_i}\omega)^| = (s_{b_{i+1}}s_{a_{i+1}}\omega)^\bullet$, we have $c \in (s_{b_{i+1}}s_{a_{i+1}})^\bullet$. As above, there is no triplet $b_{i+1} \prec_c b'$ in $s_{a_{i+1}}\omega$ because $c \in (t\omega)^|$. Hence $c \in (s_{a_{i+1}}\omega)^|$. Since $c \in (s_{b_{i+1}}s_{a_{i+1}})^\bullet$, we have $\eta_{s_{a_{i+1}}}(c) = b_{i+1}$.

Especially, we have shown $b' = \eta_{s_{\rho(\omega)}\omega}(c)$. Using $\mathcal{P}_i$ $(i \leq p)$ we also have $b = \lambda_c(\rho(\omega), b')$.
Whence $b = \eta_{t\omega}(c) = \lambda_c\Big(\rho(\omega), \eta_{s_{\rho(\omega)}\omega}(c)\Big)$. □

Lemma 2.3

Let ω be a n-epiphyte, then $(ts_{\rho(\omega)}\omega)^\bullet = (tt\omega)^\bullet$. Moreover, for all $c \in (tt\omega)^\bullet$, $s_c ts_{\rho(\omega)}\omega = s_c tt\omega$.

Proof.

Nodes :

When showing *neatness* in the proof of Lemma 2.2, we have already shown the inclusion $(t\omega)^| \subseteq (s_{\rho(\omega)}\omega)^|$. We then show the converse. Let $c \in (s_{\rho(\omega)}\omega)^|$ and suppose there is a triplet $b \prec_c b'$ in $t\omega$. Then there are two paths

$$\succ_{b=b_0} a_0 \succ_{b_1} \cdots \succ_{b_p} a_p = a \qquad \text{and} \qquad \succ_{b'=b'_0} a'_0 \succ_{b'_1} \cdots \succ_{b'_q} a'_q = a$$

such that $c \in (s_{a_i}\omega)^|$, $\eta_{s_{a_i}\omega}(c) = b_i$ for $i < p$ and $b'_i = \rho(s_{a'_i}\omega)$ for $i < q$, together with a triplet $b_p \prec_c b'_q$ in $s_a\omega$. Extending the first path down to the root, we get to see $c \in (s_a\omega)^|$, hence there can not be a triplet $b_p \prec_c b'_q$ in $s_a\omega$. So there is no triplet $b \prec_c b'$ in $t\omega$, hence $c \in (t\omega)^|$. This yields the inclusion $(t\omega)^| \supseteq (s_{\rho(\omega)}\omega)^|$, whence the equality $(tt\omega)^\bullet = (ts_{\rho(\omega)}\omega)^\bullet$.

Sources :

Let $c \in (tt\omega)^{\bullet}$, and let $b := \eta_{t\omega}(c)$, $a := \eta_{\omega}(b)$, $b' := \eta_{s_{\rho(\omega)}\omega}(c)$. Then using Remark 1.11, we have $s_c s_b s_a \omega = s_c s_{b'} s_{\rho(\omega)}\omega$, that is $s_c tt\omega = s_c t s_{\rho\omega}\omega$. □

Lemma 2.4

Let ω be a n-epiphyte and $b' \in (s_{a'}\omega)^{\bullet}$, $b \in (s_a\omega)^{\bullet}$, such that there is a path

$$a' = a_0 \succ_{b_1} \cdots \succ_{b_p = b} a_p = a$$

in ω. We suppose moreover that for all $i < p$, there is a path in $s_{a_i}\omega$

$$b_i = b_i^0 \succ_{c_i^1} \cdots \succ_{c_i^{q_i}} b_i^{q_i} = \rho(s_{a_i}\omega)$$

such that $\forall j < q_i$, $c_i^j = \rho\left(s_{b_i^j} s_{a_i}\omega\right)$. Then there is a path in $t\omega$.

$$\kappa(a', b') = \lambda_{\rho(s_{b^0}s_{a^0}\omega)}(a^0, b^0) \prec_{\rho(s_{b^0}s_{a^0}\omega)} \cdots \prec_{\rho\left(s_{b^{q-1}}s_{a^{q-1}}\omega\right)} \lambda_{\rho(s_{b^q}s_{a^q}\omega)}(a^q, b^q)$$

where $a^q = a$, $b^q = b$ and

$$\forall i,\ \rho(s_{b^i}s_{a^i}\omega) = \rho\left(s_{\lambda_{\rho\left(s_{b^i}s_{a^i}\omega\right)}(a^i, b^i)} t\omega\right)$$

Proof. We proceed by induction on the increasing height of $\kappa(a', b')$ in the tree structure of $t\omega$. Let

$$\succ_{b'_0} a'_0 \succ_{b'_1} \cdots \succ_{b'_s = b'} a'_s = a'$$

be the path in ω such that $b'_0 = \kappa(a, b) \in \omega^{|}$, and $b'_i = \rho(s_{a'_i}\omega)$ for $i < s$.
Then we have an extended path

$$\succ_{b'_0} a'_0 \gg\!\!\succ_{b'_s = b'} a'_s = a' \succ_{b'_{s+1} := b_1} \cdots \succ_{b'_{s+p} := b_p = b} a'_{s+p} := a_p = a$$

And we let $l := s + p$, $c := \rho(s_{b'_0}s_{a'_0}\omega)$, and $r := \max\{r' \in [\![0, l]\!] \mid \forall i < r', c \in (s_{a'_i}\omega)^{|}\}$. Note that for all $i < l$, there is a path in $s_{a'_i}\omega$

$$b'_i = b'^0_i \succ_{c'^1_i} \cdots \succ_{c'^{q'_i}_i} b'^{q'_i}_i = \rho(s_{a'_i}\omega)$$

such that $\forall j < q'_i$, $c'^j_i = \rho\left(s_{b'^j_i} s_{a'_i}\omega\right)$. Hence, if $\rho(s_{b'_i}s_{a'_i}\omega) \in (s_{a'_i}\omega)^{|}$, then by Definition 1.9, it is the root of $ts_{a'_i}\omega = s_{b'_{i+1}}s_{a'_{i+1}}\omega$, hence $c = \rho(s_{b'_i}s_{a'_i}\omega) = \rho(s_{b'_{i+1}}s_{a'_{i+1}}\omega)$. Thus, we always have $c = \rho(s_{b'_r}s_{a'_r}\omega)$.

- Either $r = l$, $\kappa(a', b') = \lambda_c(a, b) = \lambda_{\rho(s_b s_a \omega)}(a, b)$, and we have the result with $p = 0$.
- Or $r < l$, and $\kappa(a', b') = \lambda_c(a'_r, b'_r)$. We let $a'' := a'_r$. Since $r < l$, we have $(b'_r, c) \in (s_{a'_r}\omega)^{\dagger}$ and there is a triplet $b'_r \prec_c b''$ in $s_{a''}\omega$. Hence there is a triplet $\kappa(a', b') = \lambda_c(a'_r, b'_r) \prec_c \kappa(a'', b'')$ in $t\omega$. Moreover, there is still a path $a'' = a'_r \succ_{b'_{r+1}} \cdots \succ_{b'_l = b} a_l = a$ in ω, where $b'_l = b$ because $r < l$. And we have

  $$b''_0 := b'' \succ_{c''_1 := c} b''_1 := b'_r \succ_{c''_2 := c'^1_r} \cdots \succ_{c''_{q'_r+1} := c'^{q'_r}_r} b''_{q'_r+1} := b'^{q'_r}_r = \rho(s_{a'_r}\omega)$$

  where $\forall j < q'_r + 1$, $c''_j = \rho\left(s_{b''_j}s_{a'_r}\omega\right)$. Hence, we may use the induction hypothesis to extend the path from $\kappa(a'', b'')$.

  As to see the last assertion, we shall check that $c = \rho(s_{\lambda_c(a'_r, b'_r)}t\omega)$, that is: $c = \rho(s_{b'_0}s_{a'_0}\omega)$. This is the definition of c. □

Lemma 2.5

Let ω be an n-epiphyte, $b \in (s_a\omega)^\bullet$, $c = \rho(s_b s_a \omega)$ and $c' = \rho(s_{\kappa(a,b)} t\omega)$.
Then $\kappa(\lambda_c(a, b), c) = \kappa(\kappa(a, b), c')$.

Proof. By Lemma 2.4, there is a path

$$\kappa(a, b) = b_0 \prec_{c'=c_0} b_1 \prec_{c_1} \cdots \prec_{c_{p-1}} b_p = \lambda_c(a, b)$$

where $c_i = \rho(s_{b_i} t\omega)$ for all i, including $c_p := c$. Hence, $\kappa(\lambda_c(a, b), c) = \kappa(\kappa(a, b), c')$. □

Lemma 2.6

Let ω be a n-epiphyte and a triplet $b \prec_c b'$ in $s_a\omega$.

- For every leaf $d \in (s_{b'} s_a \omega)^|$ we have $\lambda_d(\lambda_c(a, b), c) = \lambda_d(\lambda_{c'}(a, b'), c')$ where $c' = \eta(d)$.
- If $c' = \rho(s_{b'} s_a \omega)$, we have $\kappa(\lambda_c(a, b), c) = \kappa(\lambda_{c'}(a, b'), c')$.

Proof.

- Let $d \in (s_{b'} s_a \omega)^|$ and $c' := \eta(d)$. Since there is a triplet $b' \succ_c b$ in $s_a\omega$, there is a triplet $\kappa(a, b') \succ_c \lambda_c(a, b)$ in $t\omega$. Hence, by Lemma 1.14, there is a path

$$\lambda_{c'}(a, b') \gg\!- \ \kappa(a, b') \succ_c \lambda_c(a, b) \qquad \text{in } t\omega.$$

  Since $d \in (s_{b'} s_a \omega)^|$, we have $d \in (s_c s_b s_a \omega)^\bullet$ because of the triplet $b \prec_c b'$. Hence $d \in (s_c s_{\lambda_c(a,b)} t\omega)^\bullet$ (this is by Remark 1.11) and it implies the desired equality.

- In this configuration, there is a triplet $\lambda_c(a, b) \prec_c \kappa(a, b')$. Using Lemma 2.4, there is a path

$$\kappa(a, b') = b_0 \prec_{c_0} b_1 \prec_{c_1} \cdots \prec_{c_{p-1}} b_p = \lambda_{c'}(a, b')$$

  where $c_i = \rho(s_{b_i} t\omega)$ for all i, including $c_p := c'$. Hence, $\kappa(\lambda_c(a, b), c) = \kappa(\lambda_{c'}(a, b'), c')$. □

Lemma 2.7

Let ω be a n-epiphyte of dimension ≥ 3, and let b, c be such that $c \in (s_b s_{\rho(\omega)} \omega)^\bullet$.

- $\forall d \in (s_c s_b s_{\rho(\omega)} \omega)^\bullet, \quad \lambda_d(b, c) = \lambda_d(\lambda_c(\rho(\omega), b), c)$.
- $\kappa(b, c) = \kappa(\lambda_c(\rho(\omega), b), c)$.

Proof.

- Let $d \in (s_c s_b s_{\rho(\omega)} \omega)^\bullet$, and let

$$\succ_{c_0} b_0 \succ_{c_1} \cdots \succ_{c_p = c} b_p = b$$

  be the path such that $\lambda_d(b, c) = c_0 \in (s_{\rho(\omega)} \omega)^|$. Then Lemma 2.6 yields

$$\lambda_d(\lambda_c(\rho(\omega), b), c) = \lambda_d(\lambda_{c_{p-1}}(\rho(\omega), b_{p-1}), c_{p-1}) = \cdots = \lambda_d(\lambda_{c_0}(\rho(\omega), b_0), c_0)$$

  and $\lambda_d(\lambda_{c_0}(\rho(\omega), b_0), c_0) = c_0$ is already a leaf of $t\omega$, because it is a leaf of $s_{\rho(\omega)}\omega$ (we use Lemma 2.3). Whence the first equality: $\lambda_d(b, c) = \lambda_d(\lambda_c(\rho(\omega), b), c)$.

- Let

$$\succ_{c_0} b_0 \succ_{c_1} \cdots \succ_{c_p = c} b_p = b$$

  be the path such that $\kappa(b, c) = c_0 \in (s_{\rho(\omega)} \omega)^|$. Then Lemma 2.6 yields

$$\kappa(\lambda_c(\rho(\omega), b), c) = \kappa(\lambda_{c_{p-1}}(\rho(\omega), b_{p-1}), c_{p-1}) = \cdots = \kappa(\lambda_{c_0}(\rho(\omega), b_0), c_0)$$

  and $\kappa(\lambda_{c_0}(\rho(\omega), b_0), c_0) = c_0$ is already a leaf of $t\omega$, because it is a leaf of $s_{\rho(\omega)}\omega$ (we use Lemma 2.3). Whence the second equality: $\kappa(b, c) = \kappa(\lambda_c(\rho(\omega), b), c)$. □

Lemma 2.8

Let ω be a n-epiphyte, and $c, c' \in (s_b s_a \omega)^\bullet$, $d \in (s_c s_b s_a \omega)^\bullet$. Then

$$\left(c \prec_d^{s_b s_a \omega} c'\right) \Leftrightarrow \left(\lambda_d(\lambda_c(a, b), c) \prec_d^{tt\omega} \kappa(\lambda_{c'}(a, b), c')\right).$$

Proof. We proceed by induction on the height of a in ω.

- When $b \in (s_a\omega)^\bullet$ is a leaf, $\lambda_d(\lambda_c(a, b), c) = \lambda_d(b, c)$, and $\kappa(\lambda_{c'}(a, b), c') = \kappa(b, c')$. Hence

$$\begin{aligned} c \prec_d^{s_b s_a \omega} c' &\overset{\text{Def. of } t\omega}{\Longleftrightarrow} c \prec_d^{s_b t\omega} c' \\ &\overset{\text{Def. of } tt\omega}{\Longleftrightarrow} \lambda_d(b, c) \prec_d^{tt\omega} \kappa(b, c') \\ &\overset{\text{Equations above}}{\Longleftrightarrow} \lambda_d(\lambda_c(a, b), c) \prec_d^{tt\omega} \kappa(\lambda_{c'}(a, b), c') \end{aligned}$$

- When there is a triplet $a \prec_b \hat{a}$ in ω,

$$\begin{aligned} c \prec_d^{s_b s_a \omega} c' &\overset{s_b s_a \omega = t s_{\hat{a}} \omega}{\Longleftrightarrow} c \prec_d^{t s_{\hat{a}} \omega} c' \\ &\overset{\text{Def. of } t s_{\hat{a}} \omega}{\Longleftrightarrow} \hat{c} \prec_d^{s_{\hat{b}} s_{\hat{a}} \omega} \hat{c}' \\ &\qquad \text{where } \tilde{b} \succ\!\!\succ_{\hat{c}} \hat{b} \text{ and } \tilde{b}' \succ\!\!\succ_{\hat{c}'} \hat{b} \text{ in } s_{\hat{a}}\omega \\ &\overset{\text{Induction}}{\Longleftrightarrow} \lambda_d(\lambda_{\hat{c}}(\hat{a}, \hat{b}), \hat{c}) \prec_d^{tt\omega} \kappa(\lambda_{\hat{c}'}(\hat{a}, \hat{b}), \hat{c}') \end{aligned}$$

  It remains to show that $\lambda_d(\lambda_{\hat{c}}(\hat{a}, \hat{b}), \hat{c}) = \lambda_d(\lambda_c(a, b), c)$ and $\kappa(\lambda_{\hat{c}'}(\hat{a}, \hat{b}), \hat{c}') = \kappa(\lambda_{c'}(a, b), c')$.

  - As to see the first equality, let

  $$\succ_{c=c_0} \tilde{b} = b_0 \succ_{c_1} \cdots \succ_{c_p=\hat{c}} b_p = \hat{b}$$

    be the path in $s_{\hat{a}}\omega$ such that $c_0 = \lambda_d(\hat{b}, \hat{c}) \in (s_{\hat{a}}\omega)^|$. Notice that since there is a triplet $a \prec_b \hat{a}$ in ω and $c \in (s_{\hat{a}}\omega)^|$, then $\lambda_c(a, b) = \lambda_c(\hat{a}, \tilde{b})$. Then Lemma 2.6 yields

  $$\lambda_d(\lambda_{c_0}(\hat{a}, b_0), c_0) = \lambda_d(\lambda_{c_1}(\hat{a}, b_1), c_1) = \cdots = \lambda_d(\lambda_{c_p}(\hat{a}, b_p), c_p) = \lambda_d(\lambda_{\hat{c}}(\hat{a}, \hat{b}), \hat{c}).$$

    Whence the equality.
  - We now see the second equality: let

  $$\succ_{c'=c'_0} \tilde{b}' = b'_0 \succ_{c'_1} \cdots \succ_{c'_q=\hat{c}} b'_q = \hat{b}$$

    be the path in $s_{\hat{a}}\omega$ such that $c'_0 = \kappa(\hat{b}, \hat{c}') \in (s_{\hat{a}}\omega)^|$. Notice that since there is a triplet $a \prec_b \hat{a}$ in ω and $c' \in (s_{\hat{a}}\omega)^|$, then $\lambda_{c'}(a, b) = \lambda_{c'}(\hat{a}, \tilde{b}')$. Then Lemma 2.6 yields

  $$\kappa(\lambda_{c'_0}(\hat{a}, b'_0), c'_0) = \kappa(\lambda_{c'_1}(\hat{a}, b'_1), c'_1) = \cdots = \kappa(\lambda_{c'_q}(\hat{a}, b'_q), c'_q) = \kappa(\lambda_{\hat{c}'}(\hat{a}, \hat{b}), \hat{c}')$$

    Whence the equality. □

Theorem 2.9

The properties $\mathcal{P}_n$ and $\mathcal{Q}_n$ hold.

Proof. Let ω be a n-epiphyte. We begin by showing that $t\omega$ is well-defined. Because of Lemma 2.2, we already know that $t\omega$ is a neat rooted tree.

$\boxed{\forall b \in (t\omega)^\bullet,\ (s_b t\omega)^\bullet = A(b)}$:

Let $b \in (t\omega)^\bullet = \omega^|$. Then $A(b) = (s_b s_{\eta(b)} \omega)^\bullet$. On the other hand, $s_b t\omega = s_b s_{\eta(b)} \omega$, whence $(s_b t\omega)^\bullet = A(b)$.

$\boxed{\forall (b \prec_c b') \text{ in } \omega,\ s_c s_b t\omega = t s_{b'} t\omega}$:

Notice that if $\dim(\omega) < 3$, there is no triplet in $t\omega$, hence this property is vacuously satisfied. Henceforth, we suppose $n \geq 3$. Let $b \prec_c b'$ be a triplet of ω. Then there are two paths

$$\succ_{b=b_0} a_0 \succ_{b_1} \cdots \succ_{b_p} a_p = a \qquad \text{and} \qquad \succ_{b'=b'_0} a'_0 \succ_{b'_1} \cdots \succ_{b'_q} a'_q = a$$

such that $c \in (s_{a_i}\omega)^{|}$, $\eta_{s_{a_i}\omega}(c) = b_i$ for $i < p$ and $b'_i = \rho(s_{a'_i}\omega)$ for $i < q$, together with a triplet $b_p \prec_c b'_q$ in $s_a\omega$. Then, according to Remark 1.11, we have $s_c s_b t\omega = s_c s_{b_p} s_a \omega$.
On the other hand, using $\mathcal{Q}_{n-1}$, we have

$$t s_{b'} t\omega = t s_{b'_0} s_{a'_0} \omega = t t s_{a'_0} \omega = t s_{b'_1} s_{b'_1} \omega = \cdots = t s_{b'_q} s_a \omega.$$

Finally, because of the triplet $b_p \prec_c b'_q$, we have $s_c s_{b_p} s_a \omega = t s_{b'_q} s_a \omega$,
whence the equality $s_c s_b t\omega = t s_{b'} t\omega$.

This completes the proof of $\mathcal{P}_n$, and we now show that $t s_{\rho(\omega)} \omega = t t\omega$.
Notice that if $n = 2$, this equality is clear, so we now suppose $n \geq 3$. Because of Lemma 2.3, we already have the equality of nodes and sources. It remains to see the equality of triplets. Suppose that there is a triplet $\lambda_d(b, c) \prec_d \kappa(b, c')$ in $t s_{\rho(\omega)} \omega$ with a triplet $c \prec_d c'$ in $s_{\rho(\omega)}\omega$. Hence, using Lemma 2.8, there is a triplet $\lambda_d(\lambda_c(\rho(\omega), b), c) \prec_d \kappa(\lambda_{c'}(\rho(\omega), b), c')$ in $t t\omega$. Finally, using Lemma 2.7, this is the triplet $\lambda_d(b, c) \prec_d \kappa(b, c')$. The other implication uses the same arguments.
Hence we have shown $\mathcal{Q}_n$, and it ends the proof. □

3 Epiphytes morphisms

3.1. In this section, we focus on the definition of epiphytes morphisms and the associated category **Epi**. Epiphytes morphisms are of two kinds:

- The *renamings*, which also are the isomorphisms. They goes between epihytes of the same dimension and leave the structure unchanged.
- The *structural maps*, which are formal inclusions of sources and targets of some epiphyte in itself.

Definition 3.2 : Rooted tree isomorphism

An *isomorphism* $f : S \to T$ from a rooted tree S to a rooted tree T corresponds to the data of

- A bijection $f : S^\bullet \to T^\bullet$.
- For each node $a \in S^\bullet$, a bijection $f_a : A(a) \to A(f(a))$.

such that for each triplet $a \prec_b a'$ in S, there is a triplet $f(a) \prec_{f_a(b)} f(a')$ in T.
If $f : S \to T$ and $g : T \to U$ are two isomorphisms, their composition is defined as $g \circ f : S^\bullet \to U^\bullet$ on nodes, and for each $a \in S^\bullet$, $(g \circ f)_a := g_{f(a)} \circ f_a$. The identity is defined as the identity map on nodes and arities. Notice that the composition defined above is associative.

Lemma 3.3

A tree isomorphism $f : S \to T$ is always invertible for the composition. Its inverse is given by $f^{-1} : T \to S$ defined by

- $f^{-1} : T^\bullet \to S^\bullet$ on nodes.
- For each $a \in T^\bullet$, $(f^{-1})_a := (f_{f^{-1}(a)})^{-1} : A(a) \to A(f^{-1}(a))$.

Moreover, such an isomorphism f satisfies the following:

- f preserves paths to the root and height in the tree structure.
- $(a, b) \in S^{|}$ iff $(f(a), f_a(b)) \in T^{|}$.

- $f(\rho(S)) = \rho(T)$.

Proof. We first prove the three last properties, then we will check that f^{-1} is indeed a rooted tree isomorphism.

- Since f sends triplets to triplets, the first point is immediate. Since the height is the length of the unique path from a node to the root, it is preserved.
- Let $b \in A(a)$ in S. If $(f(a), f_a(b))$ is a leaf, then so is (a, b) because f sends triplets to triplets. Conversely, suppose that $(a, b) \in S^{|}$, we check that $(f(a), f_a(b))$ is a leaf. Suppose that there is a triplet $f(a) \prec_{f_a(b)} \hat{a}$ in T. Then we shall have $a' = f(\hat{a})$ for somea unique $a' \in S^{\bullet}$. Since f preserves height, there is at least a triplet $a'' \prec_{b''} a'$ in S. Such a triplet is sent onto a triplet $f(a'') \prec_{f_{a''}(b'')} \hat{a}$ by f. Hence we shall have $f(a) = f(a'')$ and $f_a(b) = f_{a''}(b)$, hence $a = a''$ and $b = b''$. This implies the existence of a triplet $a \prec_b a'$ in S, which is false.
- The root is preserved because it is the only element of null height.

We now prove that f^{-1} is a rooted tree isomorphism. Suppose that there is a triplet $a \prec_b a'$ in T. Then there are two nodes $\hat{a} := f^{-1}(a)$, $\hat{a}' := f^{-1}(a') \in S^{\bullet}$ and $\hat{b} := (f_{\hat{a}})^{-1}(b) \in A(\hat{a})$. Since f preserves height, there is at least a triplet $a'' \prec_{b''} \hat{a}'$ in S, which is sent to a triplet $f(a'') \prec_{f_{a''}(b'')} a'$ in T. Hence $f(a'') = f(\hat{a})$ and $f_{a''}(b'') = f_{\hat{a}}(\hat{b})$, whence $a'' = \hat{a}$, $b'' = \hat{b}$, and a triplet $\hat{a} \prec_{\hat{b}} \hat{a}'$ in S. □

Definition 3.4 : Epiphyte renamings

Let ω and ϖ be two epiphytes of the same dimension n. We define by induction on n the *renamings* $f : \omega \to \varpi$ between them.

- When $\omega = \varpi = \blacklozenge$ are 0-dimensional, there is a unique renaming between them, called the *identity*, or the *trivial-renaming*:
$$\mathsf{id} : \omega \to \varpi$$
- Let $n \geq 1$ and suppose known the definition of $(n-1)-$epiphytes renamings and their target. Then a renaming $f : \omega \to \varpi$ consists of
  - A rooted-tree isomorphism $f : \omega \to \varpi$.
  - For each $a \in \omega^{\bullet}$, a renaming $f_a : s_a\omega \to s_{f(a)}\varpi$ such that on nodes, f_a coincide with the bijection given by the rooted tree isomorphism f. We also write $s_a f$ for f_a.

  such that for each triplet $a \prec_b a'$ and leaf c of $s_{a'}\omega$, $f_{a,b}(c) = f_{a',\eta(c)}(c)$ and $f_{a,b,c} = f_{a',\eta(c),c}$.

Definition 3.5 : Composition and inverse

Given two renamings $f : \omega \to \varpi$ and $g : \varpi \to \varrho$ between epiphytes of the same dimension n, we may compose them. The composition $g \circ f$ is defined inductively on the dimension n as follows:

- In dimension zero, $\mathsf{id} \circ \mathsf{id} := \mathsf{id}$.
- Suppose the composition of $(n-1)$-dimensional renamings is known. Then we define the composition $g \circ f$ as
  - $g \circ f : \omega \to \varrho$ as an isomorphism of rooted trees.
  - For each $a \in \omega^{\bullet}$, $(g \circ f)_a := g_{f(a)} \circ f_a : s_a\omega \to s_{g(f(a))}\varrho$.

We check below that this is well defined.
Moreover, for any epiphyte ω, there is an identity $\mathsf{id} : \omega \to \omega$ defined inductively as

- The identity on the rooted tree structure.
- For each $a \in \omega^{\bullet}$, $\mathsf{id}_a = \mathsf{id} : s_a\omega \to s_a\omega$.

We also check below that any renaming $f : \omega \to \varpi$ between two $n-$epiphytes is invertible for the composition, and its inverse f^{-1} is given by id when $n = 0$ and when $n > 0$ by

- f^{-1} on rooted trees
- For each $a \in \varpi^{\bullet}$, $f_a^{-1} := (f_{f^{-1}(a)})^{-1}$.

Proof.

Composition is well-defined :

We proceed by induction on n. When $n = 0$, the composition is well defined.
Let $n > 0$, $f : \omega \to \varpi$ and $g : \varpi \to \varrho$ be two renamings between $n-$epiphytes. Let $a \prec_b a'$ be a triplet in ω, c a leaf of $s_{a'}\omega$ and $b' = \eta(c)$. Then there is a triplet $f(a) \prec_{f_a(b)} f(a')$ in ϖ, and using Lemma 3.3, there is a leaf $f_{a',b'}(c)$ in $s_{f(a')}\varpi$. Hence, we have the equations

$$g_{f(a),f_a(b)}(f_{a',b'}(c)) = g_{f(a'),f_{a'}(b')}(f_{a',b'}(c)) \qquad g_{f(a),f_a(b),f_{a',b'}(c)} = g_{f(a'),f_{a'}(b'),f_{a',b'}(c)}$$

Because f is also a renaming, we have the identities

$$f_{a,b}(c) = f_{a',b'}(c) \qquad f_{a,b,c} = f_{a',b',c}$$

Whence the identities

$$g_{f(a),f_a(b)}(f_{a,b}(c)) = g_{f(a'),f_{a'}(b')}(f_{a',b'}(c)) \qquad g_{f(a),f_a(b),f_{a,b}(c)} = g_{f(a'),f_{a'}(b'),f_{a',b'}(c)}$$

That is

$$(g \circ f)_{a,b}(c) = (g \circ f)_{a',b'}(c) \qquad (g \circ f)_{a,b,c} = (g \circ f)_{a',b',c}$$

Inversibility :

We proceed by induction on the dimension n of the renaming $f : \omega \to \varpi$.
When $n = 0$, $f = \mathsf{id}$, $f^{-1} = \mathsf{id}$ and their composition is id.
Suppose now that the result is known in dimensions lower than n. By Lemma 3.3, we know that f is inversible as a rooted tree isomorphism. Let $a \prec_b a'$ be a triplet in ϖ, and c a leaf in $s_{a'}\varpi$ with $b' = \eta(c)$. Still by Lemma 3.3, there is a triplet $\hat{a} \prec_{\hat{b}} \hat{a}'$ and a leaf $\hat{c}$ in $s_{\hat{a}'}\omega$ where

$$\hat{a} = f^{-1}(a) \qquad \hat{a}' = f^{-1}(a') \qquad \hat{b} = (f_{\hat{a}})^{-1}(b) \qquad \hat{b}' = (f_{\hat{a}'})^{-1}(b) \qquad \hat{c} = (f_{\hat{a}',\hat{b}'})^{-1}(c)$$

and $\hat{b}' := \eta(\hat{c})$. Hence, we have the identities

$$f_{\hat{a},\hat{b}}(\hat{c}) = f_{\hat{a}',\hat{b}'}(\hat{c}) \qquad f_{\hat{a},\hat{b},\hat{c}} = f_{\hat{a}',\hat{b}',\hat{c}}$$

The first equality yields $\hat{c} = (f_{\hat{a},\hat{b}})^{-1}(c) = (f_{\hat{a}',\hat{b}'})^{-1}(c)$, that is $f^{-1}_{a,b}(c) = f^{-1}_{a',b'}(c)$.
The second one yields $f^{-1}_{\hat{a},\hat{b},\hat{c}} = f^{-1}_{\hat{a}',\hat{b}',\hat{c}'}$, that is $f^{-1}_{a,b,c} = f^{-1}_{a',b',c}$. ∎

Lemma 3.6

Let $f : \omega \to \varpi$ be a renaming of epiphytes, $c \in (s_b s_a \omega)^{\bullet}$, $b' := \lambda_c(a, b)$ and $a' = \eta(b')$, then $f_{a,b}(c) = f_{a',b'}(c)$ and $f_{a,b,c} = f_{a',b',c}$.

Proof. Let

$$\succ_{b_0=b'} a_0 = a' \succ_{b_1} a_1 \cdots \succ_{b_p} a_p$$

be such that $\forall i < p$, $(b_i, c) \in (s_{a_i}\omega)^{|}$, then by definition of a renaming of epiphytes, we have

$$f_{a_0,b_0,c} = f_{a_1,b_1,c} = \cdots = f_{a_p,b_p,c} \qquad \text{and} \qquad f_{a_0,b_0}(c) = f_{a_1,b_1}(c) = \cdots = f_{a_p,b_p}(c).$$

∎

Definition 3.7

Let $f : \omega \to \varpi$ be a renaming of n-epiphytes for some $n \geq 1$.
Then f induces a renaming $tf : t\omega \to t\varpi$ defined by id when $n = 1$, and when $n > 1$ by:

- $\forall b \in \omega^{|}, \quad tf(b) := f_{\eta(b)}(b)$.
- $\forall b \in \omega^{|}, \quad (tf)_b := f_{\eta(b), b}$.

Proof. We check that tf as defined above indeed is a renaming.

tf is a bijection : This is shown by the second point of Lemma 3.3.

tf preserves triplets : Let $\lambda_c(a, b) \prec_c \kappa(a, b')$ be a triplet in $t\omega$, and

$$\succ_{b_0} a_0 \succ_{b_1} \cdots \succ_{b_p = b} a_p = a$$

the path in ω such that $b_0 = \lambda_c(a, b)$. Then there is a path

$$\succ_{f_{a_0}(b_0)} f(a_0) \succ_{f_{a_1}(b_1)} \cdots \succ_{f_{a_p}(b_p) = f_a(b)} f(a_p) = f(a)$$

with, $\forall i < p$, $f_{a_i, b_i}(c) \in (s_{f(a_i)}\varpi)^{|}$. Hence, $f_{a_0}(b_0) = \lambda_{f_{a_0, b_0}(c)}(f(a), f(b)) = \lambda_{f_{a,b}(c)}(f(a), f(b))$ (where the second equality is by Lemma 3.6).
Let

$$\succ_{b'_0} a'_0 \succ_{b'_1} \cdots \succ_{b'_q = b'} a'_q = a$$

be the path in ω such that $b'_0 = \kappa(a, b')$. Then there is a path

$$\succ_{f_{a'_0}(b'_0)} f(a'_0) \succ_{f_{a'_1}(b'_1)} \cdots \succ_{f_{a'_q}(b'_q) = f_a(b')} f(a'_q) = f(a)$$

with $f_{a'_i}(b'_i) = \rho(s_{f(a'_i)}\varpi)$ for $i < q$ (by Lemma 3.3). Hence $f_{a'_0}(b'_0) = \kappa(f(a), f_a(b'))$.
Whence the existence of a triplet $f_{a_0}(b_0) \prec_{f_{a,b}(c)} f_{a'_0}(b'_0)$, *i.e.* a triplet $tf(\lambda_c(a, b)) \prec_{f_{a,b}(c)} tf(\kappa(a, b'))$.

Equations on $tf_{b,c}(d)$ and $tf_{b,c,d}$:

Let $\lambda_c(a, b) \prec_c \kappa(a, b')$ be a triplet in $t\omega$, and $d \in (s_{\kappa(a,b')}t\omega)^{|}$ with $c' = \eta(d)$. We shall check that $tf_{\lambda_c(a,b),c}(d) = tf_{\kappa(a,b'),c'}(d)$. That is – by keeping the above notations – that

$$f_{a_0, b_0, c}(d) = f_{a'_0, b'_0, c'}(d)$$

We prove it by the following sequence of equalities:

$$\begin{array}{rcll} f_{a_0,b_0,c}(d) & = & f_{a_1,b_1,c}(d) & \text{because } a_0 \succ_{b_1} a_1 \text{ and } c \in (s_{a_0}\omega)^{|} \\ & = & \cdots & \\ & = & f_{a_p,b_p,c}(d) & \text{because } a_{p-1} \succ_{b_p} a_p \text{ and } c \in (s_{a_{p-1}}\omega)^{|} \\ & = & f_{a,b,c}(d) & \end{array}$$

Let $\hat{c}_0 := c'$ and $(\hat{b}_i, \hat{c}_{i+1}) := \lambda_d(b'_i, \hat{c}_i)$ for $0 < i \leq p$. Lemma 3.6 yields $f_{a'_0,b'_0,c'}(d) = f_{a'_0,\hat{b}_0,\hat{c}_1}(d)$.
Since $\hat{c}_1 \in (s_{a'_0}\omega)^{|}$, we also have $f_{a'_0,\hat{b}_0,\hat{c}_1}(d) = f_{a'_1,b'_1,\hat{c}_1}(d)$. Then, continuing from this expression, we have

$$f_{a'_0,b'_0,\hat{c}_0}(d) = f_{a'_1,b'_1,\hat{c}_1}(d) = \cdots = f_{a'_p,b'_p,\hat{c}_p}(d) = f_{a,b',\hat{c}_p}(d)$$

We now show the following property by induction on i:

$$\forall i, \qquad d \in (s_{b'_i}s_{a'_i}\omega)^{|}, \quad \hat{c}_i = \eta_{s_{b'_i}s_{a'_i}\omega}(d)$$

- For $i = 0$, $d \in (s_{b'_0}s_{a'_0}\omega)^{|} = (s_{\kappa(a,b')}t\omega)^{|}$ and $c' = \eta_{s_{\kappa(a,b')}t\omega}(d)$ hold by assumption.
- Suppose that the result is known for $i < p$, then $d \in (s_{b'_i}s_{a'_i}\omega)^{|}$ and $\hat{c}_i = \eta_{s_{b'_i}s_{a'_i}\omega}(d)$. Hence we also have $d \in (ts_{a'_i}\omega)^{|}$ and $\eta_{ts_{a'_i}\omega}(d) = \lambda_d(b'_i, \hat{c}_i)$ by Lemma 2.3. That is, $\eta_{ts_{a'_i}\omega}(d) = \hat{c}_{i+1}$. Since $ts_{a'_i}\omega = s_{b'_{i+1}}s_{a'_{i+1}}\omega$, we have $d \in (s_{b'_{i+1}}s_{a'_{i+1}}\omega)^{|}$ and $\hat{c}_{i+1} = \eta_{s_{b'_{i+1}}s_{a'_{i+1}}\omega}(d)$ as expected.

We may finally use the identity $(\hat{c}_p, d) \in (s_{b'} s_a \omega)^{|}$ to have the equality $f_{a,b,c}(d) = f_{a,b',\hat{c}_p}(d)$. Whence the equality $f_{a_0,b_0,c}(d) = f_{a'_0,b'_0,c'}(d)$.

The second equation $f_{a_0,b_0,c,d} = f_{a'_0,b'_0,c',d}$ is shown by the same reasoning. □

Definition 3.8 : Flag of an epiphyte

Let ω be an epiphyte. A *flag* of ω is a sequence $\mathfrak{p} = \xi_l \xi_{l-1} \cdots \xi_1$ of formal symbols ξ_i for $1 \leq i \leq l$ and $0 \leq l \leq \mathtt{dim}(\omega)$. When the sequence is empty, we denote it by $[]$. For each i, we have either $\xi_i = t$ or $\xi_i = s_x$ for some $x \in (\xi_{i-1}\xi_{i-2} \cdots \xi_1 \omega)^\bullet$.
When $f : \omega \to \varpi$ is a morphism between two epiphytes of the same dimension and $\mathfrak{p} = \xi_l \xi_{l-1} \cdots \xi_1$ is a flag of ω, we define a flag $f_* \mathfrak{p}$ of ϖ, and a morphism $\mathfrak{p} f : \mathfrak{p}\omega \to (f_*\mathfrak{p})\varpi$ by induction on $\mathfrak{p}$.

- When $\mathfrak{p}$ is the empty flag, $f_*\mathfrak{p}$ is defined as the empty flag and $\mathfrak{p} f := f$.
- Suppose that $\mathfrak{p} = s_x \mathfrak{q}$, then $f_*\mathfrak{p} := s_{(\mathfrak{q}f)(x)} f_*\mathfrak{q}$ and $\mathfrak{p} f := (\mathfrak{q} f)_x$.
- Suppose that $\mathfrak{p} = t\mathfrak{q}$, then $f_*\mathfrak{p} := t(f_*\mathfrak{q})$ and $\mathfrak{p} f := tf$.

When $\mathfrak{q} = \xi_l \xi_{l-1} \cdots \xi_1$ is a flag of ω and $\mathfrak{q} = \xi_m \xi_{m-1} \cdots \xi_{l+1}$ is a flag of $\mathfrak{p}\omega$, we define their concatenation $\mathfrak{q} \frown \mathfrak{p}$ as $\xi_m \cdots \xi_1$. It is a flag of ω.

Lemma 3.9

We have the following identites related to flags in epiphytes:

1. Let $f : \omega \to \varpi$ and $g : \varpi \to \varrho$ be two morphisms between epiphytes of the same dimension, and $\mathfrak{p}$ a flag of ω. Then $(g \circ f)_*\mathfrak{p} = g_* f_* \mathfrak{p}$ and $\mathfrak{p}(g \circ f) = (f_*\mathfrak{p})g \circ \mathfrak{p}f$.

2. Let $f : \omega \to \varpi$ be a morphism of epiphytes of the same dimension, $\mathfrak{p}$ a flag of ω and $\mathfrak{q}$ a flag of $\mathfrak{p}\omega$. Then $f_*(\mathfrak{q} \frown \mathfrak{p}) = (\mathfrak{p}f)_*\mathfrak{q} \frown f_*\mathfrak{p}$.

Proof.

1. We proceed by induction on $\mathfrak{p}$.

   - Suppose that $\mathfrak{p}$ is empty, then there is noting to prove.
   - Suppose that $\mathfrak{p} = s_x \mathfrak{q}$ where $x \in (\mathfrak{q}\omega)^\bullet$, then

$$\begin{array}{rcl}
(g \circ f)_*\mathfrak{p} & = & s_{\mathfrak{q}(g \circ f)(x)} \frown (g \circ f)_* \mathfrak{q} \\
 & = & s_{((f_*\mathfrak{q})g \circ \mathfrak{q}f)(x)} \frown (g_* f_* \mathfrak{q}) \\
 & = & g_*(s_{(\mathfrak{q}f)(x)} \frown f_*\mathfrak{q}) \\
 & = & g_* f_* \mathfrak{p}
\end{array}$$

   and

$$\begin{array}{rcl}
\mathfrak{p}(g \circ f) & = & (\mathfrak{q}(g \circ f))_x \\
 & = & ((f_*\mathfrak{q})g \circ \mathfrak{q}f)_x \\
 & = & ((f_*\mathfrak{q})g)_{(\mathfrak{q}f)(x)} \circ (\mathfrak{q}f)_x \\
 & = & \left(s_{(\mathfrak{q}f)(x)} f_*\mathfrak{q}\right) g \circ \mathfrak{p}f \\
 & = & (f_*\mathfrak{p})g \circ \mathfrak{p}f
\end{array}$$

   - Suppose that $\mathfrak{p} = t\mathfrak{q}$, then

$$\begin{array}{rcl}
(g \circ f)_*\mathfrak{p} & = & t \frown (g \circ f)_*\mathfrak{q} \\
 & = & t \frown g_* f_* \mathfrak{q} \\
 & = & g_*(t \frown f_*\mathfrak{q}) \\
 & = & g_* f_* \mathfrak{p}
\end{array}$$

   and

$$\begin{array}{rcl}
\mathfrak{p}(g \circ f) & = & t\,(\mathfrak{q}(g \circ f)) \\
 & = & t((f_*\mathfrak{q})g \circ \mathfrak{q}f) \\
 & = & t(f_*\mathfrak{q})g \circ t(\mathfrak{q}f) \\
 & = & (f_*\mathfrak{p})g \circ \mathfrak{p}f
\end{array}$$

2. We proceed by induction on the flag $\mathfrak{q}$.

   - When $\mathfrak{q}$ is empty, there is nothing to prove.
   - Suppose that $\mathfrak{q} = s_x\mathfrak{r}$ where $x \in (\mathfrak{r}\omega)^\bullet$, then

$$\begin{array}{rcl} f_*(\mathfrak{q} \frown \mathfrak{p}) & = & s_{((\mathfrak{r}\frown\mathfrak{p})f)(x)} \frown f_*(\mathfrak{r} \frown \mathfrak{p}) \\ & = & s_{(\mathfrak{r}\mathfrak{p}f)(x)} \frown (\mathfrak{p}f)_*\mathfrak{r} \frown f_*\mathfrak{p} \\ & = & (\mathfrak{p}f)_*\mathfrak{q} \frown f_*\mathfrak{p} \end{array}$$

   - Suppose that $\mathfrak{q} = t\mathfrak{r}$, then

$$\begin{array}{rcl} f_*(\mathfrak{q} \frown \mathfrak{p}) & = & t \frown f_*(\mathfrak{r} \frown \mathfrak{p}) \\ & = & t \frown (\mathfrak{p}f)_*\mathfrak{r} \frown f_*\mathfrak{p} \\ & = & (\mathfrak{p}f)_*\mathfrak{q} \frown f_*\mathfrak{p} \end{array}$$

□

Definition 3.10 : Face of an epiphyte

Let ω be an epiphyte. The *faces* of ω are defined as the flags of ω, modulo the rewriting rules:

- $\mathfrak{q}s_ys_x\mathfrak{p} \leftrightarrow \mathfrak{q}s_yt\mathfrak{p}$ when $(x, y) \in (\mathfrak{p}\omega)^{|}$.
- $\mathfrak{q}s_ys_x\mathfrak{p} \leftrightarrow \mathfrak{q}ts_{x'}\mathfrak{p}$ when there is a triplet $x \prec_y x'$ in $\mathfrak{p}\omega$.
- $\mathfrak{q}ts_x\mathfrak{p} \leftrightarrow \mathfrak{q}tt\mathfrak{p}$ when $x = \rho(\mathfrak{p}\omega)$.

Notice that the application $\mathfrak{p} \mapsto \mathfrak{p}\omega$ descend to the quotient as an application $[\mathfrak{p}] \mapsto \mathfrak{p}\omega =: [\mathfrak{p}]\omega$. The length of flags is also invariant by rewritings, and the length of a (representant of a) face will be called the *codimension* of that face. Given a face $[\mathfrak{p}]$, we also define $s_x[\mathfrak{p}] := [s_x\mathfrak{p}]$ and $t[\mathfrak{p}] := [t\mathfrak{p}]$.

Lemma 3.11

Suppose $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{q}$ as flags of an epiphyte ω, and $f : \omega \to \varpi$ a renaming. Then $\mathfrak{p}f = \mathfrak{q}f$.

Proof. It suffice to show the result in the case $\mathfrak{p} = \xi\xi' \leftrightarrow \zeta\zeta' = \mathfrak{q}$.

- Suppose $\xi\xi' = s_ys_x \leftrightarrow s_yt = \zeta\zeta'$. Then $y \in \omega^{|}$ and $x = \eta(y)$. Hence for any $z \in \mathfrak{p}\omega$, $(s_ys_xf)(z) = f_{x,y}(z)$ and $(s_ytf)(z) = f_{\eta(y),x}(z) = s_{x,y}(z)$. For the same reason $(s_ys_xf)_z = (s_ytf)_z$, whence the result.

- Suppose $\xi\xi' = s_ys_x \leftrightarrow ts_{x'} = \zeta\zeta'$. Then there is a triplet $x \prec_y x'$ in ω. Hence for any $z \in \mathfrak{p}\omega$, $(s_ys_xf)(z) = f_{x,y}(z)$ and $(ts_{x'}f)(z) = f_{x',\eta(z)}(z) = s_{x,y}(z)$ because f is a renaming. For the same reason $(s_ys_xf)_z = (ts_{x'}f)_z$, whence the result.

- Suppose $\xi\xi' = ts_x \leftrightarrow tt = \zeta\zeta'$. Then $x = \rho(\omega)$. Hence for any $z \in \mathfrak{p}\omega$, $(ts_x)(z) = f_{x,\eta_{s_x\omega}(z)}(z)$ and $(ttf)(z) = f_{\eta(\eta_{t\omega}(z)),\eta_{t\omega}(z)}(z)$. According to the proof of *Neatness* in Lemma 2.2, we have $\eta_{t\omega}(z) = \lambda_z(x, \eta_{s_x\omega}(z))$. Hence, by Lemma 3.6, $(ts_xf)(z) = (ttf)(z)$. For the same reason $(ts_xf)_z = (ttf)_z$, whence the result.

□

Lemma 3.12

Let $f : \omega \to \varpi$ be a renaming and $\xi\xi' \leftrightarrow \zeta\zeta'$ in ω. Then $f_*(\xi\xi') \overset{*}{\leftrightarrow} f_*(\zeta\zeta')$ in ϖ.

Proof. We distinguish on the rewriting rule applied.

- Suppose $\xi\xi' = s_ys_x \leftrightarrow s_yt = \zeta\zeta'$. Then $y \in \omega^{|}$ and $x = \eta(y)$. Hence $f_x(y) \in \varpi^{|}$ and $\eta(f_x(y)) = f(x)$ by Lemma 3.3. Since $f_*(s_ys_x) = s_{f_x(y)}s_{f(x)}$ and $f_*(s_yt) = s_{f_x(y)}t$, we have $f_*(\xi\xi') \leftrightarrow f_*(\zeta\zeta')$.

- Suppose $\xi\xi' = s_ys_x \leftrightarrow ts_{x'} = \zeta\zeta'$. Then there is a triplet $x \prec_y x'$ in ω. Hence there is a triplet $f(x) \prec_{f_x(y)} f(x')$ in ϖ. Since $f_*(s_ys_x) = s_{f_x(y)}s_{f(x)}$ and $f_*(ts_{x'}) = ts_{f(x)}$, we have $f_*(\xi\xi') \leftrightarrow f_*(\zeta\zeta')$.

- Suppose $\xi\xi' = ts_x \leftrightarrow tt = \zeta\zeta'$. Then $x = \rho(\omega)$. Hence $f(x) = \rho(\varpi)$ by Lemma 3.3. Since $f_*(ts_x) = ts_{f(x)}$ and $f_*(tt) = tt$, we have $f_*(\xi\xi') \leftrightarrow f_*(\zeta\zeta')$.

$\square$

Lemma 3.13

Let $f : \omega \to \varpi$ be a renaming and $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{q}$ two equivalent flags of ω. Then $f_*\mathfrak{p} \overset{*}{\leftrightarrow} f_*\mathfrak{q}$ in ϖ.

Proof. It suffices to show $\mathfrak{p} \leftrightarrow \mathfrak{q} \Rightarrow f_*\mathfrak{p} \leftrightarrow f_*\mathfrak{q}$. We proceed by induction on the length of $\mathfrak{p}$ and $\mathfrak{q}$.

- If $\mathfrak{p} = \mathfrak{q} = [[]]$ the result is clear.
- Suppose now that $\mathfrak{p} = \xi\mathfrak{p}'$ and $\mathfrak{q} = \zeta\mathfrak{q}'$ with $\mathfrak{p} \leftrightarrow \mathfrak{q}$.
  - If the rewriting takes place in $\mathfrak{p}'$, we have $\mathfrak{p}' \leftrightarrow \mathfrak{q}'$, hence $f_*\mathfrak{p}' \leftrightarrow f_*\mathfrak{q}'$ by induction hypothesis. Whence $f_*\mathfrak{p} = (\mathfrak{p}'f)_*\xi \frown f_*\mathfrak{p} \leftrightarrow (\mathfrak{q}'f)_*\zeta \frown f_*\mathfrak{q} = f_*\mathfrak{q}$, because $\xi = \zeta$ and $\mathfrak{p}'f = \mathfrak{q}'f$ by Lemma 3.11.
  - If the rewriting takes place in the leftmost position, we write $\mathfrak{p} = \xi\xi'\mathfrak{r}$ and $\mathfrak{q} = \zeta\zeta'\mathfrak{r}$. Then $f_*\mathfrak{p} = (\mathfrak{r}f)_*(\xi\xi') \frown f_*\mathfrak{r} \leftrightarrow (\mathfrak{r}f)_*(\zeta\zeta') \frown f_*\mathfrak{r} = f_*\mathfrak{q}$ accoring to Lemma 3.12. $\square$

Definition 3.14 : Epiphytes morphisms

For each epiphyte ω and face $[\mathfrak{p}]$ of ω, there is by definition a *structural map*

$$\iota_{[\mathfrak{p}]} : \mathfrak{p}\omega \to \omega$$

More, generally, we define the morphisms of epiphytes $\omega \to \varpi$ as the pairs $(f, [\mathfrak{p}])$ where $[\mathfrak{p}]$ is a face of ϖ of length $(\mathsf{dim}(\varpi) - \mathsf{dim}(\omega))$, and $f : \omega \to \mathfrak{p}\varpi$ is a renaming of epiphytes (of the same dimension), as defined in Definition 3.4. And $\iota_{[\mathfrak{p}]}$ is a short for $(\mathsf{id}, [\mathfrak{p}])$.
Given two morphisms $(f, [\mathfrak{p}]) : \omega \to \varpi$ and $(g, [\mathfrak{q}]) : \varpi \to \varrho$, we define their composition as

$$(g, [\mathfrak{q}]) \circ (f, [\mathfrak{p}]) := (\mathfrak{p}g \circ f, [g_*\mathfrak{p} \frown \mathfrak{q}]) : \omega \to \varrho$$

This is well defined according to lemmas 3.11 and 3.13. We let $\mathsf{id}_\omega = (\mathsf{id}_\omega, [[]])$, it is neutral for $\circ$.

Lemma 3.15 : Associativity of $\circ$

The composition of morphisms of epiphytes (as defined in Definition 3.14) is associative.

Proof. Let $(f, [\mathfrak{p}]) : \omega \to \varpi$, $(g, [\mathfrak{q}]) : \varpi \to \varrho$, $(h, [\mathfrak{r}]) : \varrho \to \varsigma$ be three composable morphisms of epiphytes. Then we have, using Lemma 3.9:

$$\begin{aligned} ((h, [\mathfrak{r}]) \circ (g, [\mathfrak{q}])) \circ (f, [\mathfrak{p}]) &= (\mathfrak{q}h \circ g, [h_*\mathfrak{q} \frown \mathfrak{r}]) \circ (f, [\mathfrak{p}]) \\ &= (\mathfrak{p}(\mathfrak{q}h \circ g) \circ f, [(\mathfrak{q}h \circ g)_*\mathfrak{p} \frown (h_*q \frown \mathfrak{r})]) \\ &= ((g_*\mathfrak{p} \frown \mathfrak{q})h \circ \mathfrak{p}g \circ f, [(\mathfrak{q}h)_*g_*\mathfrak{p} \frown h_*\mathfrak{q} \frown \mathfrak{r}]) \end{aligned}$$

and, on the other hand

$$\begin{aligned} (h, [\mathfrak{r}]) \circ ((g, [\mathfrak{q}]) \circ (f, [\mathfrak{p}])) &= (h, [\mathfrak{r}]) \circ (\mathfrak{p}g \circ f, [g_*\mathfrak{p} \frown \mathfrak{q}]) \\ &= ((g_*\mathfrak{p} \frown \mathfrak{q})h \circ (\mathfrak{p}g \circ f), [h_*(g_*\mathfrak{p} \frown \mathfrak{q}) \frown \mathfrak{r}]) \\ &= ((g_*\mathfrak{p} \frown \mathfrak{q})h \circ \mathfrak{p}g \circ f, [(\mathfrak{q}h)_*g_*\mathfrak{p} \frown h_*\mathfrak{q} \frown \mathfrak{r}]) \end{aligned}$$

Whence the associativity of $\circ$. $\square$

Definition 3.16 : Epi

The category of epiphytes **Epi** has the epiphytes as objects and the epiphytes morphisms as arrows.

4 Normal form of faces

4.1. Every face admits two distinguished representants, which will be called their *normal forms*. The existence of these normal forms allow us to decide easily if two given flags induce the same face. We begin by characterising the 3-flags inducing a same face.

Definition 4.2 : Type of a 3-flag

We assign to each 3-flag $\mathfrak{p}$ of an epiphyte ω a *type* $\tau(\mathfrak{p})$, as follows:

- For each triplet $b \prec_c b'$ in $t\omega$, let

  $$\succ_{b=b_0} a_0 \succ_{b_1} \cdots \succ_{b_p} a_p = a'_q \prec_{b'_q} \cdots \prec_{b'_1} a'_0 \prec_{b'_0=b'}$$

  be the paths in ω with $c \in (s_{a_i}\omega)^{|}$, $\eta_{s_{a_i}\omega}(c) = b_i$ for $i < p$, $b'_i = \rho(s_{a'_i}\omega)$ for $i < q$ and a triplet $b_p \prec_c b'_q$ in $s_{a_p}\omega$. Then we have the following cycle of equivalent flags:

  $$s_c\underline{s_b t} \leftrightarrow \underline{s_c s_{b_0}} s_{a_0} \leftrightarrow s_c\underline{t s_{a_0}} \leftrightarrow \underline{s_c s_{b_1}} s_{a_1} \leftrightarrow \cdots \leftrightarrow \underline{s_c s_{b_p}} s_{a_p}$$
  $$\leftrightarrow t\underline{s_{b'_q} s_{a'_q}} \leftrightarrow \underline{tt} s_{a'_{q-1}} \leftrightarrow t\underline{s_{b'_{q-1}} s_{a'_{q-1}}} \leftrightarrow \cdots \leftrightarrow t\underline{s_{b'_0} s_{a'_0}} \leftrightarrow \underline{t s_{b'}} t \leftrightarrow s_c s_b t.$$

  We assign to each flag in this sequence the type $\mathrm{I}_{(b,c)}$.

- For each leaf c of $t\omega$, let

  $$\succ_{b=b_0} a_0 \succ_{b_1} \cdots \succ_{b_p} a_p = \rho(\omega)$$

  be the path in ω with $c \in (s_{a_i}\omega)^{|}$, $\eta_{s_{a_i}\omega}(c) = b_i$ for $i \leq p$. Then we have the following cycle of equivalent flags:

  $$s_c\underline{s_b t} \leftrightarrow \underline{s_c s_{b_0}} s_{a_0} \leftrightarrow s_c\underline{t s_{a_0}} \leftrightarrow \underline{s_c s_{b_1}} s_{a_1} \leftrightarrow \cdots \leftrightarrow \underline{s_c s_{b_p}} s_{a_p} \leftrightarrow s_c\underline{t s_{a_p}} \leftrightarrow \underline{s_c t} t \leftrightarrow \underline{s_c s_b} t.$$

  We assign to each flag in this sequence the type II_c.

- Since the root of $t\omega$ is $b := \kappa(\rho(\omega), \rho(s_{\rho(\omega)}\omega))$, there is a path

  $$\succ_{b=b_0} a_0 \succ_{b_1} \cdots \succ_{b_p} a_p = \rho(\omega)$$

  such that $b_i = \rho(s_{a_i}\omega)$ for $i \leq p$. Hence there is a cycle of equivalent flags:

  $$t\underline{tt} = \underline{tt} s_{a_p} \leftrightarrow t\underline{s_{b_p} s_{a_p}} \leftrightarrow \underline{tt} s_{a_{p-1}} \leftrightarrow \cdots \leftrightarrow \underline{tt} s_{a_0} \leftrightarrow t s_{b_0} s_{a_0} \leftrightarrow \underline{t s_b} t \leftrightarrow ttt.$$

  We assign to each flag in this sequence the type III.

Since every 3-flag must appears in exactly one of those cycle, τ is well defined. Moreover, for two 3-flags $\mathfrak{p}$, $\mathfrak{q}$, we have $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{q}$ iff $\tau(\mathfrak{p}) = \tau(\mathfrak{q})$.

Remark 4.3
Notice that for each of the above sequence, there are exactly two flags $\xi_1\xi_2 t$ and $\xi'_1\xi'_2 t$, and moreover $\underline{\xi_1\xi_2} t \leftrightarrow \xi'_1\xi'_2 t$. Hence, every 3-face of an epiphyte admits exactly two representants $\underline{\xi_1\xi_2} t \leftrightarrow \xi'_1\xi'_2 t$ which are called their *normal forms*. We focus now on extending this property to every face of an epiphyte.

Remark 4.4
Notice that any flag of the form $s_c s_b s_a$ is equivalent to $s_c s_{\lambda_c(a,b)} t$.
Similarly, any flag of the form $t s_b s_a$ is equivalent to $t s_{\kappa(a,b)} t$.

Lemma 4.5 : confluence

Let $\underline{\xi_4\xi_3}\xi_2\xi_1 \leftrightarrow \xi'_4\xi'_3\xi_2\xi_1$ be two 4-flags of an epiphyte ω.
Suppose $\xi_3\xi_2\xi_1 \overset{*}{\leftrightarrow} \xi_3^*\xi_2^* t$ and $\xi'_3\xi_2\xi_1 \overset{*}{\leftrightarrow} \xi_3^{**}\xi_2^{**} t$, then $\xi_4\xi_3^*\xi_2^* \overset{*}{\leftrightarrow} \xi'_4\xi_3^{**}\xi_2^{**}$ as flags of $t\omega$.

Proof. Notice first that the result does not depend on the choice of $\xi_3^*\xi_2^*$ (resp. $\xi_3^{**}\xi_2^{**}$) because they yield equivalents flags $\xi_4\xi_3^*\xi_2^*$ (resp. $\xi_4\xi_3^{**}\xi_2^{**}$) of $t\omega$. If $\xi_1 = t$, then we may choose $\xi_3^*\xi_2^* = \xi_3^{**}\xi_2^{**} = \xi_3\xi_2$, hence the result, we now suppose $\xi_1 = s_a$ for some $a \in \omega^\bullet$.
If $\xi_2 = t$ and $a = \rho(\omega)$ then $ts_a \leftrightarrow tt$ as flags of ω, so we are done because we are reduced to the case $\xi_1 = t$.
If $\xi_2 = t$ and there is a triplet $a' \prec_{b'} a$, then $ts_a \leftrightarrow s_{b'}s_{a'}$ as flags of ω.
Hence we may now suppose $\xi_2 = s_b$ for some $b \in (s_a\omega)^\bullet$.

- Suppose $\xi_4\xi_3 = s_d s_c$ and $\xi'_4\xi'_3 = s_d t$, with $d \in (s_b s_a \omega)^|$, $\eta(d) = c$. By Definition 4.2, the type of a flag of the form $s_d **$ in $t\omega$ will always by II_d, whence the result.
- Suppose $\xi_4\xi_3 = tt$ and $\xi'_4\xi'_3 = ts_c$ with $c = \rho(s_b s_a \omega)$. By Remark 4.4, we may choose $\xi_3^*\xi_2^* = ts_{\kappa(a,b)}$ and $\xi_3^{**}\xi_2^{**} = s_c s_{\lambda_c(a,b)}$. Still by Remark 4.4, $\xi_4\xi_3^*\xi_2^* = tts_{\kappa(a,b)} \leftrightarrow ts_{c'}s_{\kappa(a,b)} \overset{*}{\leftrightarrow} ts_{\kappa(\kappa(a,b),c')}t$ where $c' = \rho(s_{\kappa(a,b)}t\omega)$, and $\xi'_4\xi_3^{**}\xi_2^{**} = ts_c s_{\lambda_c(a,b)} \overset{*}{\leftrightarrow} ts_{\kappa(\lambda_c(a,b),c)}t$.
  Using Lemma 2.5, $\kappa(\kappa(a,b),c') = \kappa(\lambda_c(a,b),c)$, whence the result.
- Suppose $\xi_4\xi_3 = s_d s_c$ and $\xi'_4\xi'_3 = ts_{c'}$ with a triplet $c \prec_d c'$ in $s_b s_a \omega$. Using Remark 4.4, we may choose $\xi_3^*\xi_2^* = s_c s_{\lambda_c(a,b)}$ and $\xi_3^{**}\xi_2^{**} = s_{c'}s_{\lambda_{c'}(a,b)}$. Still by Remark 4.4, $\xi_4\xi_3^*\xi_2^* = s_d s_c s_{\lambda_c(a,b)} \overset{*}{\leftrightarrow} s_d s_{\lambda_d(\lambda_c(a,b),c)}t$ and $\xi'_4\xi_3^{**}\xi_2^{**} = ts_{c'}s_{\lambda_{c'}(a,b)} \overset{*}{\leftrightarrow} ts_{\kappa(\lambda_{c'}(a,b),c')}t$.
  By Lemma 2.8, there is a triplet $\lambda_d(\lambda_c(a,b),c) \prec_d \kappa(\lambda_{c'}(a,b),c')$ in $tt\omega$, whence the result.

□

Theorem 4.6 : Normal form of a face

Let ω be an epiphyte, and some face x of codimension $p \geq 2$. Then x admits exactly two distinguished representants of the form $\zeta_p\zeta_{p-1}t\cdots t$ and $\zeta'_p\zeta'_{p-1}t\cdots t$. Moreover, $\zeta_p\zeta_{p-1} \leftrightarrow \zeta'_p\zeta'_{p-1}$ in $t^{p-2}\omega$. We call them the *normal forms* of x. Hence a normal form is always defined up to a leftmost rewriting.

Proof. We proceed by induction on p.
Suppose $p = 2$, then x admits exactly two representants $\xi_2\xi_1 \leftrightarrow \xi'_2\xi'_1$, whence the result.
When $p = 3$, the result is known (see Remark 4.3).
Suppose now $p \geq 4$ and the result known in lower dimension. Then we write $\mathfrak{p} = \zeta_p\cdots\zeta_1 = \zeta_p\zeta_{p-1}\mathfrak{p}'$. We may first compute a normal form $\mathfrak{n}' = \xi_{p-2}\xi_{p-3}t\cdots t$ of $\mathfrak{p}'$, then find a normal form $\xi_{p-1}^*\xi_{p-2}^*t$ of $\xi_{p-1}\zeta_{p-2}\zeta_{p-3}$ in $t^{p-4}\omega$, and then find a normal form $\xi_p^{**}\xi_{p-1}^{**}t$ of $\zeta_p\xi_{p-1}^*\xi_{p-2}^*$ in $t^{p-3}\omega$. Thus a normal form $\mathfrak{n} = \xi_p^{**}\xi_{p-1}^{**}t\cdots t$ of $\mathfrak{p}$.
We now see the uniquess of $\mathfrak{n}$ (up to one leftmost rewriting). In order to do so, we show that (up to one leftmost rewriting) $\mathfrak{n}$ does not change under rewritings of $\mathfrak{p}$. Suppose $\mathfrak{p} \leftrightarrow \mathfrak{q}$.

- Either the rewriting takes place in $\zeta_{p-1}\cdots\xi_1$, hence the normal form $\xi_{p-1}^*\xi_{p-2}^*t\cdots t$ did not change up to a leftmost rewriting (by induction hypothesis), which in turn does not change $\mathfrak{n}$ up to a leftmost rewriting.
- Or the rewriting is of the form $\zeta_p\zeta_{p-1} \leftrightarrow \zeta'_p\zeta'_{p-1}$ in $\mathfrak{p}'\omega = \mathfrak{n}'\omega$, which leaves $\mathfrak{n}$ unchanged up to a leftmost rewriting, according to Lemma 4.5 applied to $t^{p-4}\omega$.

In order to have the uniqueness, suppose now that $\mathfrak{n}$ and $\mathfrak{m}$ are two normal forms of $\mathfrak{p}$. Then using the process described above we may compute the normal form of $\mathfrak{m}$, which is $\mathfrak{m}$ up to a leftmost rewriting. But by invariance under rewritings, we also have $\mathfrak{m} = \mathfrak{n}$ up to a leftmost rewriting, whence the uniqueness. □

Lemma 4.7

Let $\mathfrak{p}$ be a flag of length at least 2 in $t\omega$ for some epiphyte ω. Denoting $\mathfrak{n}$ a normal form of $\mathfrak{p}$ and $\mathfrak{m}$ a normal form of $\mathfrak{p}t$, we have $\mathfrak{m} = \mathfrak{n}t$ up to a leftmost rewriting.

Proof. This is a direct consequence of the uniqueness of normal forms. □

Lemma 4.8 : Right regularity of t

Let $\mathfrak{p}$, $\mathfrak{q}$ be two flags of $t\omega$ for some epiphyte ω, such that $\mathfrak{p}t \overset{*}{\leftrightarrow} \mathfrak{q}t$. Then $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{q}$ in $t\omega$.

Proof. If $\mathfrak{p}$, $\mathfrak{q}$ are the null flag or a 1-flag, this is a direct observation. Suppose now that they have length at least 2. Denoting n a normal form of $\mathfrak{p}$ and $\mathfrak{m}$ a normal form of $\mathfrak{q}$, we have $\mathfrak{n}t = \mathfrak{m}t$ up to a leftmost rewriting because both are normal forms of $\mathfrak{p}t$. Hence there is a sequence of rewritings $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{n} \overset{*}{\leftrightarrow} \mathfrak{m} \overset{*}{\leftrightarrow} \mathfrak{q}$ in $t\omega$. □

Definition 4.9 : $\chi(\xi)$, $\chi(\mathfrak{p})$

Let ω be an epiphte, and ξ be the formal symbol t or s_b for some $b \in \omega^{|}$. Then $\chi(\xi)$ is the following formal symbol:

$$\chi(\xi) = \begin{cases} s_{\rho(\omega)} & \text{if } \xi = t. \\ s_{\eta(b)} & \text{if } \xi = s_b \text{ for some } b \in \omega^{|}. \end{cases}$$

For $\mathfrak{p} = \mathfrak{p}'\xi$ a non empty flag, we let $\chi(\mathfrak{p}) = \chi(\xi)$, in such a way that we always have $\xi t \leftrightarrow \xi\chi(\xi)$.

Lemma 4.10

Suppose we have $\xi_2\xi_1 t \leftrightarrow \xi_2'\xi_1' t$ as flags of some epiphyte ω. Then there is a sequence of rewritings

$$\xi_2\xi_1\chi(\xi_1) = \underline{\xi_2^1\xi_1^1}\chi^1 \leftrightarrow \xi_2^2\underline{\xi_1^2\chi^1} \leftrightarrow \underline{\xi_2^2\xi_1^3}\chi^2 \leftrightarrow \cdots \leftrightarrow \underline{\xi_2^p\xi_1^{2p-1}}\chi^p \leftrightarrow \xi_2^{p+1}\xi_1^{2p}\chi^p = \xi_2'\xi_1'\chi(\xi_1').$$

such that $\forall i, \chi^i \neq t$.

Proof. This is directly seen by case analysis on the type of $\xi_2\xi_1 t$. □

Lemma 4.11

Let ω be an epiphyte, and suppose there is a sequence of rewritings $\mathfrak{p}t \overset{*}{\leftrightarrow} \mathfrak{q}t$. Then there is a sequence of rewritings $\mathfrak{p}\chi(\mathfrak{p}) = \mathfrak{p}^1\chi^1 \leftrightarrow \cdots \leftrightarrow \mathfrak{p}^q\chi^q = \mathfrak{q}\chi(\mathfrak{q})$ such that $\forall i, \chi^i \neq t$.

Proof. We proceed by induction on the sequence length. If the sequence is empty, the result is clear. We now suppose the sequence to be non-empty, and distinguish on the first rewriting.

- Assume the first rewriting to be of the form $\xi_p \cdots \underline{\xi_{i+1}\xi_i} \cdots \xi_1 t \leftrightarrow \xi_p \cdots \underline{\zeta_{i+1}\zeta_i} \cdots \xi_1 t$ for some $i > 1$. Since $\xi_1 t\omega = \xi_1\chi(\xi_1)\omega$, the rewriting $\xi_p \cdots \underline{\xi_{i+1}\xi_i} \cdots \xi_1\chi(\xi_1) \leftrightarrow \xi_p \cdots \underline{\zeta_{i+1}\zeta_i} \cdots \xi_1\chi(\xi_1)$ is licit. We then conclude by induction hypothesis.
- Assume the first rewriting to be of the form $\xi_p \cdots \underline{\xi_2\xi_1} t \leftrightarrow \xi_p \cdots \zeta_2\zeta_1 t$. Then using Lemma 4.10, we find a sequence of rewritings $\xi_p \cdots \xi_2\xi_1\chi(\xi_1) \overset{*}{\leftrightarrow} \xi_p \cdots \zeta_2\zeta_1\chi(\zeta_1)$ with no t appearing in rightmost position. We then conclude by induction hypothesis.
- Assume the first triplet to be of the form $\xi_p \cdots \xi_2\underline{\xi_1 t} \leftrightarrow \xi_p \cdots \xi_2\xi_1\chi(\xi_1)$. Let the sequence of rewriting be

$$\xi_p^1 \cdots \xi_1^1\chi^1 \leftrightarrow \xi_p^2 \cdots \xi_1^2\chi^2 \leftrightarrow \cdots \leftrightarrow \xi_p^q \cdots \xi_1^q\chi^q.$$

Let $m := \min\{i > 1 \mid \chi^i = t\}$. Then the $(i-1)$-th rewriting is necessarily of the form $\zeta_p \cdots \zeta_2\underline{\zeta_1\chi(\zeta_1)} \leftrightarrow \zeta_p \cdots \zeta_2\zeta_1 t$. Hence, picking the subsequence from the second to the $(i-1)$-th elements of the sequence yield a subsequence of rewritings of the form $\xi_p \cdots \xi_1\chi(\xi_1) \overset{*}{\leftrightarrow} \zeta_p \cdots \zeta_1\chi(\zeta_1)$. We then conclude by induction hypothesis. □

Lemma 4.12 : Right regularity of s_a

Let ω be an epiphyte, $a \in \omega^\bullet$, $\mathfrak{p}$, $\mathfrak{q}$ two flags of $s_a\omega$ and suppose $\mathfrak{p}s_a \overset{*}{\leftrightarrow} \mathfrak{q}s_a$.
Then $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{q}$ as flags of $s_a\omega$.

Proof. We proceed by induction on the length p of $\mathfrak{p}$, $\mathfrak{q}$.

- *Initialisation ($p \leq 1$).*
- *Heredity ($p \geq 2$).* Suppose now $p \geq 2$ and suppose the result known for lower lengths. First, using Lemma 4.11, we may suppose that no t appears in rightmost position along the sequence of rewritings $\mathfrak{p}s_a \overset{*}{\leftrightarrow} \mathfrak{q}s_a$. Then we may cut the sequence in subsequences of rewritings as follows:

$$\mathfrak{p}s_a = \mathfrak{p}^0\xi^0 s_{a^0} \overset{*}{\leftrightarrow} \mathfrak{p}^1\underline{\zeta^0 s_{a^0}} \leftrightarrow \mathfrak{p}^1\xi^1 s_{a^1} \overset{*}{\leftrightarrow} \cdots \overset{*}{\leftrightarrow} \mathfrak{p}^{q-1}\underline{\zeta^{q-1}s_{a^{q-1}}} \leftrightarrow \mathfrak{p}^{q-1}\xi^q s_{a^q} \overset{*}{\leftrightarrow} \mathfrak{p}^q\zeta^q s_{a^q} = \mathfrak{q}s_a.$$

  With $a^0 = a^q = a$, and each subsequence $\mathfrak{p}^i\xi^i s_{a^i} \overset{*}{\leftrightarrow} \mathfrak{p}^{i+1}\zeta^i s_{a^i}$ having no rightmost rewriting. Notice that each rightmost rewriting $\xi^i s_{a^i} \leftrightarrow \zeta^i s_{a^{i+1}}$ must have the form $ts_{a^i} \leftrightarrow s_{b^{i+1}}s_{a^{i+1}}$ or $s_{b^i}s_{a^i} \leftrightarrow ts_{a^{i+1}}$. Hence there is a zig-zag in the tree structure of ω, following the sequence of nodes $a = a^0, a^1, \cdots, a^q = a$. We now reason by induction on the length of this path.

  – Suppose it is constant on a. Hence the whole sequence of rewritings $\mathfrak{p}s_a \overset{*}{\leftrightarrow} \mathfrak{q}s_a$ is concentrated as one subsequence of the form $\mathfrak{p}^0\xi^0 s_a \overset{*}{\leftrightarrow} \mathfrak{p}^1\zeta^0 s_a$ having no roghtmost rewriting, whence the result.

  – Suppose the path is not constant. Then there must be two consecutive triplets of the form $a^i \prec_{b^i} a^{i+1} \succ_{b^{i+1}=b^i} a^{i+2} = a^i$ or $a^i \succ_{b^i} a^{i+1} \prec_{b^{i+1}=b^i} a^{i+2} = a^i$ appearing along the zig-zag. Hence $\xi^{i+1} = \zeta^{i+1}$ for some i. Thus by induction hypothesis we may find a sequence of rewritings $\mathfrak{p}^{i+1} \overset{*}{\leftrightarrow} \mathfrak{p}^{i+2}$ in $\xi^{i+1}s_{a^{i+1}}\omega$ and assume the subsequence $\mathfrak{p}^{i+1}\xi^{i+1}s_{a^{i+1}} \overset{*}{\leftrightarrow} \mathfrak{p}^{i+2}\zeta^{i+1}s_{a^{i+1}}$ to leave unchnged the two rightmost elements. We may now supress the two rewritings $\zeta^i s_{a^i} \leftrightarrow \xi^{i+1}s_{a^{i+1}}$ and $\zeta^{i+1}s_{a^{i+1}} \leftrightarrow \xi^{i+2}s_{a^{i+2}}$ in the sequence, which remains well formed. Hence we are reduced to a smaller case, and conclude by induction hypothesis. □

Theorem 4.13 : Right regularity

Let ω be an epiphyte, χ a flag of ω, and $\mathfrak{p}$, $\mathfrak{q}$ two flags of $\chi\omega$. If $\mathfrak{p}\chi \overset{*}{\leftrightarrow} \mathfrak{q}\chi$, then $\mathfrak{p} \overset{*}{\leftrightarrow} \mathfrak{q}$ in $\chi\omega$.

Proof. This is a direct induction on the length of χ using Lemma 4.8 and Lemma 4.12. □

5 From epiphytes to dendritic face structures

5.1. The aim of this section is to associate to any epiphyte ω a dendritic face structure $\mathcal{F}(\omega)$, and to show that this construction is functorial. The elements of the poset will be the faces of ω.

Definition 5.2 : face poset $\mathcal{F}(\omega)$

Let ω be an epiphyte, define its *face poset* $\mathcal{F}(\omega)$ as the following positive-to-one poset:

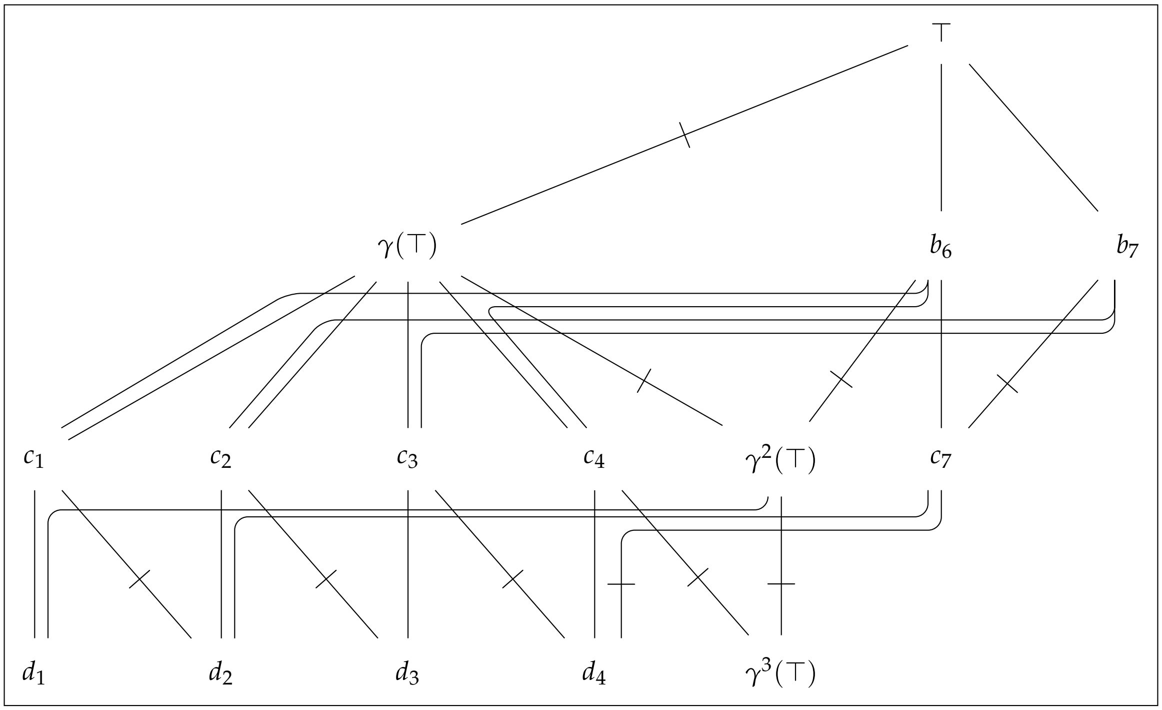
- The set of elements is given by the faces of ω, as defined in Definition 3.10.
- If $[\mathfrak{p}]$ is a face of ω with codimension q, $\mathsf{dim}(a) := \mathsf{dim}(\omega) - q$.
- We define $\prec^-$ and $\prec^+$ by

$$[\mathfrak{p}] \prec^- [\mathfrak{q}] \Leftrightarrow (\exists x, [\mathfrak{p}] = s_x[\mathfrak{q}]) \qquad \text{and} \qquad [\mathfrak{p}] \prec^+ [\mathfrak{q}] \Leftrightarrow [\mathfrak{p}] = t[\mathfrak{q}].$$

Proof. We check that $\mathcal{F}(\omega)$ as defined above indeed is a positive-to-one poset. We need to see that $\prec^-$ and $\prec^+$ are disjoint, the other properties are clear from the definition. Suppose that $v \prec^+ u$ and $v \prec^- u$, then choosing a representant $\mathfrak{p}$ for u, we have $v = [s_x\mathfrak{p}] = [t\mathfrak{p}]$ for some $x \in (\mathfrak{p}\omega)^\bullet$. Hence $s_x\mathfrak{p} \overset{*}{\leftrightarrow} t\mathfrak{p}$ which is impossible, by Theorem 4.13. □

Remark 5.3
Suppose that $[p] \prec^- [q]$, then there is some x such that $\mathfrak{p} \overset{*}{\leftrightarrow} s_x\mathfrak{q}$. Such a x is uniquely defined because of Theorem 4.13.

Exemple 5.4
Let ω be the 4-epiphyte considered in Exemple 1.12. We consider its source $s_{a_1}\omega$. Then the poset of its faces is the following:



$\mathcal{F}(s_{a_1}\omega)$

Where $\top$ denotes $[[]]$, and a face which has a normal form $s_x\mathfrak{q}$ is written x for short.

Proposition 5.5 : Greatest element

The face poset $\mathcal{F}(\omega)$ of an epiphyte ω admits a greatest element $[[]]$.

Proof. It is clear from the definition of $\mathcal{F}(\omega)$ that the face asociated to the empty flag $[]$ is a greatest element. □

Proposition 5.6 : Oriented thinness

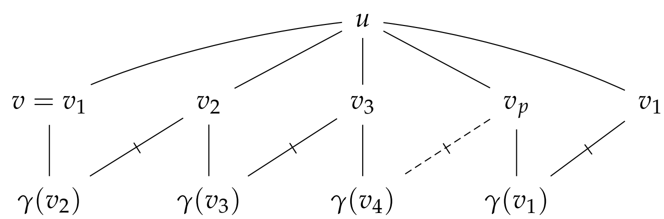
The face poset $\mathcal{F}(\omega)$ of an epiphyte ω satisfies oriented thinness.

Proof. Let $w \prec v \prec u$ in $\mathcal{F}(\omega)$. Choosing a representant $\mathfrak{p}$ of u, we may find ξ_1, ξ_2 such that $[\xi_1\mathfrak{p}] = v$ and $[\xi_2\xi_1\mathfrak{p}] = w$. The existence part of the lozenge completion follows from the definition of rewriting rules. Suppose there is a lozenge completion $w \prec v' \prec u$, then we may find ζ_1, ζ_2 such that $[\zeta_1\mathfrak{p}] = v'$ and $[\zeta_2\zeta_1\mathfrak{p}] = u$. Hence $\xi_2\xi_1\mathfrak{p} \overset{*}{\leftrightarrow} \zeta_2\zeta_1\mathfrak{p}$. Using Theorem 4.13, it implies $\xi_2\xi_1 \leftrightarrow \zeta_2\zeta_1$ in $\mathfrak{p}\omega$, whence the uniquenness. □

Proposition 5.7 : Acyclicity

The face poset $\mathcal{F}(\omega)$ of an epiphyte ω satisfies acyclicity.

Proof. Suppose there is a (non-trivial) cycle of the following form in $\mathcal{F}(\omega)$.



Then choosing a representant $\mathfrak{p}$ of u, we find $x_1, \cdots, x_p$ and $y_1, \cdots, y_p$ such that $v_i = [s_{x_i}\mathfrak{p}]$ and $\gamma(v_{i+1}) = [ts_{x_{i+1}}\mathfrak{p}] = [s_{y_i}s_{x_i}\mathfrak{p}]$ for all i (with $v_{p+1} := v_1$ and $x_{i+1} := x_1$). Using the same argument as in te proof of Proposition 5.6, we have rewritings $s_{y_i}s_{x_i} \leftrightarrow ts_{x_{i+1}}$ in $\mathfrak{p}\omega$ for all i. hence (by definition of the rewriting rules) there must be triplets $x_i \prec_{y_i} x_{i+1}$ in $\mathfrak{p}\omega$ for all i, yielding a cycle in the tree structure of $\mathfrak{p}\omega$, which is impossible. □

Theorem 5.8

The face poset $\mathcal{F}(\omega)$ of an epiphyte ω is a dendritic face complex.

Proof. This is the consequence of Proposition 5.5, Proposition 5.6, Proposition 5.7. □

Proposition 5.9

For any epiphyte ω and $[\mathfrak{p}]$ a flag of ω, we have an isomorphism $\varphi_{[\mathfrak{p}]} : \mathcal{F}(\mathfrak{p}\omega) \simeq \mathsf{cl}([\mathfrak{p}])$ given by $\phi_{[\mathfrak{p}]}([\mathfrak{q}]) = [\mathfrak{q}\mathfrak{p}]$.

Proof. Clearly, φ preserves the codimension, hence the dimension because both posets have the same dimension. Let u, v be two elements of $\mathcal{F}(\mathfrak{p}\omega)$, and let $\mathfrak{q}$ be a representant of u. Suppose $v \prec^- u$, then there is a x with $v = [s_x\mathfrak{q}]$. Hence $\varphi(v) = [s_x\mathfrak{q}\mathfrak{p}] \prec^- [\mathfrak{q}\mathfrak{p}] = \varphi(u)$. Similarly, φ preserves $\prec^+$. Suppose that $\varphi(u) = \varphi(v)$ and let $\mathfrak{q}'$ be a representant of v. Then $[\mathfrak{q}\mathfrak{p}] = [\mathfrak{q}'\mathfrak{p}]$, whence $u = [\mathfrak{q}] = [\mathfrak{q}'] = v$ by Theorem 4.13. Hence φ is injective, so it is a well defined morphism. φ is an isomorphism because both posets have the same dimension. □

Remark 5.10
Notice that we have the relations $\varphi_{[[]]} = \mathsf{id}$ and $\varphi_{[\mathfrak{p}]} \circ \varphi_{[\mathfrak{q}]} = \varphi_{[\mathfrak{q}\mathfrak{p}]}$ when it makes sense.

Proposition 5.11

Any renaming of epiphytes $f : \omega \to \varpi$ induces a morphism $\mathcal{F}(f) : \mathcal{F}(\omega) \to \mathcal{F}(\varpi)$ functorialy in f, defined by $\mathcal{F}(f)([\mathfrak{p}]) = [f_*\mathfrak{p}]$.

Proof. Notice first that the well-definedness of $\mathcal{F}(f)$ follows from Lemma 3.13.
Since $f_*(s_x\mathfrak{p}) = s_{(\mathfrak{p}f)(x)} \frown f_*\mathfrak{p}$, $\mathcal{F}(f)$ preserves $\prec^-$. And since $f_*(t\mathfrak{p}) = t \frown f_*\mathfrak{p}$, $\mathcal{F}(f)$ preserves $\prec^+$.
Since $\mathfrak{p}f$ is a bijection on nodes, $\mathcal{F}(f)$ induce a bijection between $\delta([\mathfrak{p}])$ and $\delta([f_*\mathfrak{p}])$ for any $\mathfrak{p}$. Moreover, f_* preserves codimension, and $\mathcal{F}(\omega)$ has same dimension as $\mathcal{F}(\varpi)$, hence $\mathcal{F}(f)$ preserves dimension. Whence $\mathcal{F}(f)$ a morphism of DFC.
The functoriality of $\mathcal{F}$ is given by Lemma 3.9. $\mathcal{F}(f)$ is an isomorphism because $\mathsf{dim}\,(\mathcal{F}(\omega)) = \mathsf{dim}\,(\mathcal{F}(\varpi))$. □

Theorem 5.12 : Functoriality of $\mathcal{F}$

$\mathcal{F}$ defines a functor from **Epi** to **DFC** by the formula $\mathcal{F}((f, [\mathfrak{p}])) = \varphi_{[\mathfrak{p}]} \circ \mathcal{F}(f)$.

Proof. We already have $\mathcal{F}(\mathsf{id}) = \mathcal{F}((\mathsf{id}, [[]])) = \mathsf{id} \circ \mathsf{id} = \mathsf{id}$. Let $(f, [\mathfrak{p}]) : \omega \to \varpi$ and $(g, [\mathfrak{q}]) : \varpi \to \varrho$. Then $(g, [\mathfrak{q}]) \circ (f, [\mathfrak{p}]) = (\mathfrak{p}g \circ f, [g_*\mathfrak{p} \frown \mathfrak{q}]) : \omega \to \varrho$.

$$\begin{aligned}\mathcal{F}((\mathfrak{p}g \circ f, [g_*\mathfrak{p} \frown \mathfrak{q}])) &= \varphi_{[g_*\mathfrak{p}\frown\mathfrak{q}]} \circ \mathcal{F}(\mathfrak{p}g \circ f) \\ &= \varphi_{[\mathfrak{q}]} \circ \varphi_{[g_*\mathfrak{p}]} \circ \mathcal{F}(\mathfrak{p}g) \circ \mathcal{F}(f)\end{aligned}$$

And we are done if we show the equality $\varphi_{[g_*\mathfrak{p}]} \circ \mathcal{F}(\mathfrak{p}g) = \mathcal{F}(g) \circ \varphi_{[\mathfrak{p}]}$. Let $\mathfrak{r}$ be a flag of $\mathcal{F}(\mathfrak{p}\omega)$, then we have the following commutative square:

$$\begin{array}{ccccc}
[\mathfrak{r}] & & \xmapsto{\mathcal{F}(\mathfrak{p}g)} & & [(\mathfrak{p}g)_*\mathfrak{r}] \\
\downarrow \varphi_{[\mathfrak{p}]} & & & & \downarrow \varphi_{[g*\mathfrak{p}]} \\
[\mathfrak{r} \frown \mathfrak{p}] & \xmapsto[\mathcal{F}(g)]{} & [g_*(\mathfrak{r} \frown \mathfrak{p})] & \overset{}{\underset{\text{Lemma 3.9}}{=\!=\!=}} & [(\mathfrak{p}g)_*\mathfrak{r} \frown g_*\mathfrak{p}]
\end{array}$$

Whence the result. $\square$

6 From Dendritic Face Complexes to Epiphytes

6.1. In this section, we describe a way to associate an epiphyte to every dendritic face complex. This is done by induction on the dimension of the DFC, as follows:

Definition 6.2 : $\mathcal{E}(C)$

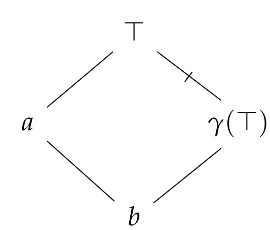
Let C be a dendritic face complex. We define an epiphyte $\mathcal{E}(C)$ of the same dimension:

- When C is $0-$dimensional, we let
$$\mathcal{E}(C) := \blacklozenge$$
- When C is $n-$dimensional with $n \geq 1$, we let $\mathcal{E}(C)$ be the $n-$epiphyte defined by:
  - $(\mathcal{E}(C))^\bullet = \delta(\top)$, where $\top$ denotes the maximal element of C.
  - The tree structure on $(\mathcal{E}(C))^\bullet$ is given by the tree structure on $\delta(\top)$.
  - For each $a \in \delta(\top)$, we let $s_a\mathcal{E}(C) := \mathcal{E}(\mathsf{cl}(a))$.

Lemma 6.3

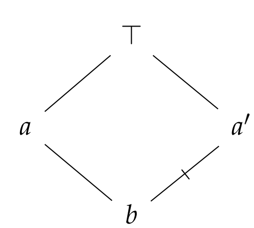
The definition above indeed defines an epiphyte.

Proof. We proceed by induction on the dimension of the poset C.
The result is clear when $\dim(C) = 0$. Suppose now $\dim(C) > 0$ and the result known in lower dimensions. First, we check that the rooted tree structure on $\mathcal{E}(C)$ is neat. If $b \in \mathcal{E}(C)$, let a be such that $b \in s_a\mathcal{E}(C)$, then by oriented thinness there is a lozenge

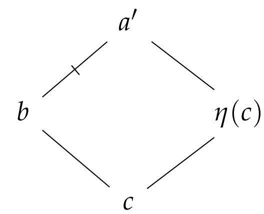


which characterise a as the lozenge completion of $b \prec^- \gamma(T) \prec^+ \top$. Whence the neatness.

Consider now a triplet $a \prec_b a'$ in $\mathcal{E}(C)$. Then there is a lozenge as above:

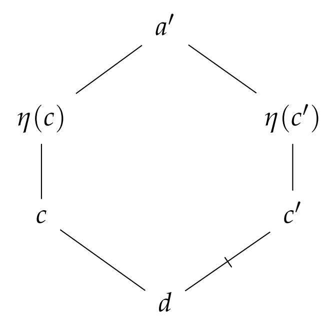


Let $c \in (s_{a'}\mathcal{E}(C))^{|}$, then there is a lozenge

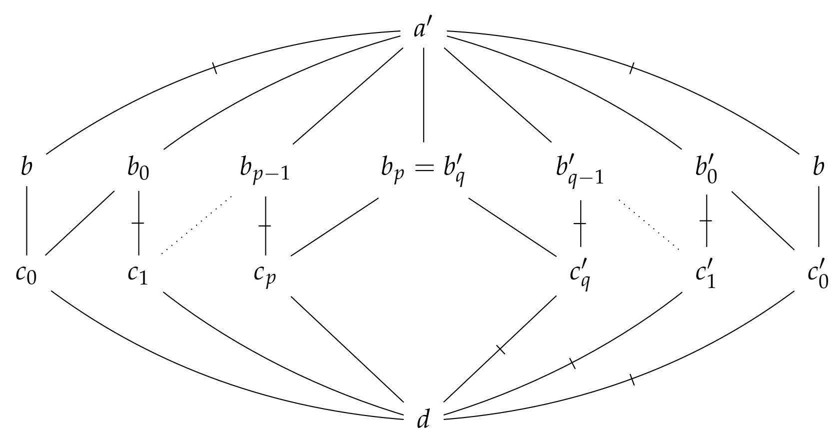


Hence $c \in (s_b s_a \mathcal{E}(C))^{\bullet}$. Conversely if $c \in (s_b s_a \mathcal{E}(C))^{\bullet}$, the completion of $c \prec^- b \prec^+ a'$ yields a lozenge of the above form, where c has to be a leaf of $s_{a'}\mathcal{E}(C)$, still by unicity of lozenge completion. Whence $(s_b s_a \mathcal{E}(C))^{\bullet} = (ts_{a'}\mathcal{E})^{\bullet}$. For $c \in (s_{a'}\mathcal{E}(C))^{|}$, we have $s_c t s_{a'}\mathcal{E}(C) = s_c s_{\eta(c)} s_{a'}\mathcal{E}(C) = \mathcal{E}(\mathsf{cl}(c)) = s_c s_b s_a \mathcal{E}(C)$. It ramains to prove that $s_b s_a \mathcal{E}(C)$ and $ts_{a'}\mathcal{E}(C)$ has same triplets.

Suppose there is a triplet $c \prec_d c'$ in $s_b s_a \mathcal{E}(C)$. Then $c,\ c' \in (s_{a'}\mathcal{E}(C))^{|}$ and there is a hexagon

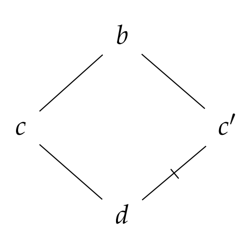


By the hexagon property, we thus have the following diagram in $\mathcal{E}(C)$:



Where $c_0 = c$, $\eta(c) = b_0$ and $c'_0 = c'$, $\eta(c') = b'_0$. That is, there is a triplet $c \prec_d c'$ in $ts_{a'}\mathcal{E}(C)$.

Conversely, given a triplet $c \prec_d c'$ in $ts_{a'}\mathcal{E}(C)$, we have a shape as above with $c_0 = c$ and $c'_0 = c'$, whence a lozenge



That is, a triplet $c \prec_d c'$ in $s_b s_a \mathcal{E}(C)$. □

Lemma 6.4 : $t\mathcal{E}(C) = \mathcal{E}(\mathsf{cl}(\gamma(\top)))$

Let C be a DFC with $\dim(C) \geq 1$, then $t\mathcal{E}(C) = \mathcal{E}(\mathsf{cl}(\gamma(\top)))$.

Proof. As seen in the proof of Lemma 6.3, because of oriented thinness the leaves of $\mathcal{E}(C)$ are the sources of $\gamma(\top)$. Whence $(t\mathcal{E}(C))^\bullet = (\mathcal{E}(\mathsf{cl}(\gamma(\top))))^\bullet$.
Let $b \in (t\mathcal{E}(C))^\bullet$, then $s_b t\mathcal{E}(C) = s_b s_{\eta(b)}\mathcal{E}(C) = \mathcal{E}(\mathsf{cl}(b)) = s_b\mathcal{E}(\mathsf{cl}(\gamma(\top)))$.

It remains to see that $t\mathcal{E}(C)$ and $\mathcal{E}(\mathsf{cl}(\gamma(\top)))$ has same triplets. This is shown using the hexagon property, exactly as in the proof of Lemma 6.3. □

Definition 6.5 : Flag of a DFC

Let C be a DFC. A *flag* of C is a sequence

$$s = \left(x_p \prec^{\alpha_p} x_{p-1} \prec^{\alpha_{p-1}} \cdots \prec^{\alpha_2} x_1 \prec^{\alpha_1} x_0 = \top\right)$$

for some $p \geq 0$. We say that this flag *ends* at x_p, denoting $\downarrow s = x_p$, and has *length* p.

Definition 6.6 : $\mathfrak{f}(s)$

Let s be a flag of C, there is flag in $\mathcal{E}(C)$ written $\mathfrak{f}(s)$ associated to s. It is defined inductively as:

- $\mathfrak{f}(\top) := []$.
- $\mathfrak{f}(x_{p+1} \prec^- s) = s_{x_{p+1}} \frown \mathfrak{f}(s)$.
- $\mathfrak{f}(x_{p+1} \prec^+ s) = t \frown \mathfrak{f}(s)$.

In order to show that this is well-defined, we will inductively check the following property.

Lemma 6.7

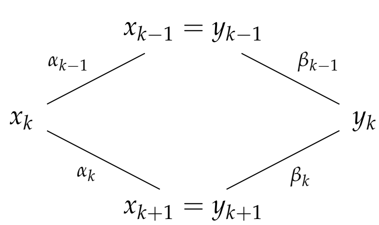
Let s be a flag of C a DFC, then $\mathfrak{f}(s)\mathcal{E}(C) = \mathcal{E}(\mathsf{cl}(\downarrow s))$.

Proof. We proceed by induction.

- $\mathfrak{f}(\top)\mathcal{E}(C) = \mathcal{E}(C) = \mathcal{E}(\mathsf{cl}(\top))$.
- $\mathfrak{f}(x_{p+1} \prec^- s)\mathcal{E}(C) = s_{x_{p+1}}\mathcal{E}(\mathsf{cl}(\downarrow s)) = \mathcal{E}(\mathsf{cl}(x_{p+1}))$.
- $\mathfrak{f}(x_{p+1} \prec^+ s)\mathcal{E}(C) = t\mathcal{E}(\mathsf{cl}(\downarrow s)) = \mathcal{E}(\mathsf{cl}(x_{p+1}))$ by Lemma 6.4. □

Lemma 6.8

Let $s = (x_p \prec^{\alpha_p} \cdots \prec^{\alpha_1} x_0 = \top)$ and $s' = (y_p \prec^{\beta_p} \cdots \prec^{\beta_1} y_0 = \top)$ be two flags of C a DFC. There is some $k \in [\![1, p-1]\!]$ with $x_i = y_i$ for $i \neq k$, $\alpha_i = \beta_i$ for $i \notin \{k-1, k\}$, and there is a lozenge



if and only if $\mathfrak{f}(s) \leftrightarrow \mathfrak{f}(s')$.

Proof. Suppose there is a lozenge as in the statement, we distinguish on the signs appearing in the lozenge. Let u be the flag $x_{k-1} \prec^{\alpha_{k-1}} \cdots \prec^{\alpha_1} x_0 = \top$, and $\mathfrak{p} := \mathfrak{f}(s)$.

- If $\alpha_{k-1} = \alpha_k = \beta_k = -$ and $\beta_{k-1} = +$, then by Lemma 6.7, $x_{k+1} \in (\mathfrak{p}\mathcal{E}(C))^|$ and $\eta(x_{k+1}) = x_k$. There is some $\mathfrak{q}$ such that $\mathfrak{f}(s) = \mathfrak{q}s_{x_{k+1}}s_{x_k}\mathfrak{p}$ and $\mathfrak{f}(s') = \mathfrak{q}s_{x_{k+1}}t\mathfrak{p}$, whence a rewriting $\mathfrak{f}(s) \leftrightarrow \mathfrak{f}(s')$.

- If $\alpha_{k-1} = \beta_{k-1} = \alpha_k = -$ and $\beta_k = +$, then by Lemma 6.7 there is a triplet $x_k \prec_{x_{k+1}} y_k$ in $\mathfrak{p}\mathcal{E}(C)$. There is some $\mathfrak{q}$ such that $\mathfrak{f}(s) = \mathfrak{q}s_{x_{k+1}}s_{x_k}\mathfrak{p}$ and $\mathfrak{f}(s') = \mathfrak{q}ts_{y_k}\mathfrak{p}$, whence a rewriting $\mathfrak{f}(s) \leftrightarrow \mathfrak{f}(s')$.
- If $\beta_{k-1} = \alpha_k = \beta_k = +$ and $\alpha_{k-1} = -$, then by Lemma 6.7, $x_k = \rho(\mathfrak{p}\mathcal{E}(C))$. There is some $\mathfrak{q}$ such that $\mathfrak{f}(s) = \mathfrak{q}ts_{x_k}\mathfrak{p}$ and $\mathfrak{f}(s') = \mathfrak{q}tt\mathfrak{p}$, whence a rewriting $\mathfrak{f}(s) \leftrightarrow \mathfrak{f}(s')$.
- The other cases are symmetric to one of the previous ones.

Conversely, suppose $\mathfrak{f}(s) \leftrightarrow \mathfrak{f}(s')$, then similarly distinguishing on the rewriting rule applied allow us to recover a lozenge in C. □

Lemma 6.9

Let s, s' be two flags in C a DFC, then $\downarrow s = \downarrow s'$ if and only if $\mathfrak{f}(s) \overset{*}{\leftrightarrow} \mathfrak{f}(s')$.

Proof. Suppose $\downarrow s = \downarrow s'$. Using Lemma 6.8, it suffices to show that we may find a sequence $s = s_0, s_1, \cdots, s_q = s'$, where each s_i differ from s_{i+1} by a lozenge. Pictorially, we seek a combinatorial homotopy between the flags s and s' in the hasse diagram of C. We achieve it by induction on the length of s, s'.

- When s, s' have length smaller than 2, they must be equal, whence the result.
- When s, s' have length 2, they are equal or differ by exactly one lozenge according to oriented thinness, whence the result.
- Suppose now s, s' of length greater than 2, and the result known for shorter flags. We write $s = \left(z = x_p \prec^{\alpha_p} \cdots \prec^{x_1} x_0 = \top\right)$ and $s' = \left(z = y_p \prec^{\beta_p} \cdots \prec^{\beta_1} y_0 = \top\right)$. Notice that, up to finding a lozenge completion for $x_2 \prec x_1 \prec x_0$ (resp. $y_2 \prec y_1 \prec y_0$), we may suppose $\alpha_1 = -$ (resp. $\beta_1 = -$). Then using Lemma 3.15 or 3.16 of [1] – according to the sign of α_2 (resp. β_2) – to the chain $x_3 \prec x_2 \prec x_1 \prec^- x_0$ (resp. $y_3 \prec y_2 \prec y_1 \prec^- y_0$), we find $u = \left(x_p \prec^{\alpha_p} \cdots \prec^{\alpha_4} x_3 \prec^{\alpha_3} x_2' \prec^- \gamma(\top) \prec^+ \top\right)$ (resp. $u' = \left(y_p \prec^{\beta_p} \cdots \prec^{\beta_4} y_3 \prec^{\beta_3} y_2' \prec^- \gamma(\top) \prec^+ \top\right)$) such that s and u (resp. s' and u') are related by a sequence of lozenges. We now have two flags $v = \left(x_p \prec^{\alpha_p} \cdots x_3 \prec^{\alpha_3} x_2' \prec^- \gamma(\top)\right)$ and $v' = \left(y_p \prec^{\beta_p} \cdots y_3 \prec^{\beta_3} y_2' \prec^- \gamma(\top)\right)$ of $\mathsf{cl}(\gamma(\top))$ ending at the same point. Hence by induction hypothesis, they are related by a sequence of lozenges. Whence s and s' being related by a sequence of lozenges.

Convrsely, suppose $\mathfrak{f}(s) \overset{*}{\leftrightarrow} \mathfrak{f}(s')$, then using Lemma 6.8 we obtain that s and s' differ by a sequence of lozenges. Since modifying a flag by a lozenge does not change its end, we have $\downarrow s = \downarrow s'$. □

Definition 6.10

Let $f : C \to D$ be an isomorphism of DFC (*i.e.* a morphism with $\mathsf{dim}(C) = \mathsf{dim}(D)$). There is a renaming of epiphytes $\mathcal{E}(f) : \mathcal{E}(C) \to \mathcal{E}(D)$ defined inductively as:

- $\mathsf{id} : \blacklozenge \to \blacklozenge$ if $\mathsf{dim}(C) = 0$.
- When $\mathsf{dim}(C) > 0$:
  - $f|_{\delta(\top_C)}^{\delta(\top_D)} : \mathcal{E}(C) \to \mathcal{E}(D)$ on the tree structure.
  - For $a \in \delta(\top_C)$, $\mathcal{E}(f)_a := \mathcal{E}\left(f|_{\mathsf{cl}(a)}^{\mathsf{cl}(f(a))}\right)$.

  where $\top_C$ (resp. $\top_D$) is the greatest element of C (resp. D).

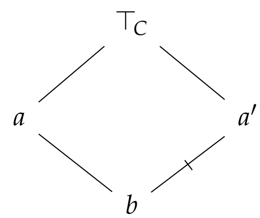
Lemma 6.11

Definition 6.10 indeed define a morphism of epiphytes.

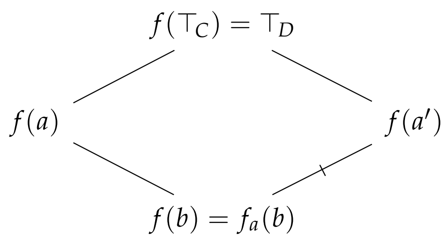
Proof. We proceed by induction on the dimension of C, D.
When $\mathsf{dim}(C) = \mathsf{dim}(D) = 0$, the result is clear.
Suppose now $\mathsf{dim}(C) = \mathsf{dim}(D) > 0$ and the result known in lower dimensions.

$\mathcal{E}(f)$ preserves triplets :

Suppose $a \prec_b a'$ in $\mathcal{E}(C)$, then there is a lozenge in C as follows:



Hence there is a lozenge as below in D.



Whence a triplet $f(a) \prec_{f_a(b)} f(a')$ in $\mathcal{E}(D)$.

relations on $\mathcal{E}(f)_{a,b}$ and $\mathcal{E}(f)_{a,b,c}$:

Suppose $a \prec_b a'$ in $\mathcal{E}(C)$, and let $c \in (s_{a'}\mathcal{E}(C))^{|}$ with $\eta(c) = b'$.
Then $\mathcal{E}(f)_{a,b}(c) = \mathcal{E}\left(f|_{\mathsf{cl}(b)}^{\mathsf{cl}(f(b))}\right)(c) = f(c) = \mathcal{E}\left(f|_{\mathsf{cl}(b')}^{\mathsf{cl}(f(b'))}\right)(c) = f_{a,b'}(c)$.
And $\mathcal{E}(f)_{a,b,c} = \mathcal{E}\left(f|_{\mathsf{cl}(c)}^{\mathsf{cl}(f(c))}\right) = \mathcal{E}(f)_{a',b',c}$. □

Remark 6.12
For $f : C \to D$ an isomorphism of DFC and $y \prec^- x_p \prec^- x_{p-1} \prec^- \cdots \prec^- x_0 = \top_C$, $\mathcal{E}(f)_{x_1,\cdots,x_p}(y) = f(y)$.

Definition 6.13

Let $f : C \to D$ be an isomorphism of DFC, and $s = (x_p \prec^{\alpha_p} \cdots \prec^{\alpha_1} x_0 = \top_C)$ a flag of C. Then its image under f is written $f_* s = (f(x_p) \prec^{\alpha_p} \cdots \prec^{\alpha_1} f(x_0))$, it is a flag of $\mathsf{cl}(f(\top_C))$.

Lemma 6.14

Let $f : C \to D$ be a isomorphism of DFC, s a flag of C and $x = \downarrow(s)$. Then $\mathfrak{f}(s)\mathcal{E}(f) = \mathcal{E}\left(f|_{\mathsf{cl}(x)}^{\mathsf{cl}(f(x))}\right)$.

Proof. By induction on s.

- If $s = (\top_C)$, it is clear.
- $\mathfrak{f}(x \prec^- s)\mathcal{E}(f) = (s_x \frown \mathfrak{f}(s))\mathcal{E}(f) = \mathcal{E}\left(f|_{\mathsf{cl}(\downarrow s)}^{\mathsf{cl}(f(\downarrow s))}\right)_x = \mathcal{E}\left(f|_{\mathsf{cl}(x)}^{\mathsf{cl}(f(x))}\right)$.
- $\mathfrak{f}(x \prec^+ s)\mathcal{E}(f) = (t \frown \mathfrak{f}(s))\mathcal{E}(f) = t\mathcal{E}\left(f|_{\mathsf{cl}(\downarrow s)}^{\mathsf{cl}(f(\downarrow s))}\right)$.
For $y \in t\mathcal{E}(\mathsf{cl}(\downarrow s))$, we have $t\mathcal{E}\left(f|_{\mathsf{cl}(\downarrow s)}^{\mathsf{cl}(f(\downarrow s))}\right)(y) = \mathcal{E}\left(f|_{\mathsf{cl}(\downarrow s)}^{\mathsf{cl}(f(\downarrow s))}\right)_{\eta(y)}(y) = f(y) = \mathcal{E}\left(f|_{\mathsf{cl}(x)}^{\mathsf{cl}(f(x))}\right)(y)$.
And this is well-defined according to Lemma 6.4. □

Remark 6.15
Hence, if $f : C \to D$ is an isomorphism of DFC, s is a flag of C and $x \in \delta(\downarrow s)$, $\mathfrak{f}(s)\mathcal{E}(f)(x) = f(x)$.

Lemma 6.16

Let $g : C \to D$ be an isomorphism of DFC and s a flag of C, $\mathfrak{f}(g_* s) = \mathcal{E}(g)_* \mathfrak{f}(s)$.

Proof. We proceed by induction on s.

- When $s = (\top_C)$, $g_* s = (g(\top_C))$. Hence $\mathfrak{f}(g_* s) = [] = \mathcal{E}(g)_* \mathfrak{f}(s)$.
- $\mathfrak{f}(g_*(x \prec^- s)) = s_{g(x)} \frown \mathcal{E}(g)_* \mathfrak{f}(s) \overset{\text{Remark 6.15}}{=} s_{\mathfrak{f}(s)\mathcal{E}(g)(x)} \frown \mathcal{E}(g)_* \mathfrak{f}(s) = \mathcal{E}(g)_* \mathfrak{f}(x \prec^- s)$.
- $\mathfrak{f}(g_*(x \prec^+ s)) = t \frown \mathcal{E}(g)_* \mathfrak{f}(s) = \mathcal{E}(g)_*(t \frown \mathfrak{f}(s)) = \mathcal{E}(g)_* \mathfrak{f}(x \prec^+ s)$. □

Definition 6.17

Let $f : C \to D$ be a morphism of DFC, then there is an associated morphism of epiphytes $\mathcal{E}(f) : \mathcal{E}(C) \to \mathcal{E}(D)$ (in the category **Epi**) defined as $\mathcal{E}(f) = \left(\mathcal{E}\left(f|^{\mathsf{Im}(f)}\right), [\mathfrak{f}(s)]\right)$, where s is any flag ending at $f(\top)$ in D. Since $\mathfrak{f}(s)$ does not depend on s (by Lemma 6.9), it is well defined.

Theorem 6.18

$\mathcal{E}$ as defined above on DFC and their morphisms yields a functor $\mathcal{E} : \mathbf{DFC} \to \mathbf{Epi}$.

Proof. Clearly, $\mathcal{E}$ preserves identities. We prove that it preserves composition.
Let $f : C \to D$ and $g : D \to E$ be two morphisms of DFC and let $f' := f|^{\mathsf{cl}(f(\top_C))}$, $g' := g|^{\mathsf{cl}(g(\top_D))}$.
Then $\mathcal{E}(g \circ f) = \left(\mathcal{E}\left((g \circ f)|^{\mathsf{cl}((g\circ f)(\top_C))}\right), [\mathfrak{f}(s)]\right)$ for any flag s with $\downarrow s = (g \circ f)(\top_C)$.
Chosing a flag s_f of D ending at $f(\top_C)$ and a flag s_g of E ending at $g(\top_D)$, we may define s as the concatenation of s_g and $g_*(s_f)$. Hence $\mathfrak{f}(s) = \mathfrak{f}(g_*(s_f)) \frown \mathfrak{f}(s_g)$ where $\mathfrak{f}(g_*(s_f))$ is computed in $\mathcal{E}(\mathsf{cl}(g(T_D)))$.
By Lemma 6.16, $\mathfrak{f}(s) = \mathcal{E}(g')_* \mathfrak{f}(s_f) \frown \mathfrak{f}(s_g)$.
On the other hand, by Remark 6.15, $\mathcal{E}\left((g \circ f)|^{\mathsf{cl}((g\circ f)(\top_C))}\right) = \mathfrak{f}(s_g)\mathcal{E}(g') \circ \mathcal{E}(f')$.
Whence, by Definition 3.14, $\mathcal{E}(g \circ f) = \left(\mathfrak{f}(s_g)\mathcal{E}(g') \circ \mathcal{E}(f'), [\mathcal{E}(g')_* \mathfrak{f}(s_f) \frown \mathfrak{f}(s_g)]\right) = \mathcal{E}(g) \circ \mathcal{E}(f)$. □

7 An equivalence of categories

Definition 7.1

Let ω be an epiphyte and $\mathfrak{p}$ a flag of ω. According to Proposition 5.9, there is an isomorphism $\varphi^{\omega}_{[\mathfrak{p}]} : \mathcal{F}(\mathfrak{p}\omega) \to \mathsf{cl}([\mathfrak{p}])$. We then define $\psi^{\omega}_{[\mathfrak{p}]}$ as $\mathcal{E}(\varphi^{\omega}_{[\mathfrak{p}]}) : (\mathcal{E} \circ \mathcal{F})(\mathfrak{p}\omega) \to \mathcal{E}(\mathsf{cl}(\mathfrak{p}))$.
By Remark 6.12, we have $\psi^{\omega}_{[\mathfrak{p}],\alpha_1,\cdots,\alpha_p}([\mathfrak{q}]) = [\mathfrak{q}\mathfrak{p}]$ whenever this expression makes sense.

Definition 7.2

Let ω be an epiphyte and $\mathfrak{p} = \xi_p \cdots \xi_1$ a flag of ω, we define $\bar{\mathfrak{p}}$ a flag of $\mathcal{F}(\omega)$ as

$$\bar{\mathfrak{p}} = \left([\xi_p \cdots \xi_1] \prec [\xi_{p-1} \cdots \xi_1] \prec \cdots \prec [[]]\right).$$

Notice that $\downarrow \bar{\mathfrak{p}} = [\mathfrak{p}]$.

Definition 7.3

Let ω be an epiphyte, there is a renaming $\theta^{\omega} : \omega \to (\mathcal{E} \circ \mathcal{F})(\omega)$, defined inductively as follows.

- When $\omega = \blacklozenge$, $\theta^{\blacklozenge} = \mathsf{id}$.
- When $\mathsf{dim}(\omega) > 0$, $\theta^{\omega}_{x_1,\cdots,x_p}(x) := [s_x s_{x_p} \cdots s_{x_1}]$.

Proof. We check that this is well-defined.
By Theorem 4.13, $\forall p$, $\forall x_1, \cdots, x_p$, $\theta^{\omega}_{x_1,\cdots,x_p} : (s_{x_p} \cdots s_{x_1}\omega)^{\bullet} \to \delta([s_{x_p} \cdots s_{x_1}])$ is injective. The surjectivity is by definition. We check that it preserves triplets. Let $x \prec_y x'$ be a triplet in $s_{x_p} \cdots s_{x_1}\omega$.
Then $[s_y s_x s_{x_p} \cdots s_{x_1}] = [t s_{x'} s_{x_p} \cdots s_{x_1}]$. We then have a lozenge

$$[s_{x_p} \cdots s_{x_1}]$$

$$[s_x s_{x_p} \cdots s_{x_1}] \qquad\qquad [s_{x'} s_{x_p} \cdots s_{x_1}]$$

$$[s_y s_x s_{x_p} \cdots s_{x_1}]$$

in $\mathcal{F}(\omega)$. Whence a triplet $\theta^{\omega}_{x_1,\cdots,x_p}(x) \prec_{\theta^{\omega}_{x_1,\cdots,x_p,x}(y)} \theta^{\omega}_{x_1,\cdots,x_p}(x')$ in $\mathcal{F}(\omega)$.

Still suppose that there is such a triplet $x \prec_y x'$, with $z \in (s_{x'} s_{x_p} \cdots s_{x_1}\omega)^{|}$ and $\eta(z) = y'$. Then we have $\theta^{\omega}_{x_1,\cdots,x_p,x,y}(z) = [s_z s_y s_x s_{x_p} \cdots s_{x_1}] = [s_z t s_x x' s_{x_p} \cdots s_{x_1}] = [s_z s_{y'} s_{x'} s_{x_p} \cdots s_{x_1}] = \theta^{\omega}_{x_1,\cdots,x_p,x',y'}(z)$.
And similarly for the second identity. $\square$

Lemma 7.4

Let ω be an epiphyte and $\mathfrak{p}$ a flag of ω. Then $\theta^{\omega}_{*}\mathfrak{p} = \mathfrak{f}(\bar{\mathfrak{p}})$ and $\mathfrak{p}\theta^{\omega} = \psi^{\omega}_{[\mathfrak{p}]} \circ \theta^{\mathfrak{p}\omega}$.

Proof. By induction on $\mathfrak{p}$.

- $\theta^{\omega}_{*}[] = \mathfrak{f}([[]])$ and $\theta^{\omega} = \mathrm{id} \circ \theta^{\omega} = \psi^{\omega}_{[[]]} \circ \theta^{\omega}$.
- $\theta^{\omega}_{*}(s_x\mathfrak{p}) = s_{(\mathfrak{p}\theta^{\omega})(x)} \frown \mathfrak{f}(\bar{\mathfrak{p}}) = s_{[s_x\mathfrak{p}]} \frown \mathfrak{f}(\bar{\mathfrak{p}}) = \mathfrak{f}(\overline{s_x\mathfrak{p}})$,
  and $s_x\mathfrak{p}\theta^{\omega} = s_x(\psi^{\omega}_{[\mathfrak{p}]} \circ \theta^{\mathfrak{p}\omega}) = \psi^{\omega}_{[\mathfrak{p}],[s_x\mathfrak{p}]} \circ \theta^{\mathfrak{p}\omega}_{x} = \psi^{\omega}_{[s_x\mathfrak{p}]} \circ \theta^{s_x\mathfrak{p}\omega}$.
- $\theta^{\omega}_{*}(t\mathfrak{p}) = t \frown \mathfrak{f}(\bar{\mathfrak{p}}) = \mathfrak{f}(\overline{t\mathfrak{p}})$,
  and for any y, $t\mathfrak{p}\theta^{\omega}(y) = (t\psi^{\omega}_{[\mathfrak{p}]} \circ t\theta^{\mathfrak{p}\omega})(y) = t\psi^{\omega}_{[\mathfrak{p}]}([s_y s_{\eta(y)}]) = [s_y s_{\eta(y)}\mathfrak{p}] = [s_y t\mathfrak{p}] = (\psi^{\omega}_{t\mathfrak{p}} \circ \theta^{t\mathfrak{p}\omega})(y)$. $\square$

Lemma 7.5

Let $f : \omega \to \varpi$ be a renaming of epiphyte, then we have $(\mathcal{E} \circ \mathcal{F})(f) \circ \theta^{\omega} = \theta^{\varpi} \circ f$.

Proof. By induction on $\mathrm{dim}(\omega) = \mathrm{dim}(\varpi)$.

- When $\mathrm{dim}(\omega) = 0$, $f = \mathrm{id}$ and $\omega = \varpi$, whence the result.
- Suppose $\mathrm{dim}(\omega) > 0$ and the result known in lower dimensions.
  For $a \in \omega^{\bullet}$,

$$\begin{aligned} (\mathcal{E}(\mathcal{F}(f)) \circ \theta^{\omega})(a) &= \mathcal{E}(\mathcal{F}(f))([s_a]) \\ &= \mathcal{F}(f)([s_a]) \\ &= [s_{f(a)}] \\ &= \theta^{\varpi}(f(a)) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{E}(\mathcal{F}(f)) \circ \theta^{\omega})_a &= \mathcal{E}(\mathcal{F}(f))_{[s_a]} \circ \theta^{\omega}_{a} \\ &= \mathcal{E}\left(\mathcal{F}(f)|^{\mathrm{cl}([s_{f(a)}])}_{\mathrm{cl}([s_a])} \circ \varphi^{\omega}_{[s_a]}\right) \circ \theta^{s_a\omega} \\ &= \mathcal{E}(\varphi^{\varpi}_{[s_{f(a)}]} \circ \mathcal{F}(f_a)) \circ \theta^{s_a\omega} && \text{by } (*) \\ &= \psi^{\varpi}_{[s_{f(a)}]} \circ \theta^{s_{f(a)}\varpi} \circ f_a && \text{by induction} \\ &= \theta^{\varpi}_{s_{f(a)}} \circ f_a && \text{by Lemma 7.4.} \end{aligned}$$

where $(*)$ is given by the equality $\mathcal{F}(f)| \circ \varphi^{\omega}_{[s_a]} = \varphi^{\varpi}_{[s_{f(a)}]} \circ \mathcal{F}(f_a)$, as it may be directly checked. $\square$

Proposition 7.6

The isomorphism of epiphytes θ^ω is natural in ω.

Proof. Let $(f, [\mathfrak{p}]) : \omega \to \varpi$ be a morphism of epiphytes.

$$\begin{array}{rcll} \theta^\varpi \circ (f, [\mathfrak{p}]) & = & (\mathfrak{p}\theta^\varpi \circ f, [\theta^\varpi_* \mathfrak{p}]) & \text{by Definition 3.14} \\ & = & (\psi^\varpi_{[\mathfrak{p}]} \circ \theta^{\mathfrak{p}\varpi} \circ f, [\mathfrak{f}(\bar{\mathfrak{p}})]) & \text{by Lemma 7.4} \end{array}$$

On the other hand, $\mathcal{E}(\mathcal{F}(f, [\mathfrak{p}])) = \Big(\mathcal{E}(\varphi^\varpi_{[\mathfrak{p}]} \circ \mathcal{F}(f)), [\mathfrak{f}(\bar{\mathfrak{p}})]\Big)$ (because $\downarrow\bar{\mathfrak{p}} = [p] = \mathcal{F}(f)(\top)$). And $\mathcal{E}(\varphi^\varpi_{[\mathfrak{p}]} \circ \mathcal{F}(f)) = \psi^\varpi_{[\mathfrak{p}]} \circ \mathcal{E}(\mathcal{F}(f))$. Then $(\mathcal{E} \circ \mathcal{F})(f, [\mathfrak{p}]) = \Big(\psi^\varpi_{[\mathfrak{p}]} \circ \mathcal{E}(\mathcal{F}(f)) \circ \theta^\omega, [\mathfrak{f}(\bar{\mathfrak{p}})]\Big)$. The equation $\mathcal{E}(\mathcal{F}(f)) \circ \theta^\omega = \theta^{\mathfrak{p}\varpi} \circ f$ hold by Lemma 7.5, whence the commutative diagram:

$$\begin{array}{ccc} \omega & \xrightarrow{\quad f \quad} & \varpi \\ \Big\downarrow{\scriptstyle \theta^\omega} & & \Big\downarrow{\scriptstyle \theta^\varpi} \\ (\mathcal{E} \circ \mathcal{F})(\omega) & \xrightarrow[(\mathcal{E}\circ\mathcal{F})(f)]{} & (\mathcal{E} \circ \mathcal{F})(\varpi) \end{array}$$

□

Definition 7.7

Let C be a DFC, we let $\tau^C : C \to (\mathcal{F} \circ \mathcal{E})(C) : x \mapsto [\mathfrak{f}(s)]$ where $\downarrow s = x$. This is well-defined according to Lemma 6.9. It is an isomorphism of *DFC*.

Proof. Notice that C as same dimension as $(\mathcal{F} \circ \mathcal{E})(C)$. We only need to wheck that τ^C is a morphism.

- For any $x \in C$, its codimension is the length of s for any flag with $\downarrow s = x$, and it is also the length of $\mathfrak{f}(s)$. Hence τ^C preserves the codimension, thus the dimension.
- Let $y \prec^- x$ in C, and s a flag of C with $\downarrow s = x$. Let $s' = (y \prec^- s)$.
  We have $[\mathfrak{f}(s')] = [s_y \frown \mathfrak{f}(s)] \prec^- [\mathfrak{f}(s)]$. Whence $\tau^C(y) \prec^- \tau^C(x)$. Similarly, τ^C preserves $\prec^+$.
- Using Lemma 6.9 we obtain the injectivity of τ^C.
  Let $x \in C$ and s a flag in C with $\downarrow s = x$, it remain to check that $\tau^C|^{\delta([\mathfrak{f}(s)])}_{\delta(x)}$ is surjective. We have $\delta([\mathfrak{f}(s)]) = \{[s_y \frown \mathfrak{f}(s)]\}_{y \in (\mathfrak{f}(s)\mathcal{E}(C))^\bullet} = \{[\mathfrak{f}(y \prec^- s)]\}_{y \prec^- x}$, whence the surjectivity. □

Proposition 7.8

The isomorphism τ^C is natural in C.

Proof. Let $g : C \to D$ be a morphism of DFC, $x \in C$ and s a flag of C with $\downarrow s = x$, we have:

$$\begin{array}{rcll} ((\mathcal{F} \circ \mathcal{E})(g) \circ \tau^C)(x) & = & \mathcal{F}(\mathcal{E}(g))([\mathfrak{f}(s)]) & \text{by Definition 7.7} \\ & = & \mathcal{F}(\mathcal{E}(g'), \mathfrak{f}(u))([\mathfrak{f}(s)]) & \text{where } g' = g|^{\mathsf{Im}(g)}, u \text{ a flag in } D \text{ with } \downarrow u = g(\top_C) \\ & = & (\varphi_{[\mathfrak{f}(u)]} \circ \mathcal{F}(\mathcal{E}(g')))([\mathfrak{f}(s)]) & \\ & = & \varphi_{[\mathfrak{f}(u)]}(\mathcal{E}(g')_*[\mathfrak{f}(s)]) & \\ & = & \varphi_{[\mathfrak{f}(u)]}([\mathfrak{f}(g'_* s)]) & \text{by Lemma 6.16.} \\ & = & [\mathfrak{f}(g'_* s) \frown \mathfrak{f}(u)] & \\ & = & [\mathfrak{f}(v)] & \text{for } v \text{ the concatenation of } g'_* s \text{ and } u \\ & = & \tau^D(g(x)) & \text{because } \downarrow v = g(x). \end{array}$$

□

Theorem 7.9 : Epi $\simeq$ DFC

There is an equivalence of categories

$$\mathbf{Epi} \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathbf{DFC}$$

Proof. This is by Proposition 7.6 and 7.8. □

References

[1] Louise Leclerc. "A poset-like approach to positive opetopes". working paper or preprint. Oct. 2023. URL: https://hal.science/hal-04231623.