## Definition of rooted tree (reformulation of Kock's definition)

A tree is a quintuple $\underline{T} = (T^0, T^1, \underset{\emptyset}{|}_T, i_T, o_T)$ where

- $T^0$ is a finite set of nodes
- $(T^1, \underset{\emptyset}{|})$ is a pointed set of edges; $\underset{\emptyset}{|}$ is called root edge
- $o : T^0 \to T^1$ is an injective function. We write $\prec_e x$ for $e = o(x)$
- $i : T^1 \setminus \{\underset{\emptyset}{|}\} \to T^0$ is a function. We write $x \prec_e$ for $x = i(e)$

and hence $x \prec_e y$ if $x = i(e)$ and $e = o(y)$

satisfying the following axiom:

(tree) $\forall x \in T^0$ $\exists$ a sequence $\prec_{\underset{\emptyset}{|}} x_1 \prec_{e_1} \cdots x_{n-1} \prec_{e_{n-1}} x_n = x$ $(n \geq 1)$

(Note that such a sequence is necessarily unique since we have $e_i = o(x_i)$ and $i(e_i) = x_{i-1}$.)

We write $T^{\overset{\emptyset}{1}} = \{e \in T^1 \mid \nexists\, x \text{ s.t. } \prec_e x\}$ (leaves of $T$)

We write $T^1_0 = T^1 - \{\underset{\emptyset}{|}\}$ and $T^1_x$ for $i^{-1}(x) = \{e \mid x \prec_e\}$

We can organize the data in a polynomial functor:

$$\begin{tikzcd} T^1_0 \arrow[r, "i"] \arrow[d, "\subseteq"'] & T^0 \arrow[d, "o"] \\ T^1 & T^1 \end{tikzcd}$$

# Inductive definition of trees

- We note that if $T^0$ is empty, then $T^1 = \{\underset{\emptyset}{|}\}$ as otherwise $i$ would not be defined.

This defines the empty tree $|$ with

$$(|)^0 = \emptyset \quad (\underset{\emptyset}{|})^1 = \{\underset{\emptyset}{|}\},\ o = \text{the empty map, and } i = id.$$

- If $T^0$ is not empty, then it has a root node $\bot_T$ characterized by $\prec_{\emptyset} \bot$. ($\bot_T$ exists by (tree) and is unique by injectivity of $o$). We can then write

$$\underline{T} = x\langle e \leftarrow \underline{T_e} \mid e \in T^1_x \rangle \quad \text{where } x = \bot_T$$

and $\underline{T_e}$ is obtained by restriction of $\underline{T}$:

i.e., the path to the root goes through $e$

$$T_e^0 = \{y \in T^0 \mid \prec_{\emptyset} x \prec_e \cdots \prec y_n = y\} \subseteq T^0$$

$$T_e^1 = i^{-1}(T_e^0) \subseteq T^1$$

$$\emptyset_{T_e} = e \qquad \text{We also have } T_e^{\overset{\emptyset}{1}} \subseteq T^{\overset{\emptyset}{1}}$$

$i_{T_e}, o_{T_e}$ defined by restriction of $i$ and $o$.

- In summary $\boxed{\underline{T} ::= | \mid x\langle e \leftarrow \underline{T_e} \mid e \in T^1_x \rangle}$

Abbreviation: $\boxed{x\langle | \dots | \rangle = x\langle (e \leftarrow |) \mid e \in T^1_x \rangle}$

# Grafting, inductively

For $\underline{T}$ and a collection $(\underline{S_e})_{e\in T^{\overset{\emptyset}{|}}}$, we define

$$\underline{T}\langle e \leftarrow \underline{S_e} \mid e \in T^{\overset{\emptyset}{|}} \rangle \quad \text{(grafting)} \quad \text{as follows:}$$

$\langle\ \rangle$ structural

$\langle\ \rangle$ defined

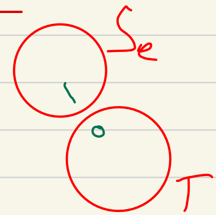
- $| \langle \underset{\emptyset}{|} \leftarrow \underline{T} \rangle = \underline{T}$

- $(x\langle e' \leftarrow \underline{T_{e'}} \mid e' \in T^{\underset{x}{|}} \rangle)\langle e \leftarrow \underline{S_e} \mid e \in T^{\overset{\emptyset}{|}} \rangle =$

$$x\langle e' \leftarrow \underline{T_{e'}}\langle e \leftarrow \underline{S_e} \mid e \in (T_{e'})^{\overset{\emptyset}{|}} \rangle \mid e' \in T^{\underset{x}{|}} \rangle$$

There is no harm in identifying and , as we have

$$x\langle e' \leftarrow \underline{T_{e'}} \mid e' \in T^{\underset{x}{|}} \rangle = x\langle e' \leftarrow \underset{\emptyset}{|} \mid e' \in T^{\underset{x}{|}} \rangle \langle e' \leftarrow \underline{T_{e'}} \mid e' \in T^{\underset{x}{|}} \rangle$$




# Grafting, non inductively (for $\underline{T}$ proper)

$$\underset{\sim}{T} \langle e \vdash \underline{S_e} \mid e \in T^{\substack{\emptyset \\ 1}} \rangle = \underline{U}$$

$$U^0 = T^0 \sqcup \Big( \bigsqcup_{e \in T^{\substack{\emptyset \\ 1}}} S_e^0 \Big) \qquad U^1 = \big(T^1 - T^{\substack{\emptyset \\ 1}}\big) \sqcup \Big( \bigsqcup_{e \in T^{\substack{\emptyset \\ 1}}} S_e^1 \Big)$$

$$\binom{1}{\emptyset}_U = \binom{1}{\emptyset}_T$$

$$o_U : U^0 \to U^1 \qquad o_U(x) = o_T(x)$$

$$o_U(e, y) = o_{S_e}(y)$$

$$i : U^{\substack{1 \\ 0}} \to U^0 \qquad i_U(e) = i_T(e)$$

$$i_U(e, u) = \begin{cases} i_{S_e}(u) & \text{if } u \in S_e^{\substack{1 \\ 0}} \\ i_T(e) & \text{if } u = \binom{1}{\emptyset}_{S_e} \end{cases}$$

glueing occurs here!

# Substitution

For $\underline{T}$ and a collection $(\underline{S_x})_{x \in T^0}$ such that for all $x$ we have $\boxed{(\underline{S_x})^{\overset{\emptyset}{!}} = T^!_x}$, we define

$\underline{T}\{y \leftarrow \underline{S_y} \mid y \in T^0\}$ (substitution) as follows:

- $1\{\} = 1$

- $(x\langle e \leftarrow \underline{T_e} \mid e \in T^!_x \rangle)\{y \leftarrow \underline{S_y} \mid y \in T^0\}$

$$= \underline{S_x}\langle e \leftarrow T_e\{y \leftarrow \underline{S_y} \mid y \in (T_e)^0\} \mid e \in T^!_x \rangle$$

$\shortparallel \; S_x^{\emptyset}\,!$

# Substitution, non inductively (for $I$ proper)

$$\underline{I}\{x \leftarrow \underline{S_x} \mid x \in T^0\} = U$$

$$U^0 = \bigsqcup_{x \in T^0} S_x^0 \qquad U^1_0 = \bigsqcup_{x \in T^0} S^1_{x\,0} \qquad \binom{|}{\emptyset}_U = \binom{|}{\emptyset}_{S_{\perp_T}}$$

$$o_U : U^0 \to U^1$$

$$o_U(x,y) = \begin{cases} (x, o_{S_x}(y)) & \text{if } y \neq \perp_{S_x} \\ (x', u) & \text{if } y = \perp_{S_x} \text{ and } x' \prec^T_u x \\ \binom{|}{\emptyset}_U & \text{if } y = \perp_{S_x} \text{ and } x = \perp_T \end{cases}$$

$$i_U(x,u) = (x, i_{S_x}(u))$$

# Notion of $(X,Y)$-tree

- An $(X,Y)$-tree is a triple $(\underline{T}, f : T^0 \to X, g : T^1 \to Y)$

- Grafting: Let $(\underline{T}, f, g)$ and $(\underline{S_e}, f_e, g_e)$, $e \in T^{\phi}$ be given with respective $(X,Y)$-structures.

  We want a $(X,Y)$-structure on $\underline{U} = \underline{T}\langle e \leftarrow S_e \mid e \in T^{\phi} \rangle$, i.e. $(\underline{U}, h, k)$

  We require the following condition $\boxed{\forall e \in T^{\phi} \quad g_e((\tfrac{!}{\phi})_{S_e}) = g(e)}$

  Then we set

  $h(x) = f(x) \quad h(e, y) = f_e(y)$

  $k(e) = g(e) \quad (e \in T^1 - T^{\phi}) \qquad k(e, u) = g_e(u)$

  The condition says $k(e, \tfrac{!}{\phi} S_e)$ "simplifies" as $g(e)$, respecting the geometry of glueing

- Substitution of $(X,Y)$-trees on a tree

  Let $\underline{T}$, $(\underline{S_x}, f_x, g_x)$ be given. We require the following condition

  $$\boxed{\text{If } x \prec_u y \text{ on } \underline{T}, \text{ then } \underbrace{g_x(u)}_{\text{leaf of } S_x} = g_y((\tfrac{!}{\phi})_{S_y})}$$

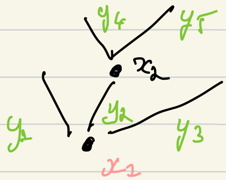
  We want an $(X,Y)$-structure on $\underline{U} = \underline{T}\{x \leftarrow S_x \mid x \in T^0\}$, i.e. $(\underline{U}, h, k)$:

  $h(x, y) = f_x(y) \quad k(\tfrac{!}{\phi}_U) = g_{\perp_T}(\tfrac{!}{\phi}_{S_{\perp_T}}) \qquad k(x, u) = g_x(u)$

  The condition says that for $x \prec_u y$ and hence $u \in S_x^{\phi}$ we have also $k(x, u) = g_y((\tfrac{!}{\phi})_{S_y})$, respecting geometry again

# Drawing $(X,Y)$-trees

$y \in Y$




$y_i \in Y \quad x_i \in X$

$x_1 = x_2$ allowed

$y_i = y_j$ allowed

Notation $\left|(\underline{T}, \ell, g)\right| = \underset{\sim}{T}$