FACETS of COHERENCE

on
- CW-complexes
- polyhedral complexes
- nestohedra

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joint work with

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PROLOGUE

Coherence in monoidal categories

$$
\begin{tikzcd}[column sep=small]
 & (X\otimes Y)\otimes Z \otimes (U\otimes V) \arrow[dl, "\alpha"'] \arrow[dr, "\alpha"] & \\
(X\otimes(Y\otimes Z))\otimes(U\otimes V) \arrow[d, "\alpha"] & & (X\otimes Y)\otimes(Z\otimes(U\otimes V)) \arrow[d, "\alpha^{-1}"] \\
X\otimes((Y\otimes Z)\otimes(U\otimes V)) \arrow[d, "\alpha"] & & (X\otimes Y)\otimes((Z\otimes U)\otimes V) \arrow[ddl, "\alpha"] \\
X\otimes(Y\otimes(Z\otimes(U\otimes V))) \arrow[dr, "\alpha^{-1}"] & & \\
 & X\otimes(Y\otimes((Z\otimes U)\otimes V)) &
\end{tikzcd}
$$

VARIOUS ITERATES of $\otimes$

- Read all vertices as functors $C^5 \to C$
- Read all α as instances in context

of $\quad \alpha : (-\otimes =)\otimes \equiv \to - \otimes (= \otimes \equiv)$

Pentagon

$$
\begin{tikzcd}[column sep=small]
 & ((X\otimes Y)\otimes Z)\otimes U \arrow[dl] \arrow[dr] & \\
(X\otimes(Y\otimes Z))\otimes U \arrow[d] & & (X\otimes Y)\otimes(Z\otimes U) \arrow[ddl] \\
X\otimes((Y\otimes Z)\otimes U) \arrow[dr] & & \\
 & X\otimes(Y\otimes(Z\otimes U)) &
\end{tikzcd}
$$

Hexagon

$$
\begin{tikzcd}[column sep=small]
 & (X\otimes Y)\otimes Z \arrow[dl] \arrow[dr] & \\
X\otimes(Y\otimes Z) \arrow[d] & & (Y\otimes X)\otimes Z \arrow[d] \\
(Y\otimes Z)\otimes X \arrow[dr] & & Y\otimes(X\otimes Z) \arrow[dl] \\
 & Y\otimes(Z\otimes X) &
\end{tikzcd}
$$

Non-symmetric Categorified operads (weak)

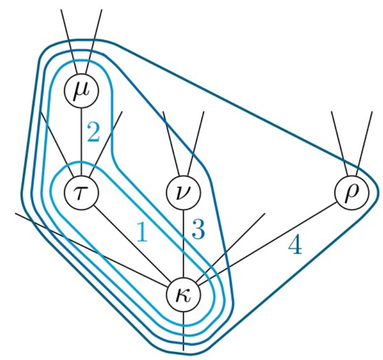
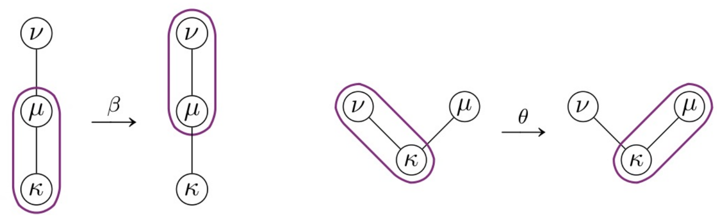


FIGURE 2. A fully nested planar tree.

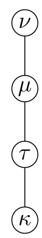


$\rightsquigarrow$ 3 pentagons and 2 hexagons

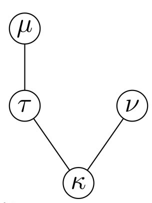
Došen and Petrić

Coherence for categorified operads

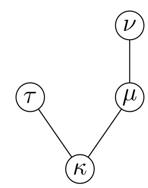
$$
\begin{tikzcd}[column sep=small]
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\beta_{\kappa,\tau,\mu} \circ 1_\nu"'] \arrow[ddr, "\beta_{\kappa\circ\tau,\mu,\nu}"] & \\
(\kappa \circ (\tau \circ \mu) \circ \nu) \arrow[dd, "\beta_{\kappa,\tau\circ\mu,\nu}"'] & & \\
& & (\kappa \circ \tau) \circ (\mu \circ \nu) \arrow[ddl, "\beta_{\kappa,\tau,\mu\circ\nu}"] \\
\kappa \circ ((\tau \circ \mu) \circ \nu) \arrow[dr, "1_\kappa \circ \beta_{\tau,\mu,\nu}"'] & & \\
& \kappa \circ (\tau \circ (\mu \circ \nu)) &
\end{tikzcd}
$$



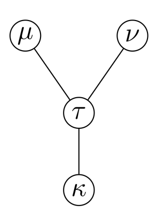
$$
\begin{tikzcd}[column sep=small]
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\theta_{\kappa\circ\tau,\mu,\nu}"'] \arrow[ddr, "\beta_{\kappa,\tau,\mu} \circ 1_\nu"] & \\
((\kappa \circ \tau) \circ \nu) \circ \arrow[dd, "\theta_{\kappa,\tau,\nu} \circ 1_\mu"'] & & \\
& & (\kappa \circ (\tau \circ \mu) \circ \nu \arrow[ddl, "\theta_{\kappa,\tau\circ\mu,\nu}"] \\
((\kappa \circ \nu) \circ \tau) \circ \mu \arrow[dr, "\beta_{\kappa\circ\nu,\tau,\mu}"'] & & \\
& (\kappa \circ \nu) \circ (\tau \circ \mu) &
\end{tikzcd}
$$



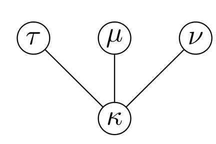
$$
\begin{tikzcd}[column sep=small]
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\theta_{\kappa,\tau,\mu} \circ 1_\nu"'] \arrow[ddr, "\beta_{\kappa\circ\tau,\mu,\nu}"] & \\
((\kappa \circ \mu) \circ \tau) \circ \nu \arrow[dd, "\theta_{\kappa\circ\mu,\tau,\nu}"'] & & \\
& & (\kappa \circ \tau) \circ (\mu \circ \nu) \arrow[ddl, "\theta_{\kappa,\tau,\mu\circ\nu}"] \\
((\kappa \circ \mu) \circ \nu) \circ \tau \arrow[dr, "\beta_{\kappa,\mu,\nu} \circ 1_\tau"'] & & \\
& (\kappa \circ (\mu \circ \nu)) \circ \tau &
\end{tikzcd}
$$



$$
\begin{tikzcd}[column sep=small]
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\beta_{\kappa,\tau,\mu} \circ 1_\nu"'] \arrow[dr, "\theta_{\kappa\circ\tau,\mu,\nu}"] & \\
(\kappa \circ (\tau \circ \mu)) \circ \nu \arrow[d, "\beta_{\kappa,\tau\circ\mu,\nu}"'] & & ((\kappa \circ \tau) \circ \nu) \circ \mu \arrow[d, "\beta_{\kappa,\tau,\nu} \circ 1_\mu"] \\
\kappa \circ ((\tau \circ \mu) \circ \nu) \arrow[dr, "1_\kappa \circ \theta_{\tau,\mu,\nu}"'] & & (\kappa \circ (\tau \circ \nu)) \circ \mu \arrow[dl, "\beta_{\kappa,\tau\circ\nu,\mu}"] \\
& \kappa \circ ((\tau \circ \nu) \circ \mu) &
\end{tikzcd}
$$



$$
\begin{tikzcd}[column sep=small]
& ((\kappa \circ \tau) \circ \mu) \circ \nu \arrow[dl, "\theta_{\kappa,\tau,\mu} \circ 1_\nu"'] \arrow[dr, "\theta_{\kappa\circ\tau,\mu,\nu}"] & \\
((\kappa \circ \mu) \circ \tau) \circ \nu \arrow[d, "\theta_{\kappa\circ\mu,\tau,\nu}"'] & & ((\kappa \circ \tau) \circ \nu) \circ \mu \arrow[d, "\theta_{\kappa,\tau,\nu} \circ 1_\mu"] \\
((\kappa \circ \mu) \circ \nu) \circ \tau \arrow[dr, "\theta_{\kappa,\mu,\nu} \circ 1_\tau"'] & & ((\kappa \circ \nu) \circ \tau) \circ \mu \arrow[dl, "\theta_{\kappa\circ\nu,\tau,\mu}"] \\
& ((\kappa \circ \nu) \circ \mu) \circ \tau &
\end{tikzcd}
$$



How to prove coherence theorems

- Rewriting techniques [Mac Lane 63 "revisited" by Squier-Otto-Kobayashi 94]
(More on this later in this talk)

- Strictification (Joyal-Street 1993)
(based on the existence of a faithful functor from a monoidal category to an associated strict monoidal category)
(Not covered here)

- "Instant one-step proof" (Kapranov 1993)
Translate the data into geometry/topology)
→ general coherence theorem for CW-complexes [C-Laplante-Anfossi 2022]
(Coming next)

PART I

Regular CW-complexes

The low-dimensional data of a CW-complex

- A set K_0 of 0-cells (points) $x, y \dots$
- A set K_1 of 1-cells (edges) $\alpha : x \to y$ (and formal $\alpha^{-1} : y \to x$)

$\rightsquigarrow$ free category K_1^*
(morphisms = paths $x \xrightarrow{\alpha} x_1 \xrightarrow{\beta^{-1}} x_2 \xrightarrow{\beta} x_1$)
(CELLULAR, COMBINATORIAL)

- A set of 2-cells A, given by attachment of 2-balls along a closed walk of 1-cells
(if edges are distinct = path $\rightsquigarrow$ regular)

Thus with a 2-cell A, up to the choice of $x \in K_0 \cap A$ there is an associated boundary $\gamma_A : x \to x$ in K_1^*.

- $K_3 \dots$

(2006)

From Ronnie Brown's Topology and groupoids...

Notation: X top. space, $Y \subseteq X$ $\rightsquigarrow$ $\pi(X)Y$

full subcategory of the fundamental groupoid of X spanned by Y

Combinatorial characterisation of $\pi(K)K_0$ (for K CW-complex):

Notation
- $\mathcal{F}(K) = K_1^* / \alpha\alpha^{-1} = id$ (free groupoid)
- $C(K) = \mathcal{F}(K) / \gamma_A = id$

Three steps in the characterisation

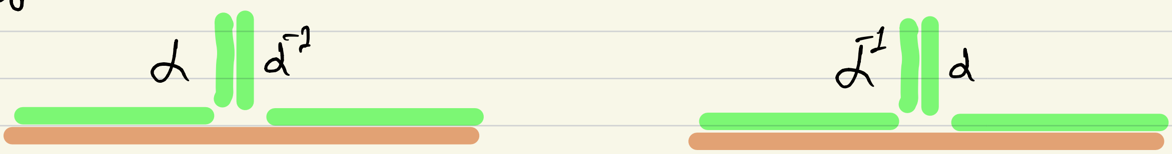
(all taken from BROWN'S book Topology and groupoids)

- $\pi(K_1)K_0 \simeq \mathcal{F}(K)$
- $\pi(K_2)K_0 \simeq \boxed{C(K)}$
  $\|$
- $\boxed{\pi(K)K_0}$ $\approx$

Proof: repeated use of Van Kampen

... to a general coherence theorem

Say that two parallel paths $\gamma, \gamma' \in K_1^{\rightarrow}(x, y)$ are combinatorially homotopic if they get equated in $C(K)$. Concretely, this means that one can go from γ to γ' by repeated use of



where $\gamma_1(\gamma_1')^{-1} = \gamma_A$ for some A
or
γ_A^{-1}

Theorem 1 ("instant one step coherence")

The following holds in a (regular) CW-complex K:

- (1) all parallel combinatorial paths are combinatorially homotopic.

$\Uparrow$

- (2) all path components are simply connected.

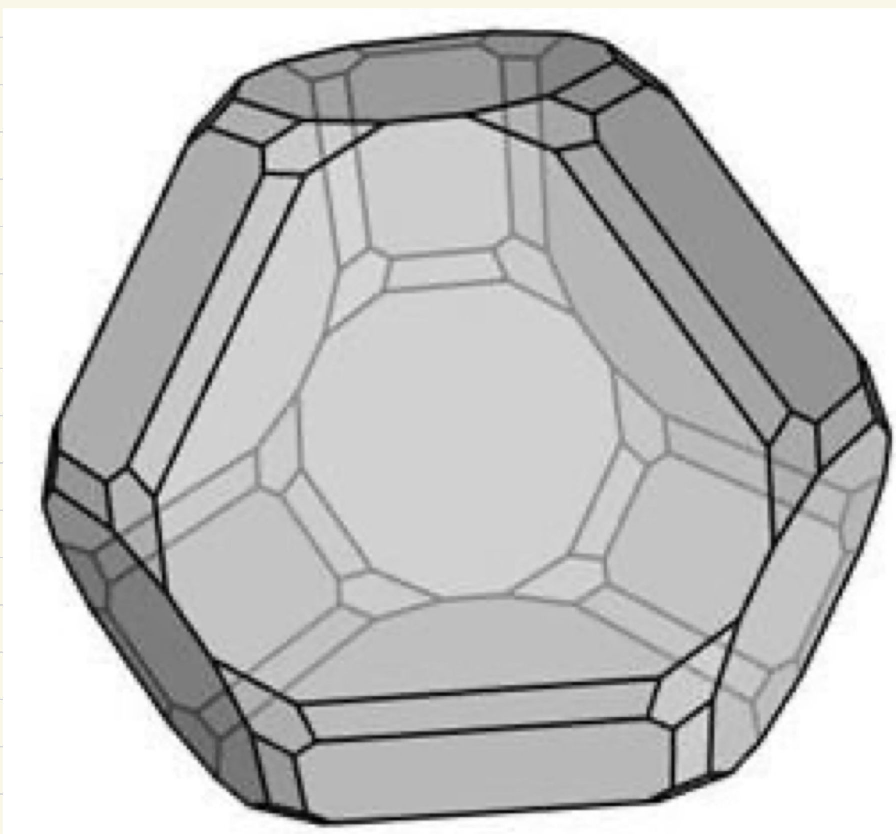
Proof: (1) reformulates as $C[K](x,y)$ is a singleton
(2) reformulates as $\Pi[K](x,y)$ is a singleton

What goes well without saying,
goes even better when you say it!

Coherence for α and σ

(non-unital symmetric monoidal)

All the data decorate the vertices, the edges, and the 2-faces of a polytope:



simple permuto-associahedron

(Baralić - Ivanović - Petrić 2019)

PART II

polyhedral complexes

Polyhedral complexes

We now turn to a (less) general proof of coherence in polyhedral complexes that retains most aspects of Mac Lane's original proof.

Polyhedron = intersection of closed half spaces

Definition 5.1. A *polyhedral complex* C is a finite collection of polyhedra in $\mathbb{R}^d$ such that

(i) the empty polyhedron is in C,

(ii) if $P \in C$, then all the faces of P are also in C,

(iii) the intersection $P \cap Q$ of two polyhedra $P, Q \in C$ is a face both of P and of Q.

$\bigcup \{P \mid P \in C\}$ = the space described

ONE CAN VIEW POLYHEDRAL COMPLEXES AS CW-COMPLEXES

- A generic vector $\vec{v} \in \mathbb{R}^d$ is a vector s.t. $\forall\, \alpha : x \to y$ edge of C $\langle \vec{v} | x \rangle \neq \langle \vec{v} | y \rangle$. This provides an orientation of edges: $x \longrightarrow y$ if $\langle \vec{v} | x \rangle < \langle \vec{v} | y \rangle$.

Outgoing link of a vertex

For each vertex x, choose $\varepsilon > 0$ small enough so that for all outgoing edge $\alpha : x \to y$ from x, x and y are separated by the hyperplane

$$\{y \mid \langle \vec{v}, z \rangle = \langle \vec{v}, x \rangle + \varepsilon\}$$

The outgoing link of x is the intersection of the complex with that hyperplane.

Adapting from Ziegler (vertex figure), we have

- The outgoing link is a polyhedral complex $C \backslash x$
- There is a (subface preserving) bijective correspondence between the k-dimensional faces of $C \backslash x$ and the $(k+1)$-dimensional faces of C that contain x and have a non-empty intersection with the outgoing link (for all k).

Coherence based on orientation

Theorem 2 (C-Laplante-Anfossi 2022)

THESE CONDITIONS HOLD if C is a POLYTOPE

Let C be a polyhedral complex. If

- There is a unique global sink (NORMAL FORM) (i.e., a vertex without any outgoing edge), and
- the 1-skeleton of the outgoing link of every vertex is connected.

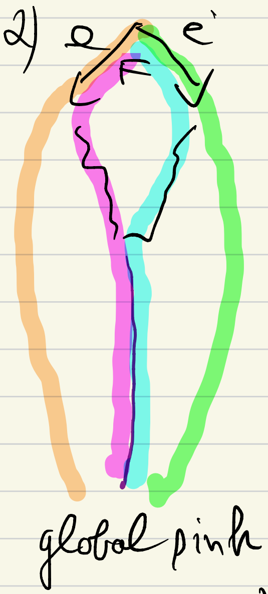
Then every two parallel combinatorial paths are combinatorially homotopic.

Proof plan

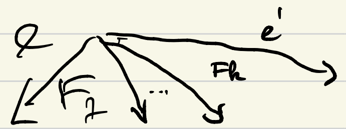
- A) Assume paths oriented + go to the sink.

1) $e = e'$ induction

2) $e \neq e'$

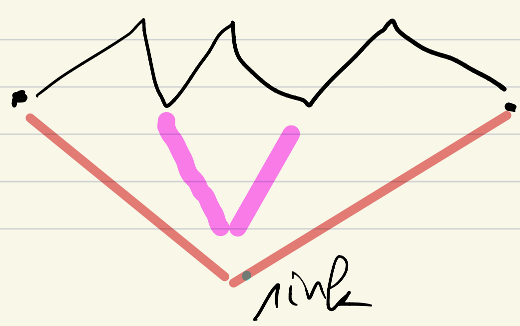


3) Otherwise, by the second assumption, there exists a sequence of faces



and we apply 2) repetitively

- B) General case: note



[cf. MacLane]

Theorem 2 is strictly less general than Theorem 1

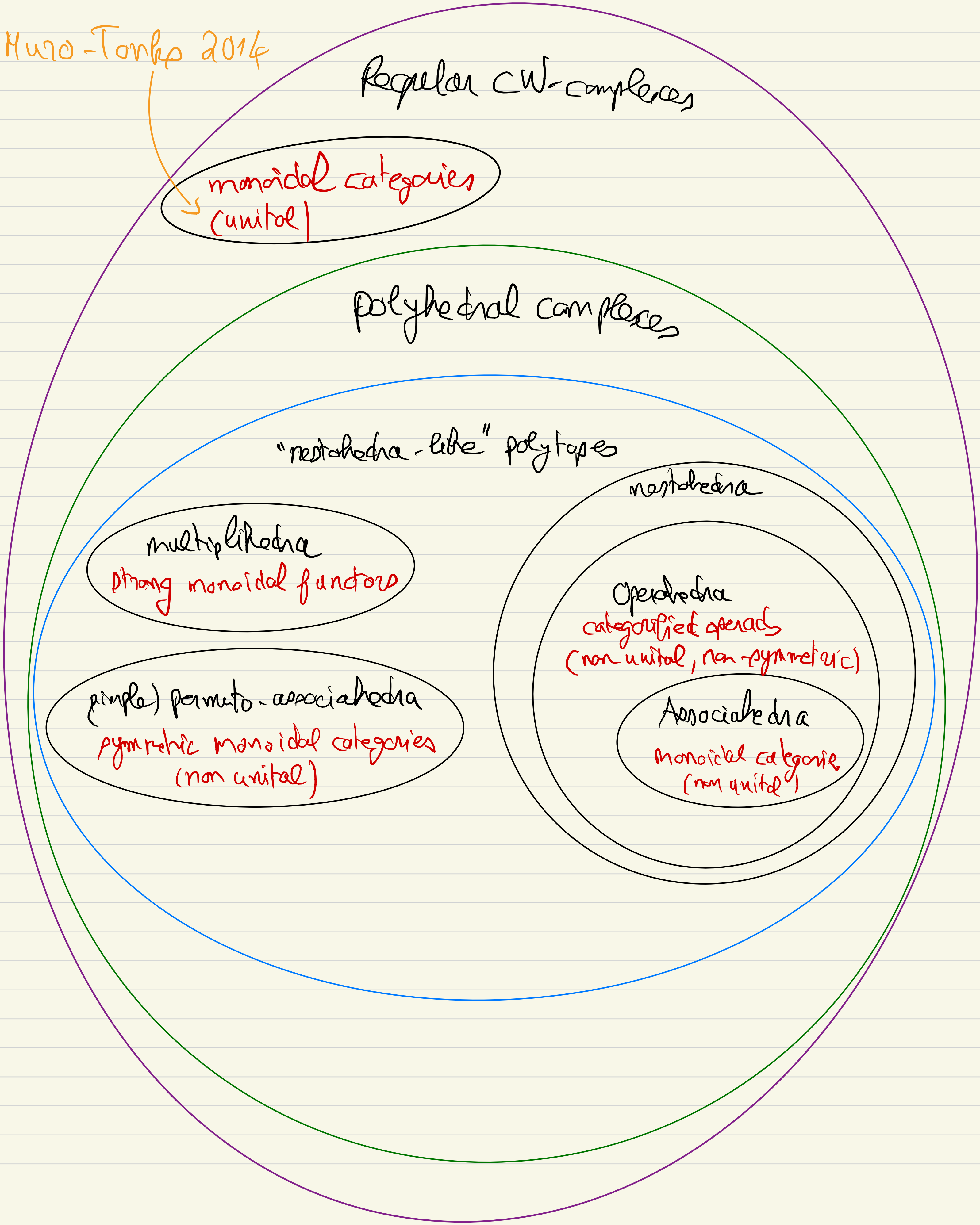
Proposition If the 1-skeleton of the outgoing link of every vertex is connected, then every path connected component of C is simply connected.

Proof idea: $\langle \vec{v} | - \rangle$ defines a Morse function.

Let $C_t = \{ x \in C \mid \langle \vec{v} | x \rangle \geq t \}$.

- At each h s.t. $\langle \vec{v} | x \rangle = h$ for some vertex, we get that C_h is homotopically equivalent to the

$$\underbrace{\text{pushout of } C_{h+\varepsilon} \text{ with } \underbrace{\text{the cone from } x \text{ to the } \underbrace{\text{outgoing link of } x}_{\text{connected}}}_{\text{simply connected}}}_{\text{simply connected (VAN KEMPEN)}}$$


Muro-Tonks 2014
Regular CW-complexes
monoidal categories
(unital)
polyhedral complexes
"nestohedra-like" polytopes
nestohedra
multiplihedra
strong monoidal functors
(simple) permuto-associahedra
symmetric monoidal categories
(non unital)
Operahedra
categorified operads
(non unital, non-symmetric)
Associahedra
monoidal categories
(non unital)

PART III

"nestohedra-like" polytopes

multiplihedra

strong monoidal functors

(simple) permuto-associahedra

symmetric monoidal categories
(non unital)

nestohedra

operahedra

categorified operads
(non unital, non-symmetric)

Associahedra

monoidal categories
(non unital)

Two flavours of rewriting

We have seen • the "one-line proof"

• an oriented proof, depending on the choice of an orientation vector (a Morse function), that "looks like" Mac Lane's proof. But there is still a distance to that proof = the one which separates

$\boxed{\text{abstract rewriting}}$ from $\boxed{\text{term rewriting}}$

In term rewriting, we have • [1] rewriting rules

$$(x \otimes y) \otimes z \to x \otimes (y \otimes z)$$

(preferably a finite number of them) which are

• [2] instantiated $\quad ((A \otimes B) \otimes C) \otimes D) \otimes (E \otimes F)$

• [3] in context $(Z \otimes (((A \otimes B) \otimes C) \otimes (D \otimes (E \otimes F)))) \otimes G$

The same remarks apply to Mac Lane's pentagons

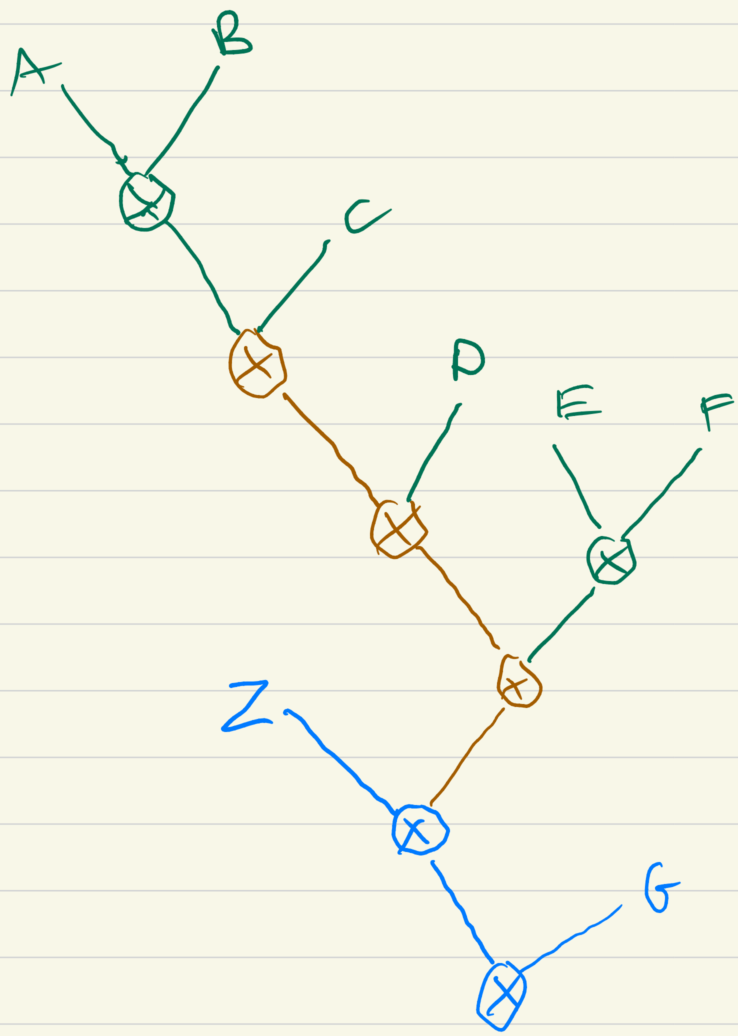
- 1 one generic coherence condition — CRITICAL PAIR

$$\begin{tikzcd}[column sep=small]
& ((x\otimes y)\otimes z)\otimes u \arrow[dl, dash] \arrow[dr, dash] & \\
(x\otimes(y\otimes z))\otimes u \arrow[d, dash] & & (x\otimes y)\otimes(z\otimes u) \arrow[dd, dash] \\
x\otimes((y\otimes z)\otimes u) \arrow[drr, dash] & & \\
& & x\otimes(y\otimes(z\otimes u))
\end{tikzcd}$$

- 2 instantiated : $x \mapsto A\otimes B$, $y \mapsto C$, $z \mapsto D$, $u \mapsto E\otimes F$
- 3 in context

$$\begin{tikzcd}[column sep=small]
& (Z\otimes \text{———})\otimes G \arrow[dl, dash] \arrow[dr, dash] & \\
(Z\otimes \text{———})\otimes G \arrow[d, dash] & & (Z\otimes(((A\otimes B)\otimes C)\otimes(D\otimes(E\otimes F))))\otimes G \arrow[dd, dash] \\
(Z\otimes \text{———})\otimes G \arrow[drr, dash] & & \\
& & (Z\otimes \text{———})\otimes G
\end{tikzcd}$$

$$(Z \otimes ((((A \otimes B) \otimes C) \otimes D) \otimes (E \otimes F))) \otimes G$$



Looking for polytopes "displaying instantiation and context"

DOŠEN - PETRIĆ

We'll see that in the class of nestohedra, there is a flavour of ●^{1} and ●^{2}, but not of ●^{3} in general.

We give a counterexample.

We give a condition on nestohedra, which we call the

<u>contextuality condition</u>, ^{3}

- that accounts for ●^{3}
- that is satisfied by operahedra

All of this makes sense also without orientation

Building sets

A <u>building set</u> $\mathbb{H}$ (a.k.a as atomic and saturated hypergraph) is given by a finite set H of vertices and a subset $\mathbb{H} \subseteq \mathcal{P}^*(H)$ s.t.

non-empty

$$\boxed{\begin{array}{l} \bullet\ \forall x \in H,\ \{x\} \in \mathbb{H} \\ \bullet\ \text{If } E_1, E_2 \in \mathbb{H} \text{ and } E_1 \cap E_2 \neq \emptyset, \text{ then } E_1 \cup E_2 \in \mathbb{H} \end{array}}$$

It is called connected if $H \in \mathbb{H}$.

If it is not connected, the maximal elements of $\mathbb{H}$ form the connected components of $\mathbb{H}$.

Two notions of restriction are relevant. Let $X \subseteq H$.

PLAIN RESTRICTION

- $\mathbb{H}_X = \{ E \mid E \in \mathbb{H} \text{ and } E \subseteq X \}$ (we write $\mathbb{H} \setminus X$ for $\mathbb{H}_{H \setminus X}$)

  For a connected building set, we write $\mathbb{H}, X \rightsquigarrow H_2 \cdots H_n$ where $H_2, \dots, H_n$ are the connected components of $\mathbb{H} \setminus X$

- $\mathbb{H}_{\cap X} = \{ E \cap X \mid E \in \mathbb{H} \}$ RECONNECTED RESTRICTION

<u>Example</u>: The set $\mathrm{Conn}(G)$ of connected subsets of a graph is a building set. Let $G = x - y - z$

$$\mathrm{Conn}(G) = \{\{x\}, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{x,y,z\}\}$$

$$\mathrm{Conn}(G)_{\{x,z\}} = \{\{x\}, \{z\}\} = \mathrm{Conn}(x \quad z)$$

$$\mathrm{Conn}(G)_{\cap \{x,z\}} = \{\{x\}, \{z\}, \{x,z\}\} = \mathrm{Conn}(x - z)$$

reconnected

Nestohedra (a.k.a hypergraph polytopes)

- Let $\mathbb{H}$ be a connected building set.

$\neq H$ and $\{x\}$

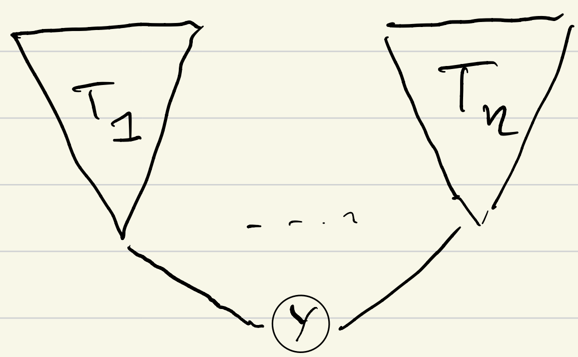
The elements of $\mathbb{H}$ (hyperedges) are interpreted as instructions for truncating a simplex. Specifically, naming the facets of a simplex by the elements of H, $E \in \mathbb{H}$ asks for truncating the face of the simplex which is the intersection of all facets x s.t. $x \in E$.

- Combinatorially, the faces of the resulting polytope, called <u>nestohedron</u> are described and named by <u>constructs</u>:

Let $\emptyset \neq Y \subseteq H$:

ALSO CALLED TUBINGS

- $(Y = H)$ H is a construct of $\mathbb{H}$ $\quad$ H
- $(Y \neq H)$ Let $H_1, \ldots, H_n$ be the connected components of $\mathbb{H}_{H \setminus Y}$. If $T_1 \ldots T_n$ are constructs of $\mathbb{H}_{H_1}, \ldots, \mathbb{H}_{H_n}$, then $Y(T_1, \ldots, T_n)$ is a construct of $\mathbb{H}$:

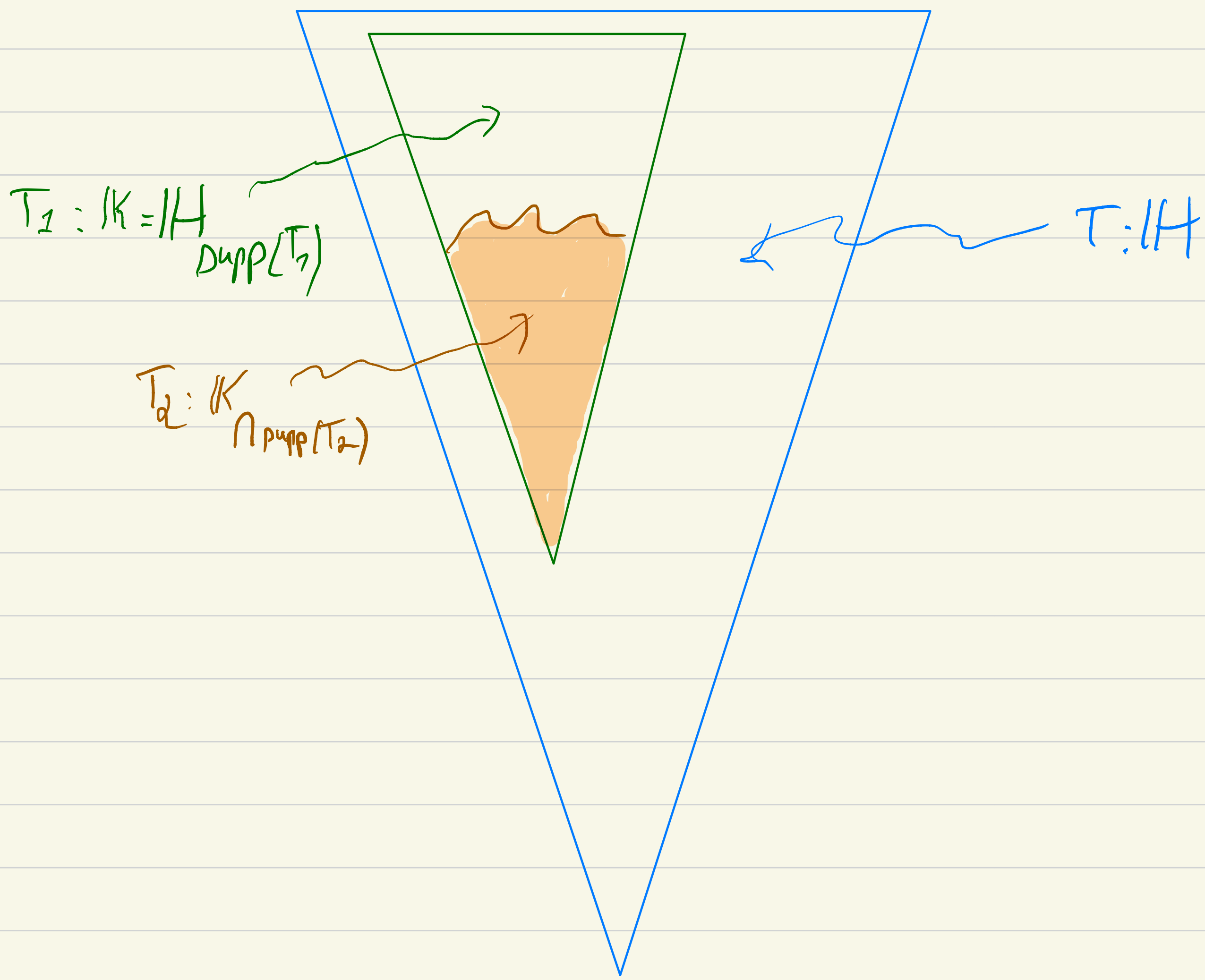


<u>Notation $T : \mathbb{H}$</u>

A construct all of whose labels are singletons is called a <u>construction</u>.

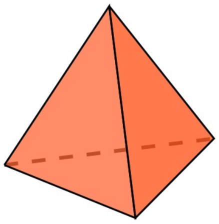
Subtrees of constructs

If T is a tree whose vertices are labeled by (disjoint) subsets of some set H, we write $\mathrm{Supp}(T)$ for the union of these labels.



Think of T as "T_1 in some context" and of T_1 as "an instance of T_2".

Simplices



$\{\{1\},\{2\},\{3\},\{4\},\{1,2,3,4\}\}$

Let $\mathcal{X}$ be a (finite) set. We take

$$\boldsymbol{S}_{\mathcal{X}} = \{\{x\} \mid x \in \mathcal{X}\} \cup \{\mathcal{X}\}$$

$\{y_1\}$ ... $\{y_p\}$

X

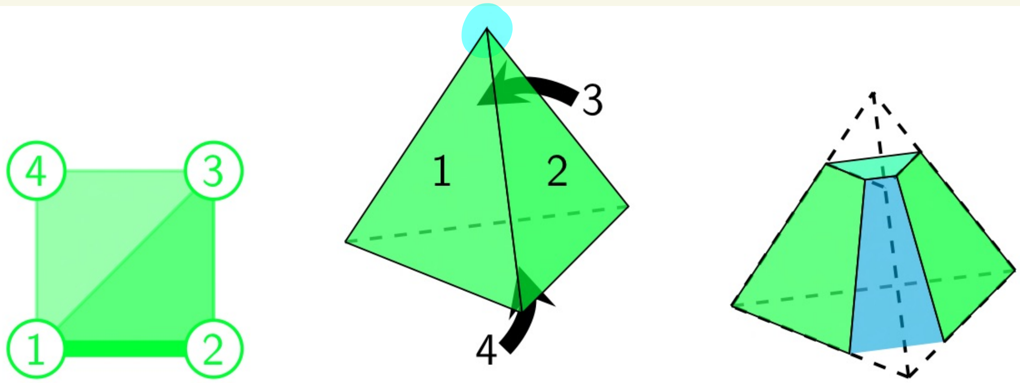
All of the constructs have the form $X(y_1, \ldots, y_p)$, where $\{y_1, \ldots, y_p\} = \mathcal{X} \setminus X$.

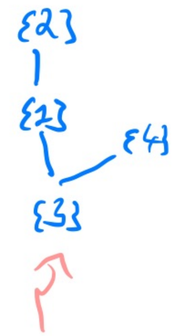
Hence they are in one-to-one correspondence with non-empty subsets $X \subseteq \mathcal{X}$.

Each x "is" a vertex, and its opposite facet is $\mathcal{X} \setminus \{x\}$

(which we also identify as x)

Truncating pimplices in action





Let $\boldsymbol{H} = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$.

The vertex $4(1,2,3)$ (resp. $3(1,2,4)$) of the simplex splits into

$4(2(1,3))$, $4(3(2(1)))$, $4(3(1(2)))$, $4(1(2,3))$ (resp. $3(2(1),4)$, $3(1(2),4)$)

The other faces of the truncation of edge $\{3,4\}(1,2)$ are

$4(3(\{1,2\}))$, $\{3,4\}(1(2))$, $3(\{1,2\},4)$, $\{3,4\}(2(1))$, and $\{3,4\}(\{1,2\})$

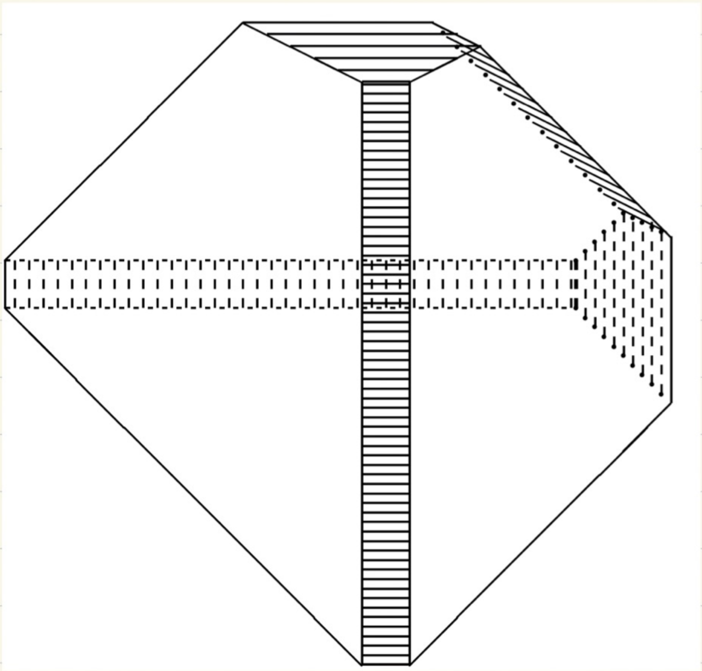
WHAT WE GET HERE is the $\boxed{\text{3-CUBE}}$

Associahedra

Let $X = \{x_1 < \cdots < x_n\}$. Then

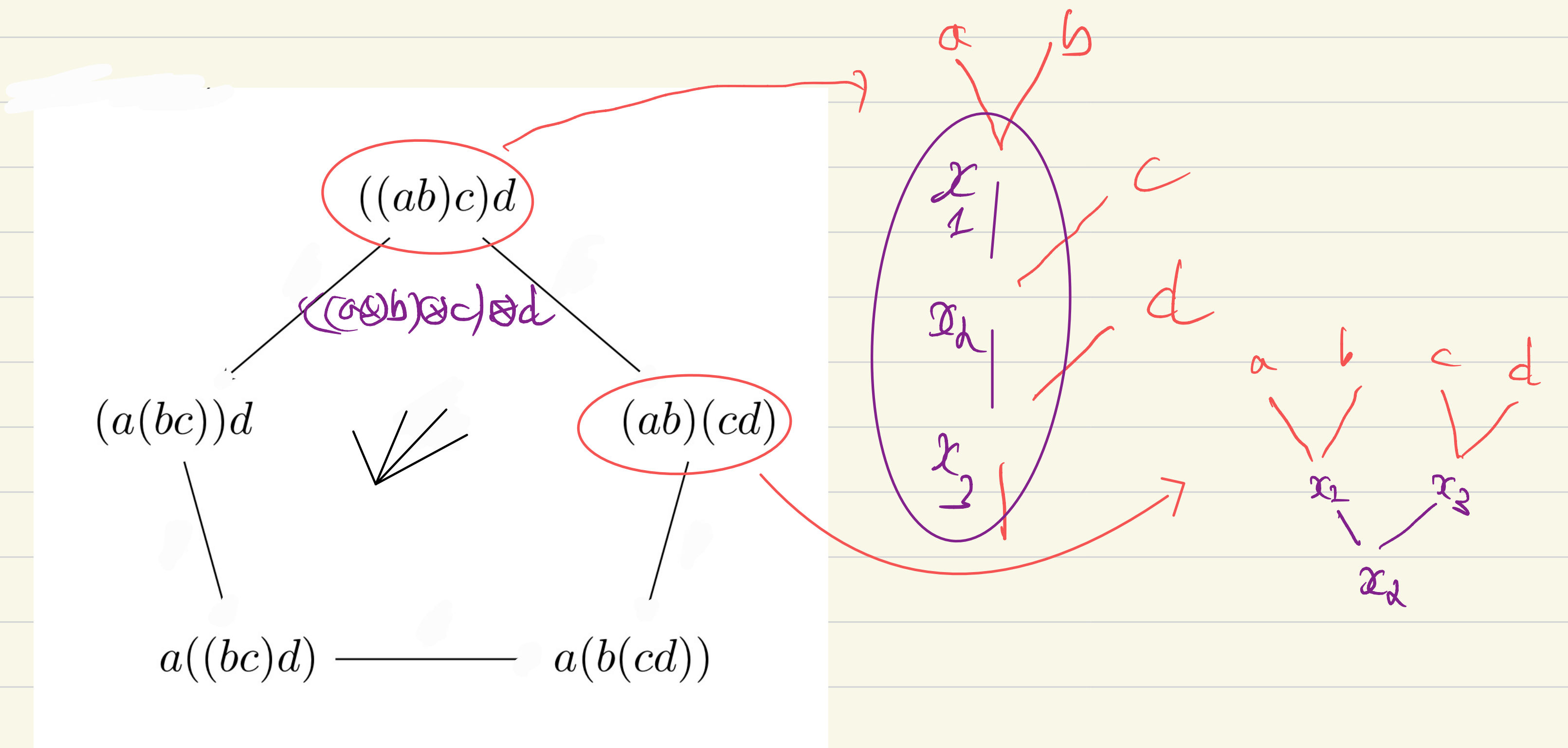
$$\boxed{K_X = \{\{x_1\}, \cdots, \{x_n\}, \{x_1, x_2\}, \cdots\cdots, \{x_{n-1}, x_n\}, \{x_1, \cdots, x_n\}\}}$$

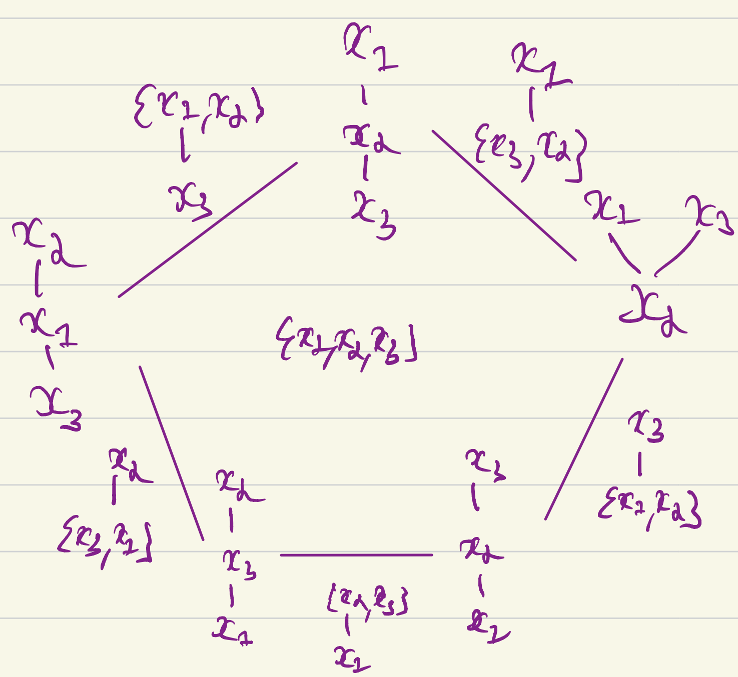
(linear graph $x_1 - x_2 - \cdots - x_n$)



Recasting Mac Lane's pentagon in the language of constructs

Dictionary: $\overset{a}{\ } \ x_1 \overset{b}{—} x_2 \overset{c}{—} x_3 \ \overset{d}{\ }$



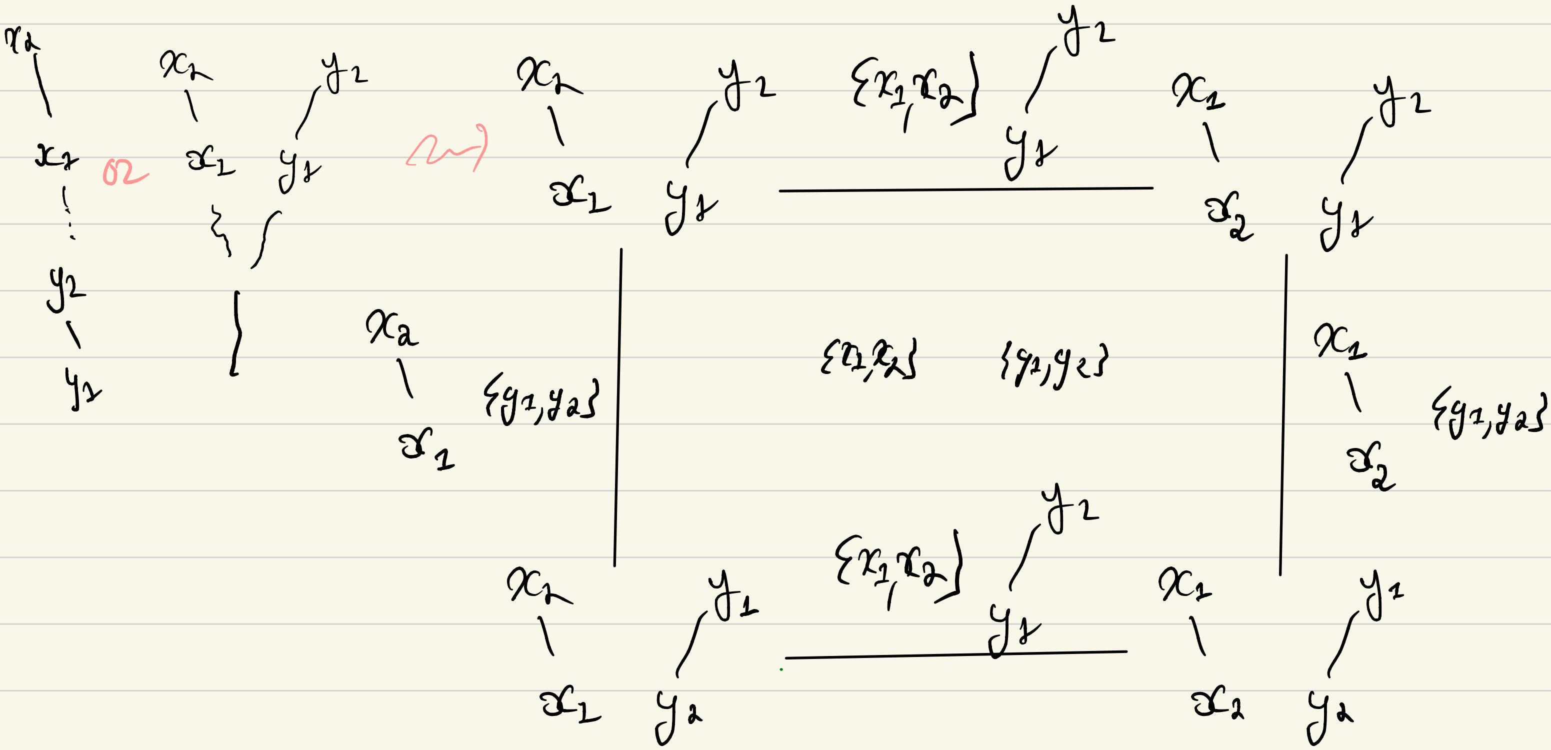


0- and 1-faces of nestohedra

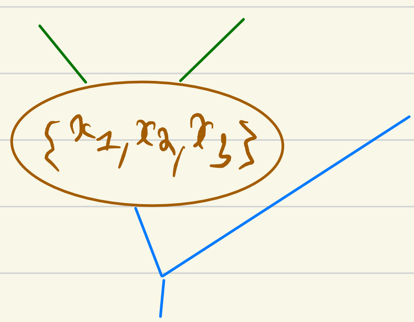
- One can read the dimension of (the face associated to) a construct as the sum of the cardinals $|X|-1$ of all decorations X appearing in the construct. In particular
- Every construction is of dimension 0.
- Edges have a single non-singleton node which has cardinal 2: it is obtained by contracting an edge in a construction.

The two kinds of 2-faces of nestohedra

- the "boring" ones with exactly two non-singleton nodes, each of cardinal 2. Schematically,



- the "interesting" ones, with exactly one non-singleton node, of cardinal 3. There are four shapes of such 2-faces (next two slides)



Towards charting the 2-faces of nestohedra

Notation if $x, y, z \in H$, then

- if y, z are in different connected components of $\mathbb{H}_{H \setminus \{e\}}$,

then we write

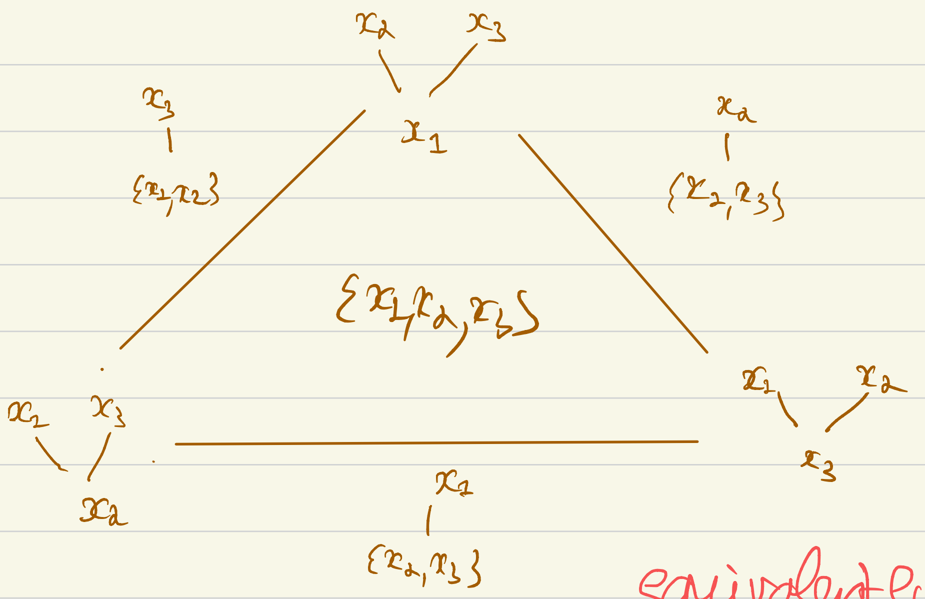
$$\boxed{x \underset{\mathbb{H}}{\rightsquigarrow} y, z}$$

- otherwise we write

$$\boxed{x \underset{\mathbb{H}}{\rightsquigarrow} \{y, z\}}$$

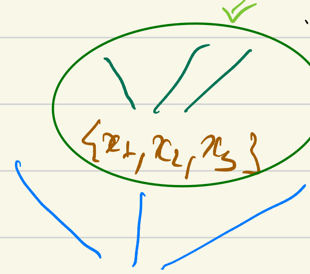
(000)

(0)



$$\boxed{\begin{array}{l} x_1 \rightsquigarrow x_2, x_3 \\ x_2 \rightsquigarrow x_1, x_3 \\ x_3 \rightsquigarrow x_1, x_2 \end{array}}$$

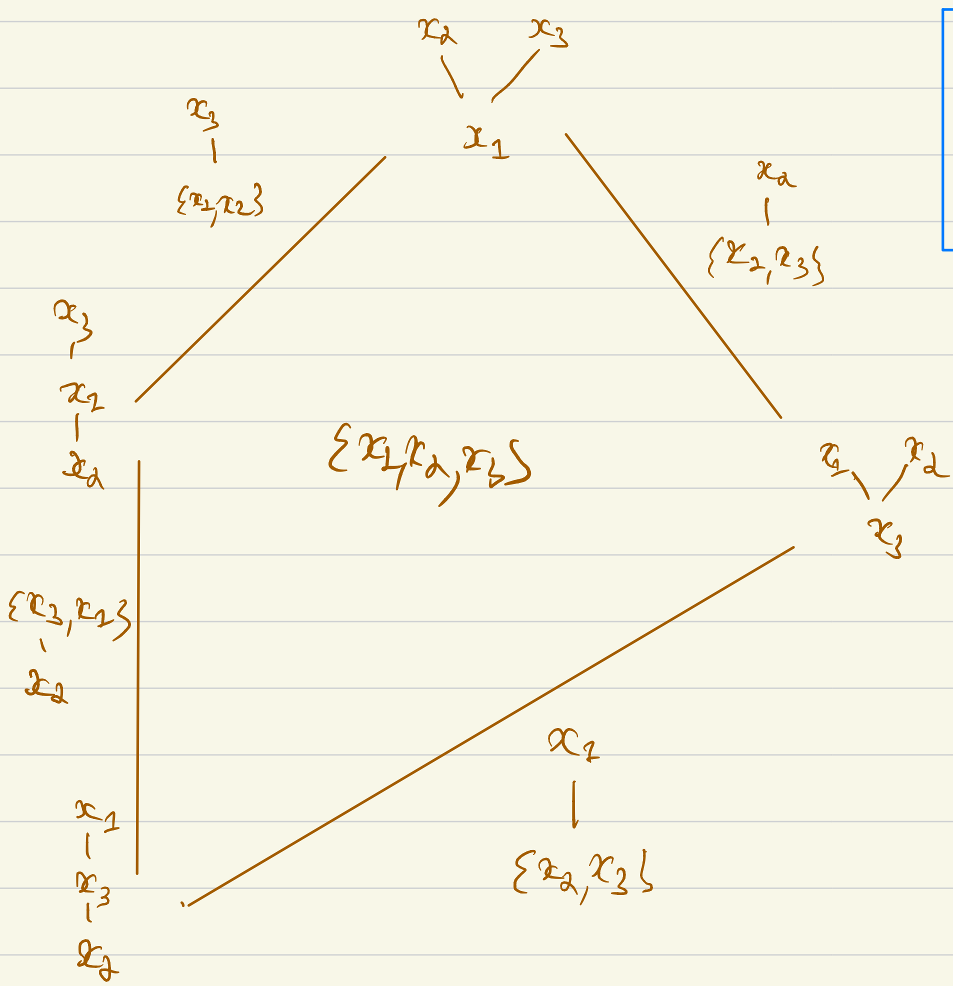
relative to $\mathbb{K}$



equivalently, relative to $\mathbb{K} \cap \{x_1, x_2, x_3\}$

(010)

(1)

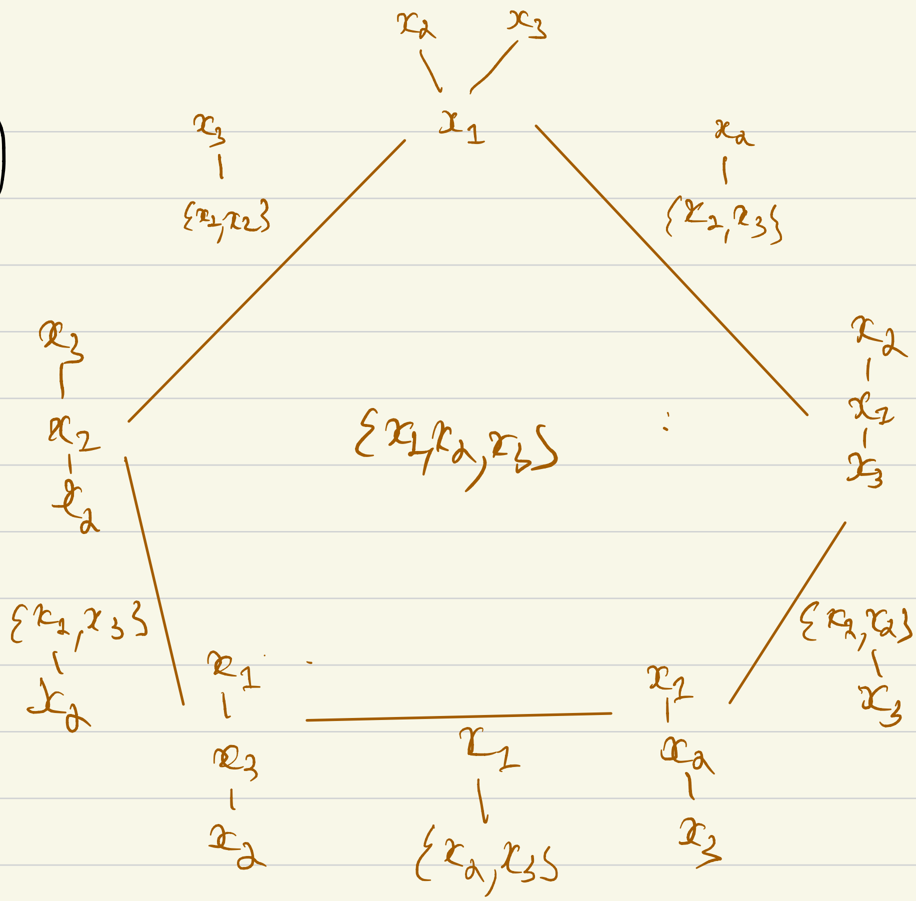


$$\boxed{\begin{array}{l} x_1 \rightsquigarrow x_2, x_3 \\ x_2 \rightsquigarrow \{x_1, x_3\} \\ x_3 \rightsquigarrow x_1, x_2 \end{array}}$$

(001), (100) are the same by permutation

(011)

(2)



$x_1 \rightsquigarrow x_2, x_3$

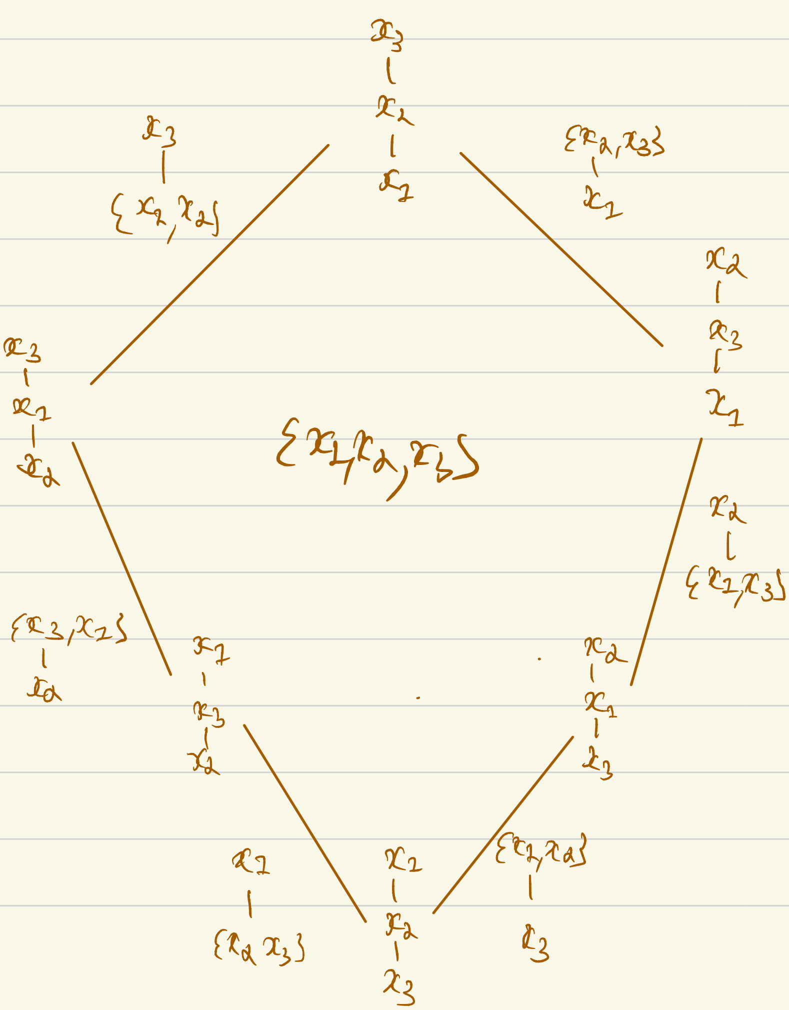
$x_2 \rightsquigarrow \{x_1, x_3\}$

$x_3 \rightsquigarrow \{x_1, x_2\}$

(101), (110) are the same by permutation

(111)

(3)



$x_1 \rightsquigarrow \{x_2, x_3\}$

$x_2 \rightsquigarrow \{x_1, x_3\}$

$x_3 \rightsquigarrow \{x_1, x_2\}$

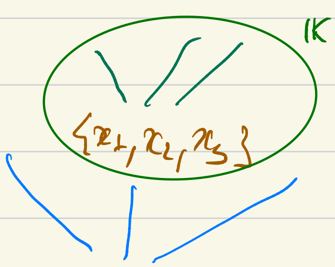
The contextuality issue

So we have only four shapes of 2 faces (coherence conditions)

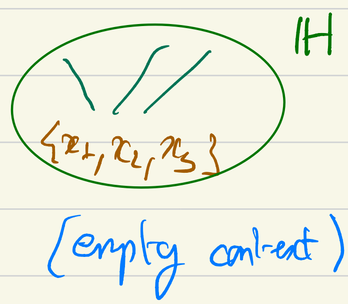
- triangles (think of the simplex!)
- cubes (both as "boring" and "interesting") (think of hypercubes (which are nestohedra)
- pentagons
- hexagons (think of operahedra)

However, two 2-dimensional constructs with the same label $\{x_1, x_2, x_3\}$, e.g.

WE CALL THEM $\{x_1, x_2, x_3\}$-FACES



and



do not have the same shape in general!

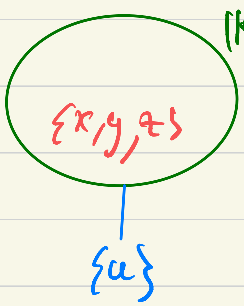
Counter-example

Consider

$$\mathbb{H} = \{ \{x\}, \{y\}, \{z\}, \{u\}, \{x,y\}, \{y,z\}, \{x,y,z\}, \{u,x,z\}, \{u,x,y,z\} \}$$

$$= \mathrm{Conn}\left(\boxed{\begin{tikzcd} & y \arrow[dr, dash] & \\ x \arrow[ur, dash] \arrow[dr, dash] & & z \\ & u \arrow[ur, dash] & \end{tikzcd}} \right)$$

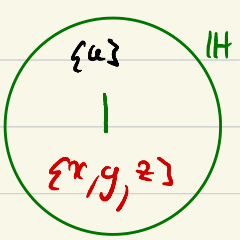
Consider the construct



$$\mathbb{K} = \mathbb{H}_{H \setminus \{u\}} = \{ \{x\}, \{y\} \{z\}, \{x,y\}, \{y,z\}, \{x,y,z\} \}$$

$$y \underset{\mathbb{K}}{\rightsquigarrow} x, z$$

$\rightsquigarrow$ pentagonal face



$$y \underset{\mathbb{H}}{\rightsquigarrow} \{x, z\}$$

$\rightsquigarrow$ hexagonal face

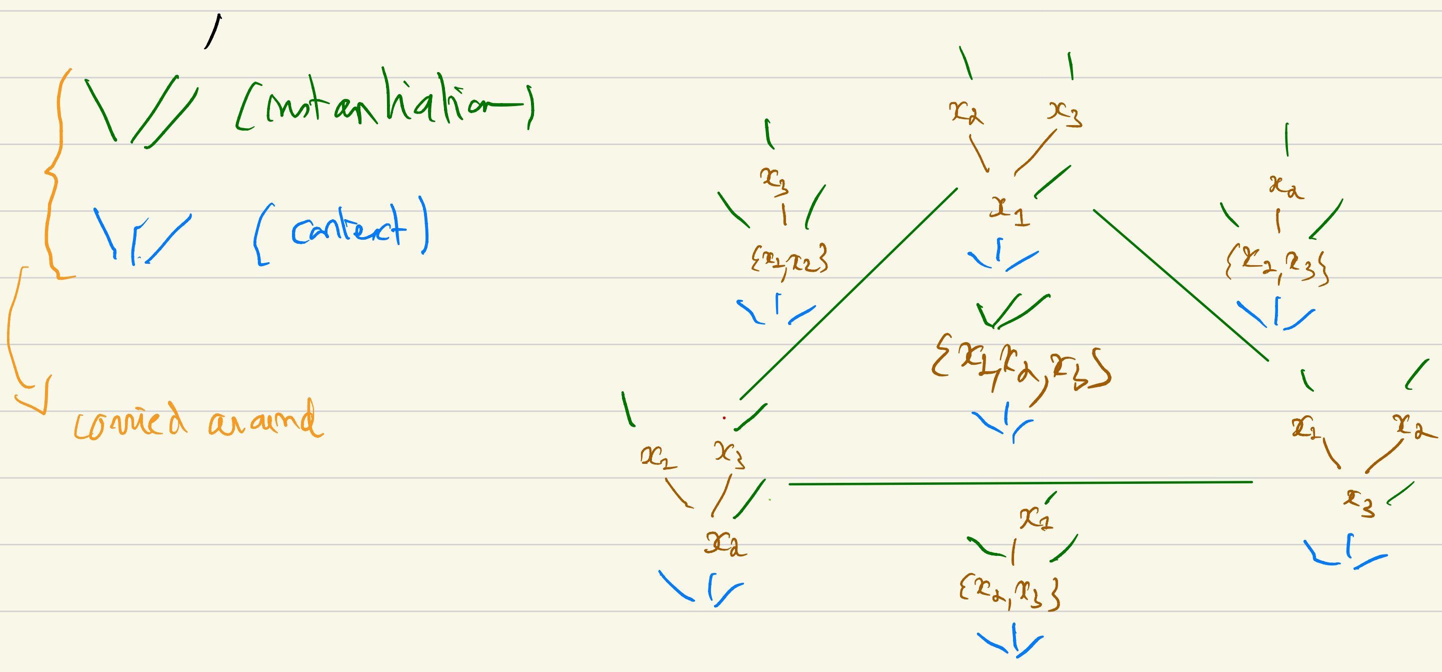
Contextual building sets

A building set $\mathbb{H}$ is called contextual (connected) if $\forall E \in \mathbb{H}, \forall x,y,z \in E$

$$\boxed{x \underset{\mathbb{H}_E}{\rightsquigarrow} \{y,z\} \iff x \underset{\mathbb{H}}{\rightsquigarrow} \{y,z\}}$$

(and hence $x \underset{\mathbb{H}_E}{\rightsquigarrow} y,z \Rightarrow x \underset{\mathbb{H}}{\rightsquigarrow} y,z$)

- This condition ensures that for any $\{x,y,z\} \subseteq H$ all $\{x,y,z\}$-faces are instances in context of $\mathbb{H}_{\cap\{x,y,z\}}$



- Moreover, we have $x \underset{\mathbb{H}_E}{\rightsquigarrow} \{y,z\}$ if and only if $\underbrace{\mathbb{H}_{\cap\{x,y,z\}}}_{\text{The "rules"}} \rightsquigarrow \{y,z\}$

($\forall E$ connected)

$x \underset{\mathbb{H}}{\rightsquigarrow} \{y,z\}$

by definition

The bestiary of contextual nestohedra

The four shapes of 2-faces correspond to (and are determined by) the four connected building sets on three vertices (up to permutation)

simplex	triangle	$\{\{x\},\{y\},\{z\},\{x,y,z\}\}$
cube	rectangle	$\{\{x\},\{y\},\{z\},\{x,y\},\{x,y,z\}\}$
associahedron	pentagon	$\{\{x\},\{y\},\{z\},\{x,y\},\{y,z\},\{x,y,z\}\}$
permutohedron	hexagon	$\{\{x\},\{y\},\{z\},\{x,y\},\{y,z\},\{x,z\},\{x,y,z\}\}$

complete graph

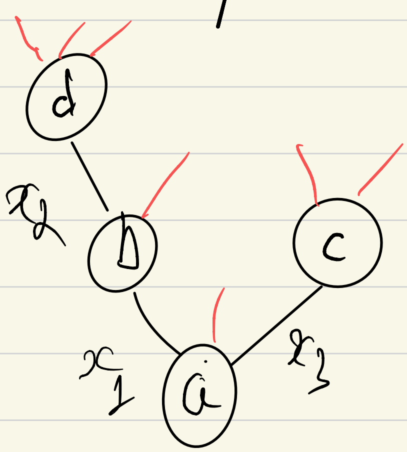
Operahedra as building sets

Given an operadic tree T, we can define a hypergraph (actually a graph) $G(T)$ as follows:

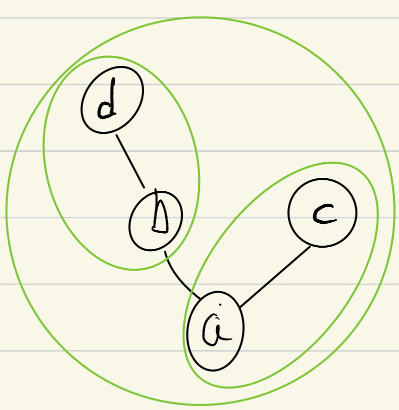
- vertices of $G(T)$ are internal edges of T
- edges of $G(T)$ witness the incidence of those internal edges.

Proposition. Given T, the constructs of $G(T)$ are in one-to-one correspondence with nestings on T.

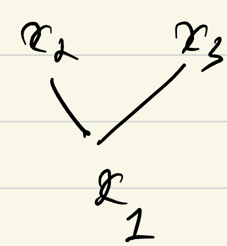
Example



$\longmapsto \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_1, x_2\}\}$



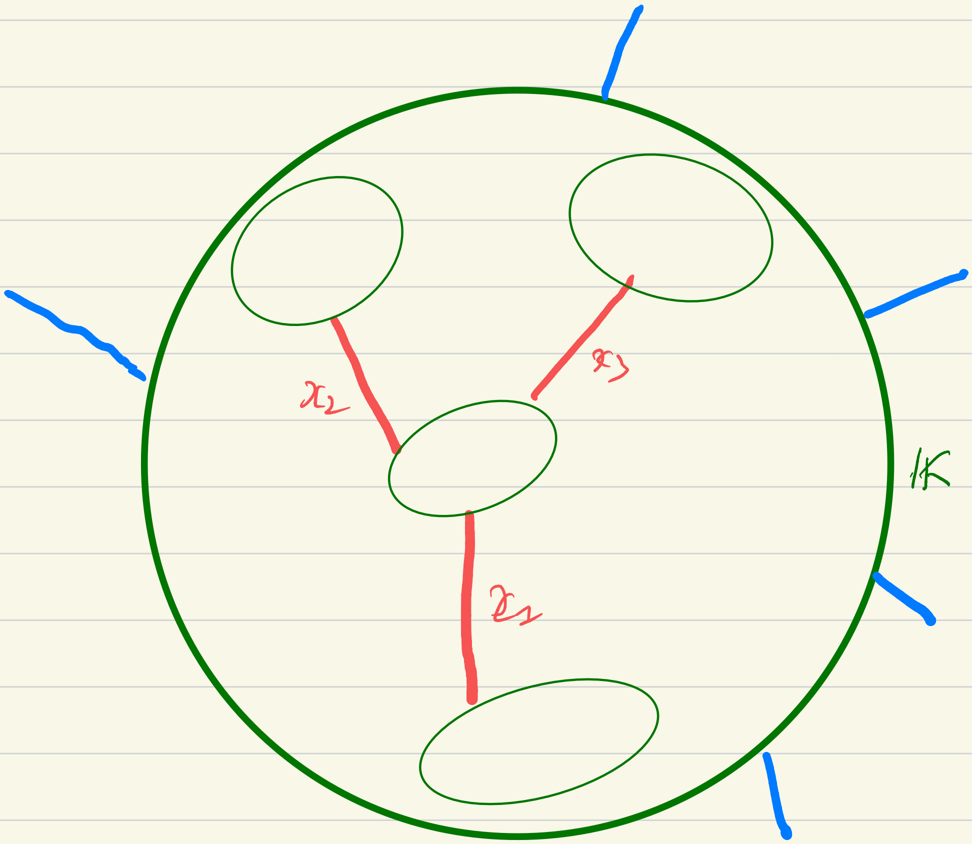
$\longmapsto$



Remark. If T_1 and T_2 have the same underlying (non planar) tree, then $G(T_1) = G(T_2)$.

Operahedra are contextual

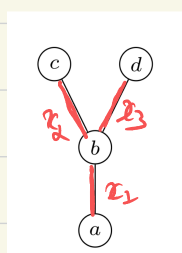
Proof "by picture"



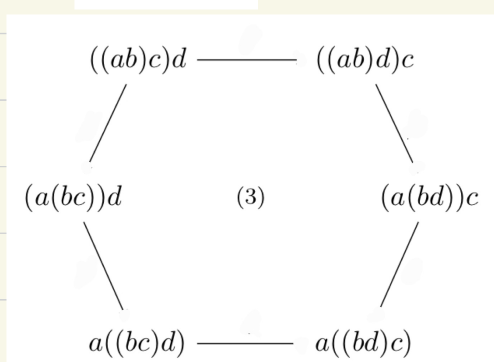
The relative position of x_1, x_2, x_3 in the subtree of $\mathcal{T}$ corresponding to $\mathbb{K}$ is the same as in $\mathcal{T}$

Moreover, with these positions, we have

$$\mathbb{H}_{\cap\{x_1,x_2,x_3\}} = \mathbb{K}_{\cap\{x_1,x_2,x_3\}} = G\Big($$



$\Big)$, corresponding to the coherence condition



Another example of contextual nestohedra

$\mathbb{H}_n = \{\{1\} \dots \{n\}, \{1,2\}, \{1,2,3\} \dots \{1 \dots n\}\}$

One sees easily that E is connected in $\mathbb{H}_n$ iff $E \in \mathbb{H}_n$

(follows from $e_1, e_2 \in \mathbb{H}$, $e_1 \cap e_2 \neq \phi \Rightarrow e_1 \subseteq e_2$ or $e_2 \subseteq e_1$)

Therefore $E = \{1 \dots m\}$ for some $m \leq n$ and $\mathbb{H}_E = \mathbb{H}_m$

then pick $i, j, k \leq m$. We have

for all $p \geq m$, $k \underset{\mathbb{H}_p}{\rightsquigarrow} \{i,j\}$ iff $i < k$ and $j < k$

and hence in particular

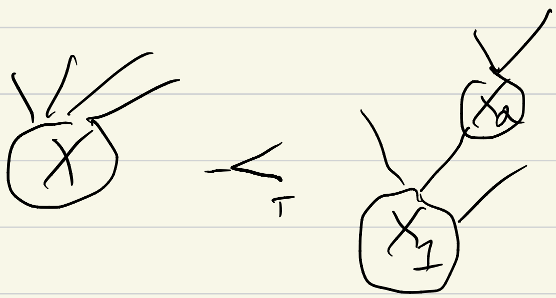
$$k \underset{\mathbb{H} = \mathbb{H}_n}{\rightsquigarrow} \{i,j\} \quad \text{iff} \quad k \underset{\mathbb{H}_E = \mathbb{H}_m}{\rightsquigarrow} \{i,j\}$$

For the road: a generalized Tamari order

Let $\mathbf{H}$ be a building set, and let $(H, \leq)$ be a total order on H.

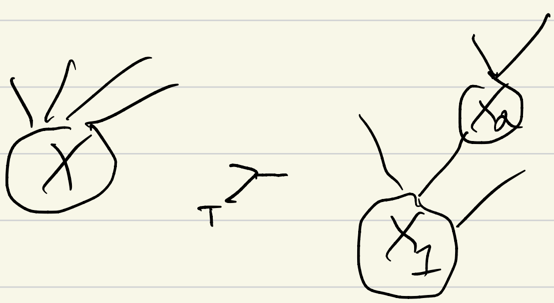
The Generalized Tamari Order (GTO) is the transitive closure $\leq_T$ of the relation $\prec_T$ defined locally on constructs by

For $X = X_1 \cup X_2$ (disjoint)



if $\max(X) \in X_2$

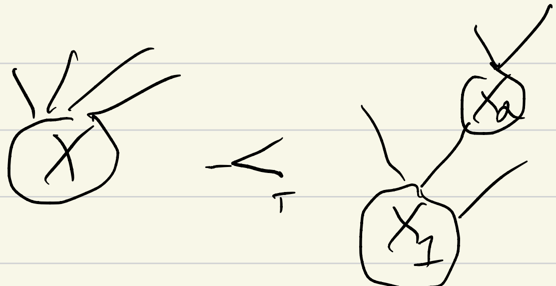
RULE 1



if $\max(X) \in X_1$

This definition is not as symmetric as it looks at first sight. If we view $S \prec T$ as a rule applied to S, the first case above, written more carefully, is

Construct of some $\mathbf{H}$

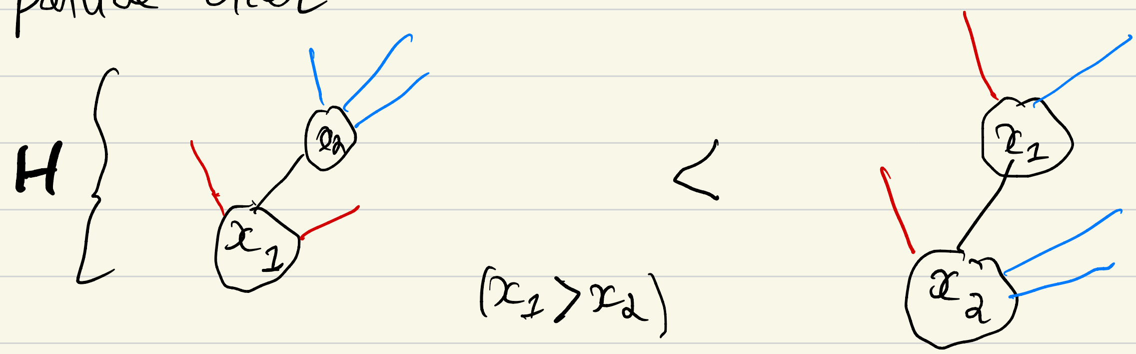


RULE 2

if
- X splits as $X = X_1 \sqcup X_2$
- $\max(X) \in X_2$, **AND**
- X_2 lies in a connected component of $\mathbf{H} \setminus X_1$

Restriction to vertices

The restriction of GTO on vertices is the flip partial order



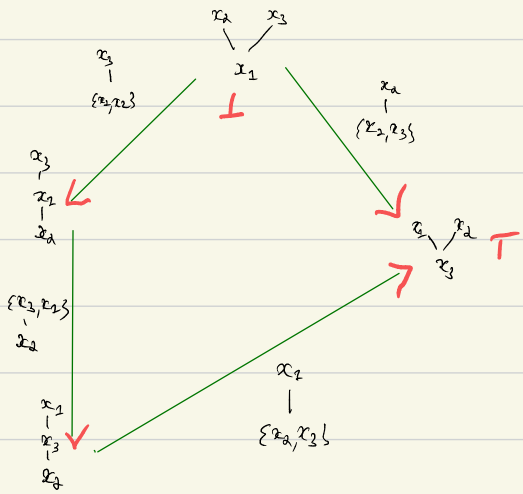
(with some permutation on the | and | edges dictated by the convectivity of **H**)

Thanks to this order, we can define a confluent and terminating rewriting system on constructions (for contextual nestohedra)

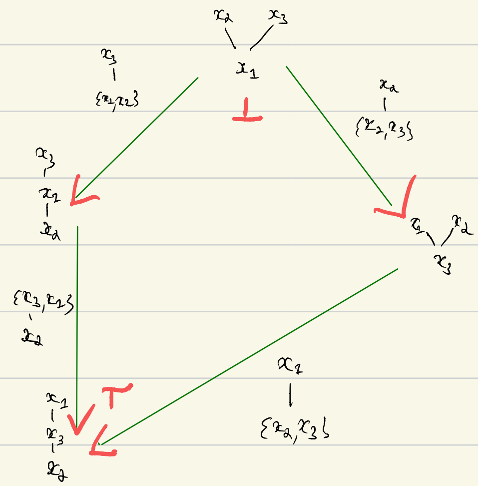
= adding directions in the considerations

We do this next in detail for situation (1).

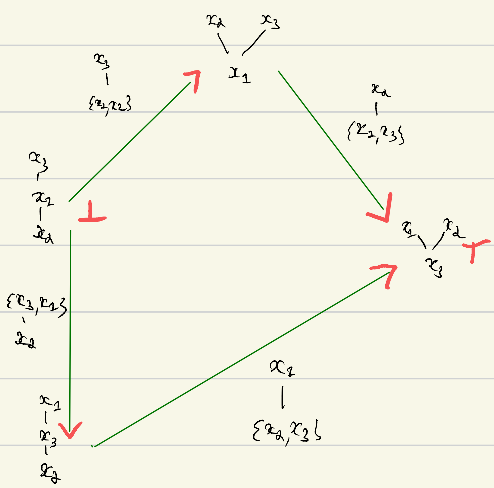
$x_1 > x_2 > x_3$



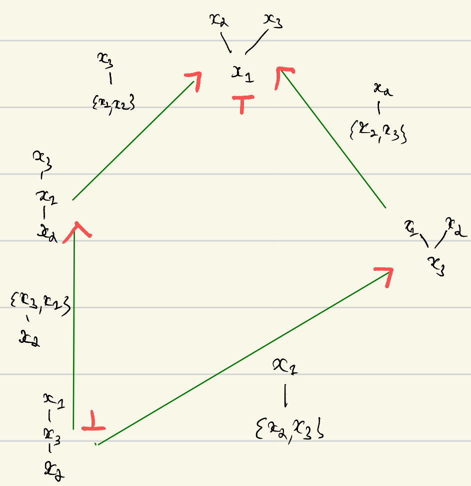
$x_1 > x_3 > x_2$



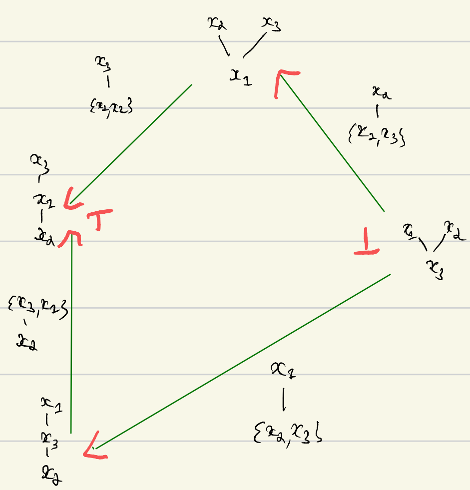
$x_2 > x_1 > x_3$



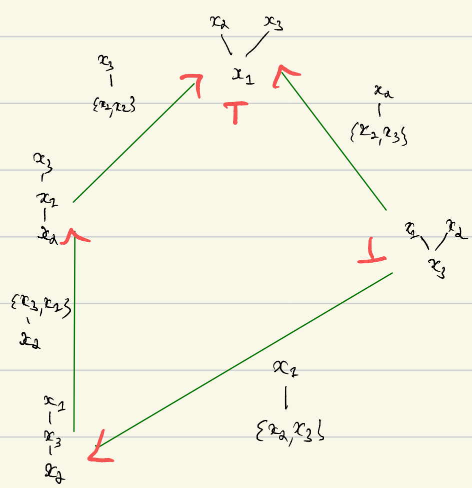
$x_2 > x_3 > x_1$



$x_3 > x_1 > x_2$



$x_3 > x_2 > x_1$



A question

Can we recover the generalised Tamari order on constructions for any nestohedron from some orientation vector?

The answer is yes (Favillare me) for operahedra and their Loday-type realisation.

Can we get a "Tamari" orientation vector for

- the Došen-Petrić realisation?
- the Postnikov realisation?