Opetopes, opetopic sets and polygraphs

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Combinatorics and geometry of OPETOPES

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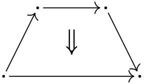
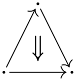
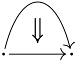
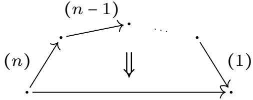
n-opetopes (for $n \leq 2$)

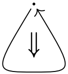
- There is a unique 0-dimensional opetope: the point (an operation with no input).
- There is a unique tree of 0-opetopes, yielding the unique arrow-shaped 1-opetope.
- 1-opetopes can assemble only as linear trees, and hence 2-opetopes are in one-to-one correspondence with natural numbers:

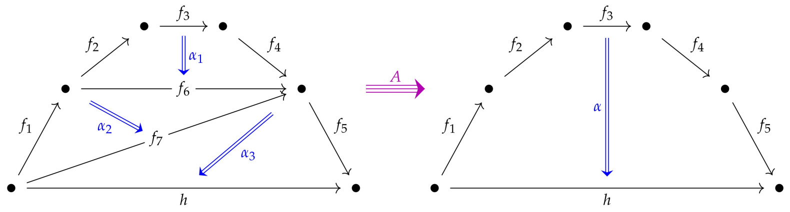
$\mathbf{3}$ = 

$\mathbf{2}$ = 

$\mathbf{1}$ = 

$\mathbf{n}$ = 

$\mathbf{0}$ = 



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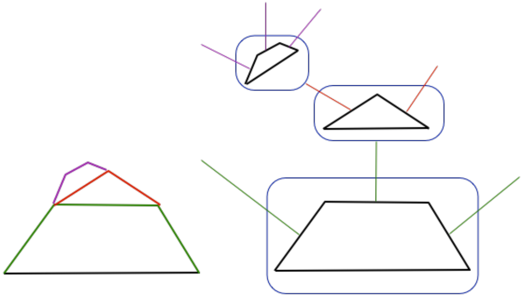
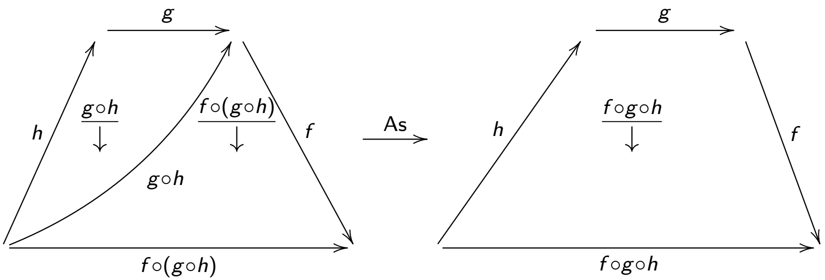
3-opetopes as trees



3-opetopes as unbiased associators

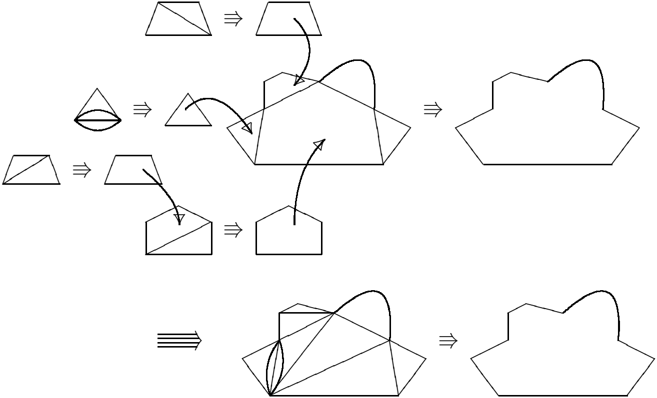


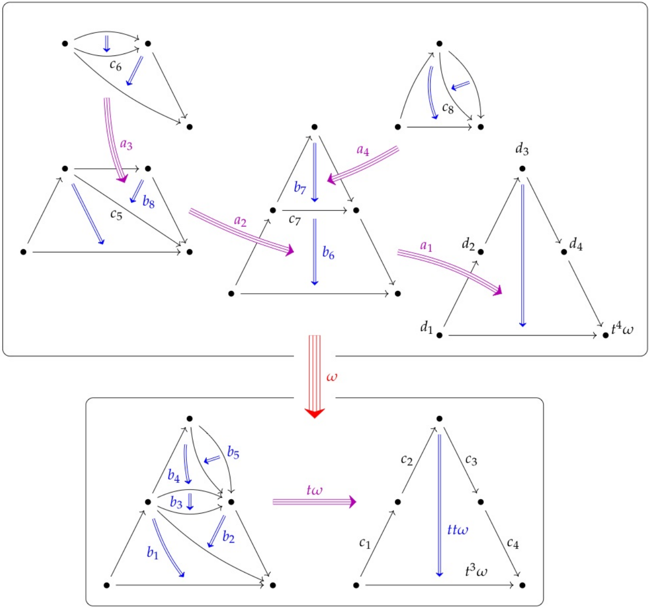
This picture features (decorated)

- 0-opetopes (unnamed)
- 1-opetopes ($f, g, h, g \circ h$,...)
- 2-opetopes (witnesses of unbiased composition $\underline{f \circ g \circ h}$,...)
- one 3-opetope (unbiased associativity)

Contrast with the biased one: $f \circ (g \circ h) = (f \circ g) \circ h$

(taken from the beautiful Lauda-Cheng notes)





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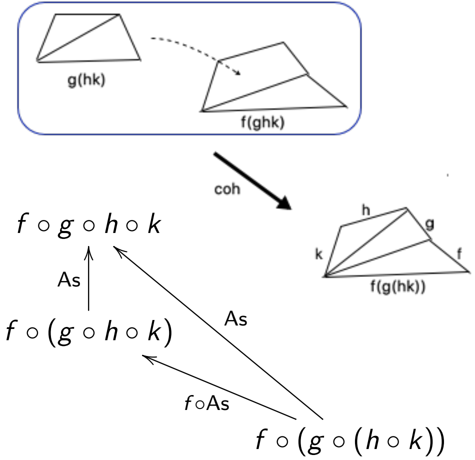
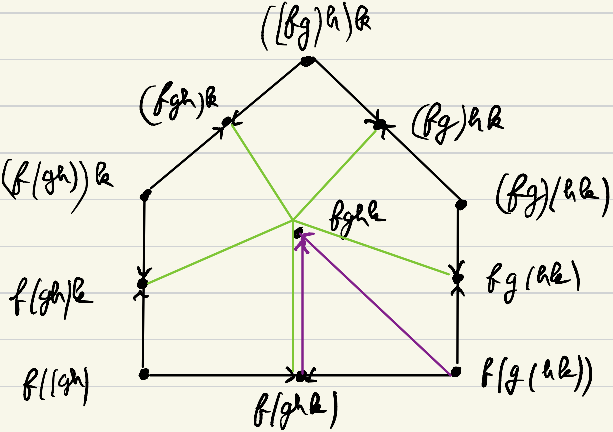
Unbiased coherence via 4-opetopes

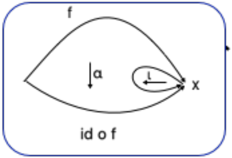


5-opetopes, etc. feature higher coherences (trees of trees of...)

Work in progress: (positive) opetopes as a triangular subdivision of (the Boardman-Vogt cubical subdivision of) associahedra



Identities via degenerate opetopes



Unit-left

This (poor) picture features

- the 2-opetope ι as a witness of the degeneracy promoting x to id_x
- the 2-opetope α as a witness of $\mathrm{id}_x \circ f$
- the 3-opetope Unit-left as the unit law $\mathrm{id}_x \circ f \to f$

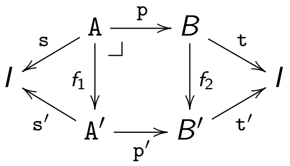
Note that ι has no sources (tree reduced to a leaf edge).

Polynomial functors (standard presentation)

Polynomial functors are triples of maps

$$I \xleftarrow{\mathtt{s}} \mathtt{A} \xrightarrow{\mathtt{p}} B \xrightarrow{\mathtt{t}} J$$

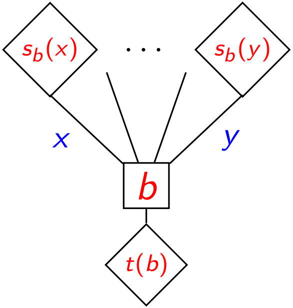
We are interested in polynomial endofunctors, i.e. $I = J$. A morphism of polynomial endofunctors is given by maps f_1, f_2 as below:



The pullback ensures that an operation b with arity $p^{-1}(b)$ is mapped to an operation with equipotent arity.

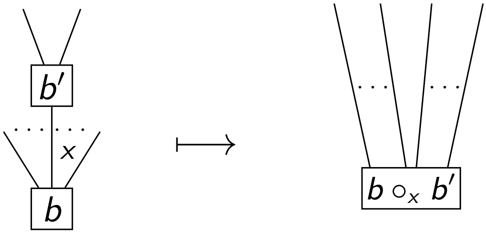
Polynomial functor (pictorially)

- We view B as a set of operations.
- For each operation b, we view $\mathtt{A}(b) = p^{-1}(b)$ as the arity of b.
- We view B as a set of colours, or of sorts (set of incoming edges).



Note the difference between names and decorations: the latter can be repeated, while the former are in bijection with the number of wires going into the operation.

Polynomial monad



Polynomial monads versus operads

Polynomial monads are a version of (set) operads that are

- Σ-free (the action of the symmetric group is free)
- non-skeletal (inputs are named, rather than numbered)
- described in the partial or "circle i" style
- coloured (or multisorted)

Note that the mechanics of polynomial functors dictates that the renaming of wires after composition be specified as part of the data defining the structure (cf. map f_1 above).

Polynomial monads are exactly the version of multicategories given by Hermida, Makkai and Power.

Free polynomial monad (trees)

Let P be a polynomial endofunctor on I. We define a new polynomial endofunctor $P^\star$ on I.

The operations are P-trees, i.e. trees with leaf edges where

- nodes are decorated by operations of P,
- incoming edges of a node decorated by b are in one-to-one correspondence with $\mathtt{A}(b)$,
- edges are decorated by colours of I

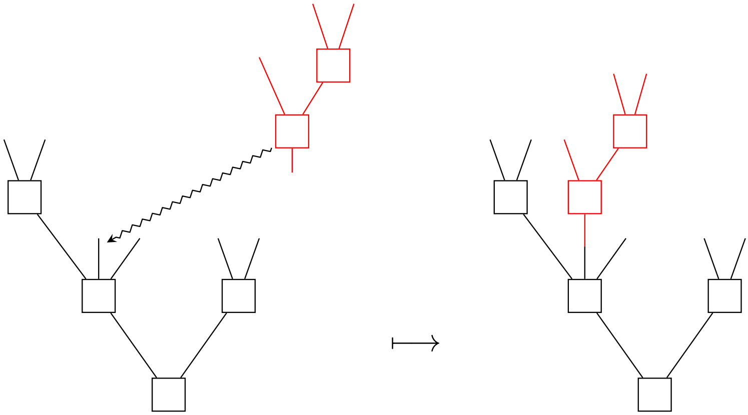
In $P^\star$:

- the arity of a tree T is the set of the occurrences of its **leaves**
- the target colour of T is the colour of the root of T

A P-tree may be reduced to a leaf (no node): we call it then degenerate.

Composition is defined by *grafting*.

The star multiplication (pictorially)



Another monad on trees: the + construction (Baez-Dolan)

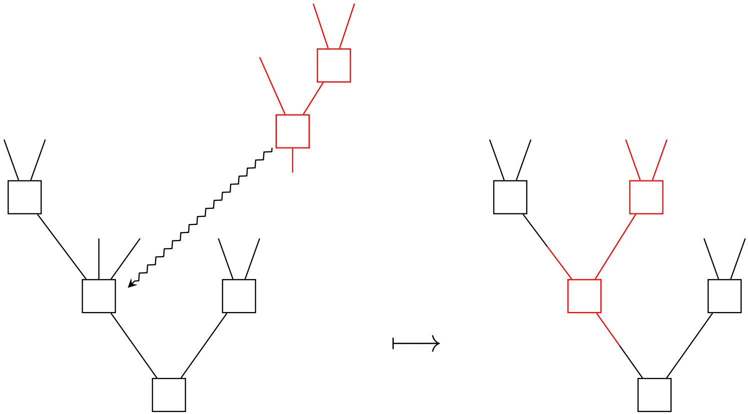
Here we follow Kock-Joyal-Batanin-Mascari 2010.

We now suppose that P is a polynomial monad on some I. Then the same P-trees give rise to another polynomial monad P^+, not on I, but on $B = B^P$:

- The arity of a tree is not its set of leaves anymore, but its set of **nodes**
- The target colour of T is $[\![T]\!]^{P^\star}$, where $[\![T]\!]$ is the evaluation of T according to the monad structure of P.
- Composition is by zooming in and *substituting* in nodes.

By iterating this construction, we shall get trees of trees of . . . !

The plus multiplication (pictorially)



Opetopes

Opetopes are defined by iteration of the $+$ construction.

- Basis $=$ identity polynomial functor $\mathcal{O}^0$on a singleton set

$$\{\blacklozenge\} \longleftarrow \{*\} \longrightarrow \{\blacksquare\} \longrightarrow \{\blacklozenge\}$$

  There is only one 0-opetope $\blacklozenge$, and there is only one 1-opetope $\blacksquare$ which has only one input $*$, decorated by the unique 0-opetope $\blacklozenge$.

- Induction: We set

$$\mathcal{O}^n = (\mathcal{O}^{n-1})^+$$

  and we write $\mathcal{O}^n$ as

$$\mathbb{O}_n \longleftarrow \mathbb{O}^\bullet_{n+1} \longrightarrow \mathbb{O}_{n+1} \longrightarrow \mathbb{O}_n$$

  (the operations of $\mathcal{O}^{n-1}$ become the colours of $\mathcal{O}^n$)

A hierarchy of shapes

An n-opetope (for $n \geq 2$) is an oriented n-dimensional volume whose boundary is divided into a pasting scheme of source $(n-1)$-opetopes and a single target $(n-1)$-opetope.

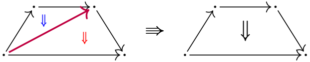
The target is determined by the pasting scheme of sources. Therefore, n-opetopes can be identified with pasting schemes of $(n-1)$-opetopes.

Pasting schemes of $(n-1)$-opetopes are described by trees whose nodes are decorated by $(n-1)$-opetopes and whose edges are decorated by $(n-2)$-opetopes.

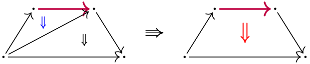
The category Ope

It has as objects all opetopes, and morphisms by generators s_x (for each node of the tree) and t, and relations

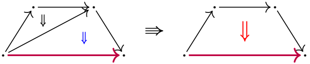
(Inner) $\quad \mathsf{s}_x \, \mathsf{s}_u = \mathsf{s}_y \, \mathsf{t} \quad$ (all edges)



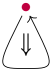
(Glob↑) $\quad \mathsf{t} \, \mathsf{s}_u = \mathsf{s}_x \, \mathsf{s}_u \quad$ (all leaves, ω non degenerate)



(Glob↓) $\quad \mathsf{s}_x \, \mathsf{t} = \mathsf{t} \, \mathsf{t} \quad$ ($x =$ root, ω non degenerate)



(Degen) $\quad \mathsf{t} \, \mathsf{s}_* = \mathsf{t} \, \mathsf{t} \quad$ (ω degenerate)



Opetopic sets are presheaves over Ope.

Polygraphs (a.k.a. computads)

A polygraph is (a presentation of) a strict ω-category (i.e. all truncations are strict n-categories). It is given by the following data:

- a set $\mathcal{P}_0$ of generating 0-cells,
- a set $\mathcal{P}_1$ of generating 1-cells, each coming with specified source and target in $\mathcal{P}_0$. This gives rise to a free strit 1-category $\mathcal{P}_1^*$ over these generators.

  ⋮

- a set $\mathcal{P}_{n+1}$ of $(n+1)$-generating cells, each coming with a specified source and target in $\mathcal{P}_n^*$. This gives rise to a free strict $(n+1)$-category $\mathcal{P}_{n+1}^*$ over these generators.

  ⋮

Many-to-one polygraphs

A polygraph is called many-to-one if for all n and $x \in \mathcal{P}_n$, we have $\mathfrak{t}\, x \in \mathcal{P}_{n-1}$ (all generating cells have as target a generating cell).

Theorem. Many-to-one polygraphs are the same thing as opetopic sets (giving rise to an equivalence of categories).

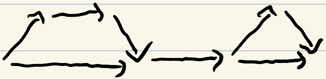
Proved independently by

- Harnik - Makkai - Zawadowski
- Cédric Ho-Thanh (relying on Henry)
- Myself (unpublished) (explicit proof!)

Henry showed that many-to-one polygraphs form a presheaf category Set[?] without an explicit description of ??

Opetopic cardinals (Zawadowski)

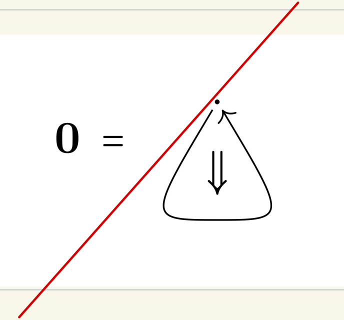
One can "concatenate" opetopes, e.g.



Thus formed opetopic cardinals form a strict ω-category

Theorem (Zawadowski) This ω-category is the terminal many-to-one polygraph

Positive opetopes



Positive-to-one only

Some results below apply (for the moment) to positive opetopes only

The many guises of opetopes

- $\bullet^0$ Abstract definition (above) $\qquad \bullet^1 \Leftrightarrow \bullet^0$ ("by construction")
- $\bullet^1$ Epiphytes (Curien – Leclerc): trees of trees of ....
- $\bullet^2$ Dendritic face complexes (Leclerc): posets of faces satisfying some (rather simple) axioms
- $\bullet^3$ (Principal) opetopic cardinals (Zawadowski): posets of faces satisfying some (quite complicated) axioms
- $\bullet^4$ Zoom complexes (Kock - Joyal - Batanin - Mascari): sequences of trees featuring

  ↑ $\bullet^4 \Leftrightarrow \bullet^0$

$\bullet^2 \Leftrightarrow \bullet^3$ (Leclerc 1) positive $\qquad \bullet^2 \Leftrightarrow \bullet^4$ (Leclerc 2) all

$\bullet^2 \Leftrightarrow \bullet^3$ (Curien – Leclerc) positive ← all (work in progress with J. Obradović)

Definition 1.1 : Rooted tree

A *rooted tree* T consist of:

- A *finite* set of nodes $T^\bullet$.
- For each node $a \in T^\bullet$, a *finite* set $A(a)$, called the arity of a.
- A (necessarily finite) set of *triplets*, denoted $a \prec_b a'$ for some $a, a' \in T^\bullet$ and $b \in A(a)$. Moreover we ask that for each $a \in T^\bullet$ and $b \in A(a)$, there is at most one triplet $a \prec_b a'$. If there is none, the pair (a, b) is said to be a *leaf* of T, and we let

$$T^{|} := \{(a, b) \text{ leaf of } T\}$$

We moreover ask for a distinguished element $\rho(T) \in T^\bullet$, called the *root* of T, satisfying the following property: for each node $a \in T^\bullet$, there is a unique *(descending) path* in T

$$a = a_0 \succ_{b_1} a_1 \succ_{b_1} \cdots \succ_{b_p} a_p = \rho(T)$$

from a to the root of T.

Definition 1.4 : neat rooted tree

Let T be a rooted tree, T will be called *neat* iff the second projection

$$\begin{array}{rrcl} \mathsf{pr}_2 : & T^{|} & \to & \bigcup_{a \in T^{\bullet}} T(a) \\ & (a, b) & \mapsto & b \end{array}$$

is injective. We then identify the leaf (a, b) with $b \in A(a)$, and let $\eta(b) := a$ (or $\eta_T(b) := a$ if needed). For a neat rooted tree T, the set $T^{|}$ will be replaced by its second projection.

POSITIVE

Definition 1.6 : Epiphyte

We define inductively *epiphytes* ω and their dimension $\mathsf{dim}(\omega)$, as follows:

- There is only one epiphyte of dimension 0, which is denoted by $\blacklozenge$. We let $\blacklozenge^{\bullet} := \emptyset$.
- Suppose that we have defined epiphytes of dimension $k \leq n$ for some $n \in \mathbb{N}$, together with their targets. Then a $(n+1)$-epiphyte ω consists in the following data:
  - A structure of neat rooted tree, which we also denote ω.
  - For each $a \in \omega^{\bullet}$, a n-epiphyte $s_a\omega$ with $(s_a\omega)^{\bullet} = A(a)$, called the *source* at a.

  Such that we have, for each triplet $a \prec_b a'$ of ω, the equality of epiphytes $s_b s_a \omega = t s_{a'} \omega$.

target derived notion!

Definition 1.5 : Positive-to-one poset

A positive-to-one poset consists of:

- A finite set of elements P.
- A gradation $\mathsf{dim} : P \to \mathbb{N}$.
- Two binary relations $\prec^-$ and $\prec^+$ on P, and we let $x \prec y$ iff $x \prec^- y$ or $x \prec^+ y$.

With the following properties:

- $\forall x, y \in P, \quad y \prec x \to \mathsf{dim}(x) = \mathsf{dim}(y) + 1$.
- $\forall x, y \in P, \quad \neg (y \prec^- x \wedge y \prec^+ x)$.
- $\forall x \in P, \quad \mathsf{dim}(x) \geq 1 \to (\exists! y, y \prec^+ x) \wedge (\exists y, y \prec^- x)$.

In particular: $\prec$, $\prec^-$ and $\prec^+$ are asymmetric, and the reflexive transitive closure of $\prec$ equips P with a structure of partially ordered set, such that dim is an increasing map.
Following the conventions of [9], for $x \in P$, we denote

$$\delta(x) := \{y \in P \mid y \prec^- x\}$$

and when $\mathsf{dim}(x) \geq 1$,

$$\gamma(x) := \{y \in P \mid y \prec^+ x\}$$

because of the third property, $\gamma(x)$ is always a singleton, hence we sometimes identify $\gamma(x)$ with its unique element, which we call the *target* of x. For $k \in \mathbb{N}$, we also denote

$$P_k := \mathsf{dim}^{-1}(\{k\}), \; P_{\geq k} := \bigcup_{i \geq k} P_i, \; P_{>k} := \bigcup_{i > k} P_i$$

and we let $\mathsf{dim}(P) := \max\{\mathsf{dim}(x)\}_{x \in P}$ be the *dimension* of P.

Definition 1.7 : Dendritic face complex

A dendritic face complex is a positive-to-one poset C, satisfying the following extra axioms:

- (*greatest element*)
  There is a greatest element in C, for the partial order induced by $\prec$.
- (*oriented thinness*)
  For $z \prec y \prec x$ in P, there is a unique $y' \neq y$ in P such that $z \prec y' \prec x$. Hence there is a lozenge as in Figure 1.1 below. Moreover, we ask for the *sign rule* $\alpha\beta = -\alpha'\beta'$ to be satisfied. When finding such a y' we say that we *complete the half lozenge* $z \prec y \prec x$.
- (*acyclicity*)
  For $x \in P_1$, $\delta(x)$ is a singleton.
  Let $x \in P_{\geq 1}$, then $\delta(x) \neq \emptyset$ and there is no cycle as in Figure 1.2 below.

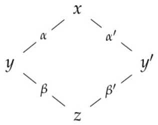


Figure 1.1: Lozenge

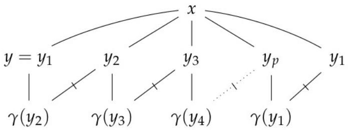
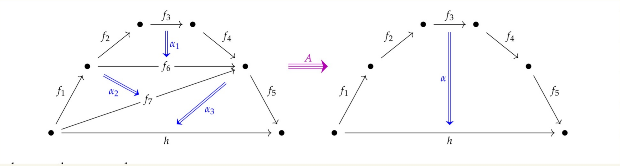
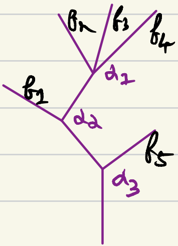
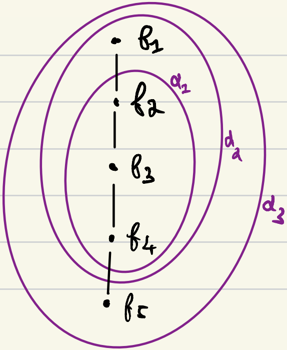


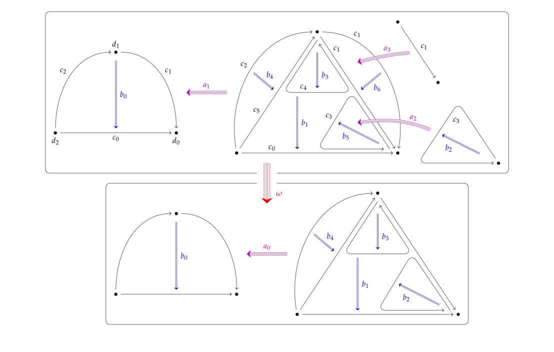
Figure 1.2: Cycle

Zoom complexes









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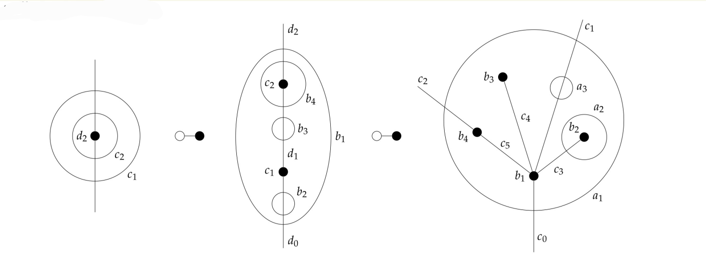

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a_2
c_4
b_2
b_4
c_5
c_3
b_1
a_1
c_0

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L. Leclerc[1] A poset-like approach to positive opetopes (submitted)

L. Leclerc[2] Two equivalent descriptions of opetopes: in terms of zoom complexes and of partial orders (submitted)

P.-L. Curien and L. Leclerc

A recursive tree-shaped definition for positive opetopes. (draft)

(under upgrading to all opetopes, in collaboration with J. Obradović)