Une preuve élémentaire
de ce que
les ensembles opétopiques
sont les
polygraphes "many-to-one"
and yet another
presentation of opetopes

PL CURIEN

(IRIF et Picube, CNRS, UPC et Inria)

GT Catégories supérieures, polygraphes et homotopie

7 octobre 2022

PART I

- Opetopes as recursively decorated trees of pouces
- target reconstruction
- Link with combinatorial descriptions (Palm - Zawadowski - Hadzihasanović

[[← Ho Thanh, with a little patch of mine making it S. Henry independent]

PART II Opetopic sets ≃ many-to-one polygraphs

("Sèvres test" copyright Marek Zawadowski)

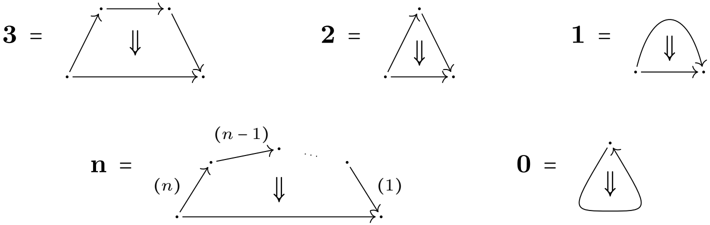
1. Picture of the Bureau International des Poids et Mesures (BIPM), Sèvres

PART I

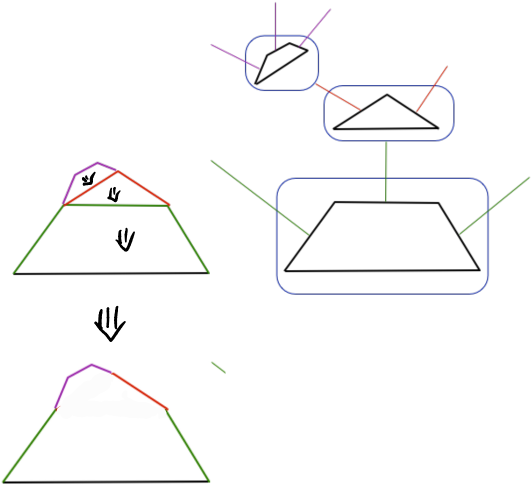
Yet another presentation of opetopes (distilled from the iterated + construction of Kock - Joyal - Batanin - Mascari)

n-opetopes (for $n \leq 2$)

- There is a unique 0-dimensional opetope: the point (an operation with no input).
- There is a unique tree of 0-opetopes, yielding the unique arrow-shaped 1-opetope.
- 1-opetopes can assemble only as linear trees, and hence 2-opetopes are in one-to-one correspondence with natural numbers:

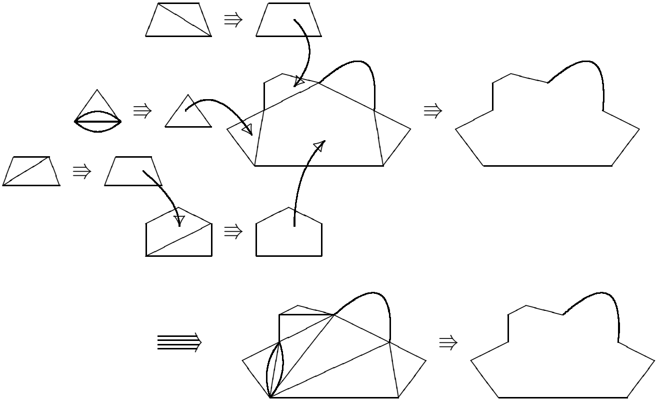


3-opetopes as trees

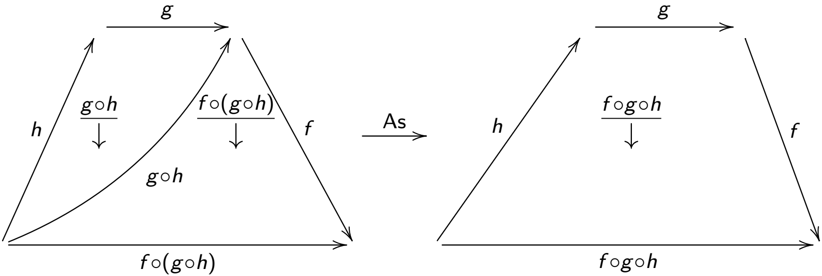


An example of 4-opetope

(taken from the beautiful Lauda-Cheng notes)



From biased to unbiased composition

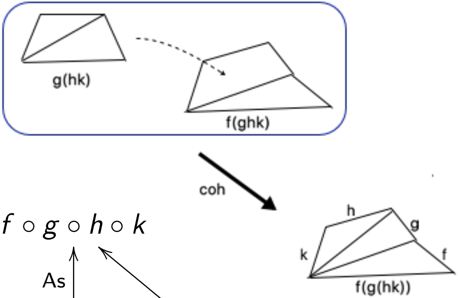


This picture features

- 0-opetopes (unnamed)
- 1-opetopes ($f, g, h, g \circ h$,...)
- 2-opetopes (witnesses of unbiased composition $\underline{f \circ g \circ h}$,...)
- one 3-opetope (unbiased associativity)

Contrast with the biased one: $f \circ (g \circ h) = (f \circ g) \circ h$

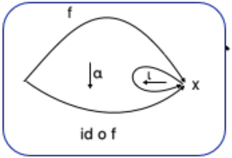
Unbiased coherence via 4-opetopes



$$\begin{tikzcd}
f \circ g \circ h \circ k & \\
f \circ (g \circ h \circ k) \arrow[u, "\mathrm{As}"] & \\
& f \circ (g \circ (h \circ k)) \arrow[ul, "f \circ \mathrm{As}"] \arrow[uul, "\mathrm{As}"']
\end{tikzcd}$$

5-opetopes, etc. feature higher coherences (trees of trees of...)

Identities via degenerate opetopes



Unit-left

This (poor) picture features

- 2-opetope ι as a witness of the degeneracy promoting x to id_x
- 2-opetope α as a witness of $\mathrm{id}_x \circ f = f$
- 3-opetope Unit-left as the unit law $\mathrm{id}_x \circ f \to f$

Note that ι has no source (tree reduced to a leaf edge).

Just like monoidal categories have many more morphisms than the canonical ones, opetopic categories will have cells with opetopic shapes, and some of them will be canonical, or universal.

A formalism of finite rooted trees with leaf edges

A rooted tree T is given by the following data:

- a non-empty finite set $\mathsf{T}^\bullet$ of nodes, containing a distinguished element, called the root node (or simply root), and denoted by $\rho(\mathsf{T})$
- for each node $\mathsf{x} \in \mathsf{T}^\bullet$, a finite set $\mathtt{A}(\mathsf{x})$, called the arity of x
- a collection of triples of the form $\mathsf{x} \prec_{\mathsf{u}} \mathsf{y}$ (called internal edges), where $\mathsf{x}, \mathsf{y} \in \mathsf{T}^\bullet$ and $\mathsf{u} \in \mathtt{A}(\mathsf{x})$
- such that each node x in $\mathsf{T}^\bullet \setminus \{\rho(T)\}$ is related to the root via a unique path of the form $\rho(\mathsf{T}) \prec_{\mathsf{u}_1} \ldots \prec_{\mathsf{u}_i} \mathsf{x}$

Note that our trees have at least one node, and may have leaf edges.

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Remarks and notation

- We define the set $\mathsf{T}^{|}$ of leaf edges (or simply leaves) of T as follows:

$$\mathsf{T}^{|} = \{(\mathsf{x},\mathsf{v}) \mid \mathsf{x} \in \mathsf{T}^{\bullet}, \mathsf{v} \in \mathtt{A}(\mathsf{x}) \setminus \{\mathsf{u} \mid \exists \mathsf{y} \;\; \mathsf{x} \prec_{\mathsf{u}} \mathsf{y}\}\}.$$

We write more visually $x \prec_{\mathsf{v}}$ for a leaf (x,v), and we often call it u, and to x as $\curlyvee(\mathsf{v})$.

- The unique path requirement entails in particular that if $\mathsf{x} \prec_{\mathsf{u}_1} \mathsf{y}$ and $\mathsf{x} \prec_{\mathsf{u}_2} \mathsf{y}$, then $\mathsf{u}_1 = \mathsf{u}_2$, so that T stripped of its leaf edges is really a tree in the usual sense of graph theory.
- We consider trees modulo renamings of their nodes and arities respecting triples.

12 Representatives of positive opetopes as iterated trees

There exists a unique positive 0-opetope, denoted by ⧫.
A representative of a positive opetope ω of dimension $n \geq 1$ is given by

- a non-empty set $\omega^\bullet$ of nodes
- the assignment of a representative of a positive $(n-1)$-opetope $\mathsf{s}_\mathsf{x}\,\omega$ for each $\mathsf{x} \in \omega^\bullet$ (the x-source of ω)
- a tree spanning $\omega^\bullet$ such that
  - $\mathtt{A}(\mathsf{x}) = (\mathsf{s}_\mathsf{x}\,\omega)^\bullet$ (for all $\mathsf{x} \in \omega^\bullet$), and
  - $\mathsf{s}_\mathsf{u}\,(\mathsf{s}_\mathsf{x}\,\omega) = \mathsf{t}\,\mathsf{s}_\mathsf{y}\,\omega$, for all triples $\mathsf{x} \prec_\mathsf{u} \mathsf{y}$ where $\mathsf{t}\,\mathsf{s}_\mathsf{y}\,\omega$ is the target of $\mathsf{s}_\mathsf{y}\,\omega$ (a derived notion defined below)

Positive opetopes

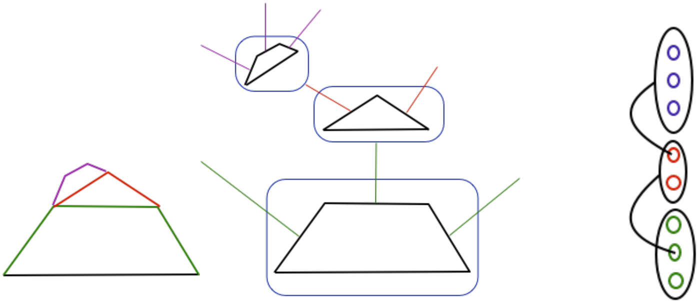
Positive opetopes are equivalence classes of representatives of positive opetopes.
We say that two representatives ω_1 and ω_2 are two witnesses of the same opetope via a bijection $\phi : \omega_1^\bullet \to \omega_2^\bullet$ if there exists a bijection

$$\psi : \bigcup_{\mathsf{u}_1 \in \omega_1^\bullet} (\mathsf{s}_{\mathsf{u}_1}\, \omega_1)^\bullet \to \bigcup_{\mathsf{u}_2 \in \omega_2} (\mathsf{s}_{\mathsf{u}_2}\, \omega_2)^\bullet$$

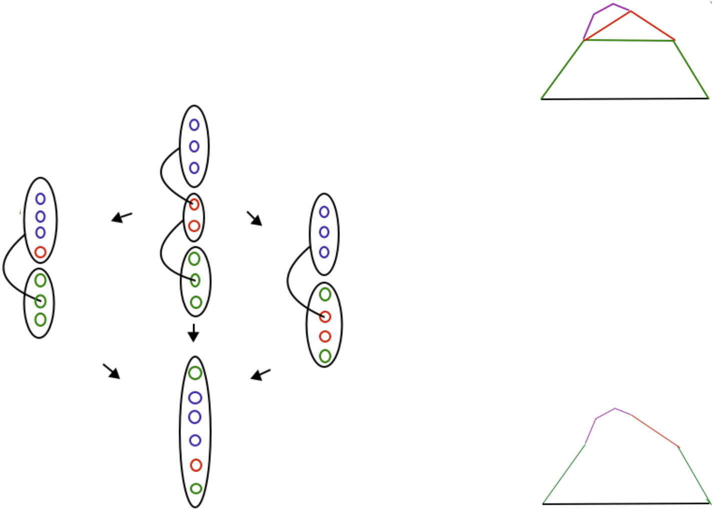
such that, for each $\mathsf{u}_1 \in \omega_1^\bullet$,

- ψ restricts and corestricts to a bijection from $(\mathsf{s}_{\mathsf{u}_1}\, \omega_1)^\bullet$ to $(\mathsf{s}_{\phi(\mathsf{u}_1)}\, \omega_2)^\bullet$, and
- $\mathsf{s}_{\mathsf{u}_1}\, \omega_1$ and $\mathsf{s}_{\phi(\mathsf{u}_1)}\, \omega_2$ are two witnesses of the same opetope via this restriction of ψ.

An example of 3-opetope

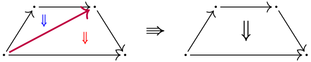


Target computation as composition (+ monad)



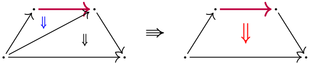
Properties of targets

(Inner) $s_u\,(s_x\,\omega) = t\,s_y\,\omega$ (all triples)

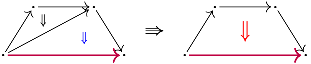


(leaf/node) $\omega^{|} = (t\,\omega)^{\bullet}$

(Glob↑) $s_u\,t\,\omega = s_u\,s_{Y^{\omega}(u)}\,\omega$ $(u \in \omega^{|})$



(Glob↓) $t\,s_{\rho(\omega)}\,\omega = t\,t\,\omega$



Target computation in terms of triplets

The target $\mathsf{t}\,\omega$ of an n-opetope $\omega = \mathsf{x}\{\mathsf{z} \leftarrow \omega_{\mathsf{z}} \mid z \in Z\}$ (which is needed for defining $(n+1)$-opetopes) is defined as follows by induction:

$$(\mathsf{t}\,\omega)^{\bullet} = ((\mathsf{s}_{\mathsf{x}}\,\omega)^{\bullet} \setminus Z) \cup (\bigcup_{\mathsf{z} \in \mathsf{Z}} (\mathsf{t}\,\omega_{\mathsf{z}})^{\bullet});$$

the assignment of $(n-2)$-opetopes to the nodes of $\mathsf{t}\,\omega$ is defined as follows:

if $\mathsf{z}' \in (\mathsf{s}_{\mathsf{x}}\,\omega)^{\bullet} \setminus \mathsf{Z}$, we set $\mathsf{s}_{\mathsf{z}'}\,\mathsf{t}\,\omega = \mathsf{s}_{\mathsf{z}'}\,\mathsf{s}_{\mathsf{x}}\,\omega$;

if $\mathsf{z}'' \in (\mathsf{t}\,\omega_{\mathsf{z}})^{\bullet}$, we set $\mathsf{s}_{\mathsf{z}''}\,\mathsf{t}\,\omega = \mathsf{s}_{\mathsf{z}''}\,\mathsf{t}\,\omega_{\mathsf{z}}$.

The triplets are those of the $\mathsf{t}\,\omega_{\mathsf{z}}$'s for z ranging over Z, plus "glueing triplets" induced by the triplets $\mathsf{z}_1 -\!\!\!\prec_{\mathsf{u}} \mathsf{z}_2$ in $\mathsf{s}_{\mathsf{x}}\,\omega$ as follows:

$$\begin{array}{ll}
\mathsf{z}_1 -\!\!\!\prec_{\mathsf{u}} \mathsf{z}_2 & \text{if } \mathsf{z}_1, \mathsf{z}_2 \notin \mathsf{Z} \\
\curlyvee^{\mathsf{t}\,\omega_{\mathsf{z}_1}}(\mathsf{u}) -\!\!\!\prec_{\mathsf{u}} \rho(\mathsf{t}\,\omega_{\mathsf{z}_2}) & \text{if } \mathsf{z}_1, \mathsf{z}_2 \in \mathsf{Z} \\
\curlyvee^{\mathsf{t}\,\omega_{\mathsf{z}_1}}(\mathsf{u}) -\!\!\!\prec_{\mathsf{u}} \mathsf{z}_2 & \text{if } \mathsf{z}_1 \in \mathsf{Z}, \mathsf{z}_2 \notin \mathsf{Z} \\
\mathsf{z}_1 -\!\!\!\prec_{\mathsf{u}} \rho(\mathsf{t}\,\omega_{\mathsf{z}_2}) & \text{if } \mathsf{z}_1 \notin \mathsf{Z}, \mathsf{z}_2 \in \mathsf{Z}
\end{array}$$

LINKING UP with opetopes à la

Palm - Zawadowski - Hadzihasanović

Oriented graded posets (Hadzihasanović)

Let $(P, \leq)$ be a finite partial order. We say that y covers x if

$$x \neq y \quad \text{and} \quad (x \leq z \leq y \Rightarrow z = x \text{ or } z = y)$$

We impose the following structure + property.

- Grading. Every $x \in P$ comes with a dimension $\dim(x)$, such that
  - minimal elements have dimension 0
  - whenever y covers x we have $\dim(y) = \dim(x) + 1$
- Orientation. Every pair s.t. y covers x is given an orientation $+$ or $-$
- Oriented thinness. Whenever we have that y covers z and z covers x, there exists a unique $z' \neq z$ filling the following lozenge

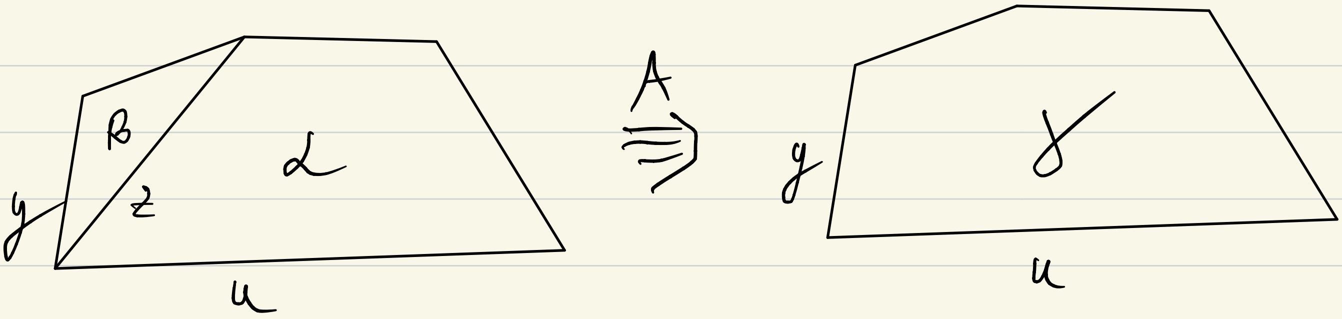
$$\begin{tikzcd} & y \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\ z \arrow[dr, "\beta_1"'] & & z' \arrow[dl, "\beta_2"] \\ & x & \end{tikzcd}$$

and moreover $\alpha_1\beta_1 = -\alpha_2\beta_2$

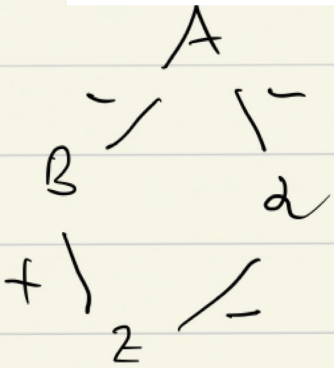
(up to symmetry, four configurations; we shall discard $\alpha_1 = + = \alpha_2$)

Understanding oriented thinness

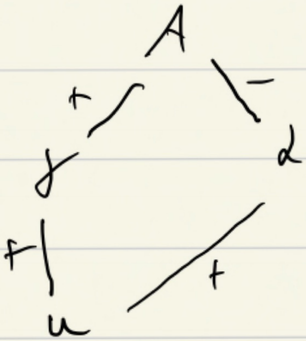
$$\begin{tikzcd}
 & y \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & \\
z \arrow[dr, "\beta_1"'] & & z' \arrow[dl, "\beta_2"] \\
 & x &
\end{tikzcd}$$



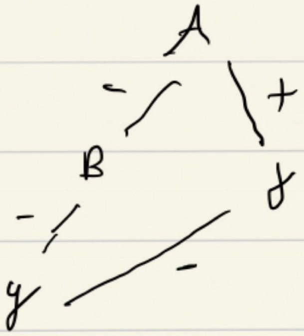
(Inner)



(Glob↑)



(Glob↓)



From iterated trees to posets (preparations)

Given an opetope ω, we define the following set recursively. The sloppy definition is

$$\omega^* = \omega^\bullet \cup \bigcup_{\mathsf{x} \in \omega^\bullet} (\mathsf{s}_\mathsf{x}\, \omega)^*$$

The more careful definition is that $\omega^* \setminus \{\omega\}$ is the colimit of the diagram (in ***Set***) formed by all pairs of inclusions

$$(\mathsf{s}_\mathsf{u}\, \omega)^* \subseteq (\mathsf{s}_\mathsf{x}\, \omega)^* \quad \text{and} \quad (\mathsf{s}_\mathsf{u}\, \omega)^* \subseteq (\mathsf{s}_\mathsf{x}\, \omega)^*$$

for each triple $\mathsf{x} \prec_\mathsf{u} \mathsf{y}$.

This ensures that all elements in ω^* name different (iterated faces) of ω.

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From iterated trees to posets (step 1)

Step 1.

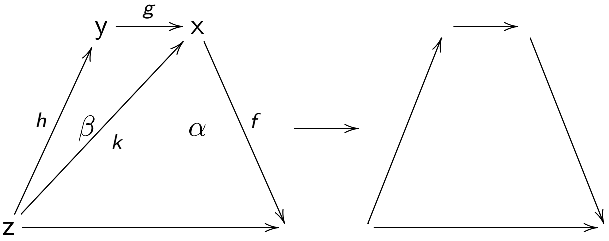
We endow ω^* with a structure of graded oriented poset where all edges have a negative orientation:

$$\mathsf{u} <^- \mathsf{x} \quad \text{when} \quad \mathsf{u} \in (\mathsf{s}_\mathsf{x}\, \omega)^\bullet$$

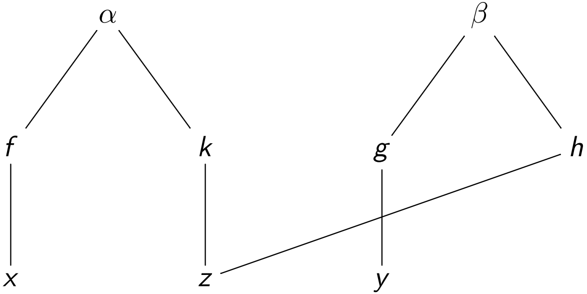
Then we define the partial order as the reflexive transitive closure of $<^-$.

We shall complete this partial order in three more steps.

Example



Example (step 1)



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From iterated trees to posets (steps 2 and 3)

Step 2. We add $n+1$ elements to ω^*, which are named $.\omega, \mathsf{t}.\omega, \mathsf{t}.\mathsf{t}.\omega \dots \mathsf{t}^n.\omega$, i.e.,

$$P_\omega = \omega^* \cup \{\mathsf{t}^i.\omega \mid 0 \leq i \leq n\},$$

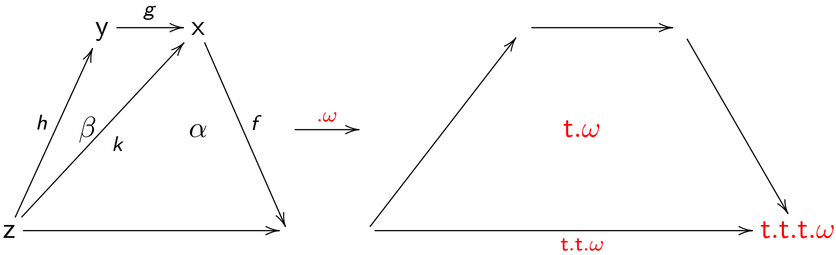
and we associate an opetope with each of the elements of P_ω as follows

$$\begin{aligned} &\mathsf{f}_{\mathsf{z}}\, \omega = \mathsf{s}_{\mathsf{z}}\, \omega && (\mathsf{z} \in \omega^*) \\ &\mathsf{f}_{.\omega}\, \omega = \omega && \\ &\mathsf{f}_{\mathsf{t}^i.\omega}\, \omega = \mathsf{t}\, \mathsf{f}_{\mathsf{t}^{i-1}.\omega}\, \omega && \end{aligned}$$

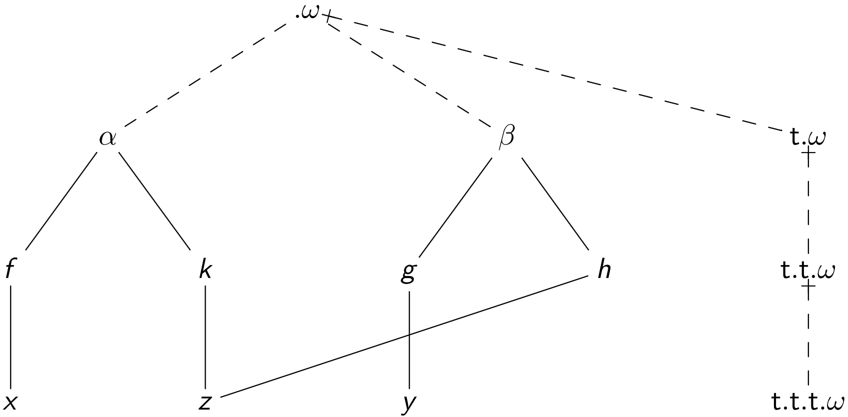
Step 3. We add the following oriented edges to the Hasse diagram of our partial order:

$$\begin{aligned} &\mathsf{t}^i.\omega <^+ \mathsf{t}^{i-1}.\omega && (1 \leq i \leq n) \\ &\mathsf{x} <^- .\omega && (\mathsf{x} \in \omega^\bullet) \end{aligned}$$

Example



Example (steps 2 and 3)



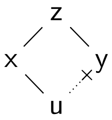
From iterated trees to posets (step 4)

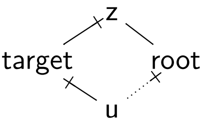
Step 4 (oriented thinness).

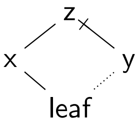
(Inner) For $z \in P_\omega$, if $x \mathrel{-}\!\!\prec^{f_z\,\omega}_{u} y$, then we add $u <^+ y$ to the oriented poset:

(Glob↓) For $z \in P_\omega$, if $u <^+ x <^+ z$ and if $y = \rho(f_z\,\omega)$, then we add $u <^+ y$:

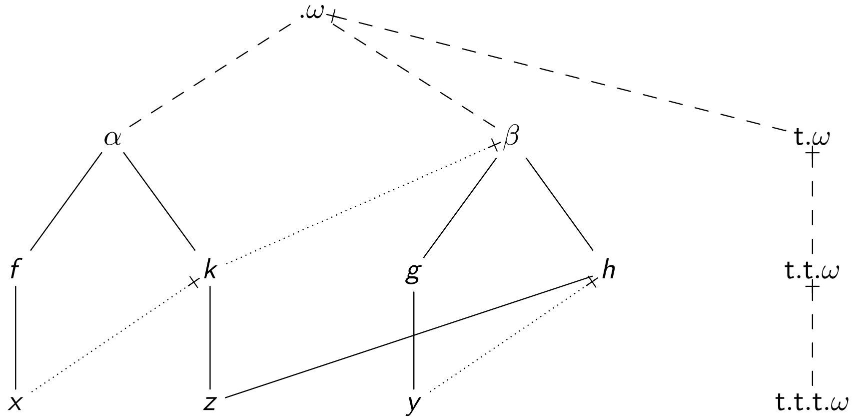
(Glob↑) For $z \in P_\omega$, if $u <^- x <^- z$ and $u \in (f_z\,\omega)^{|}$, and if $y <^+ z$, then we add $u <^- y$



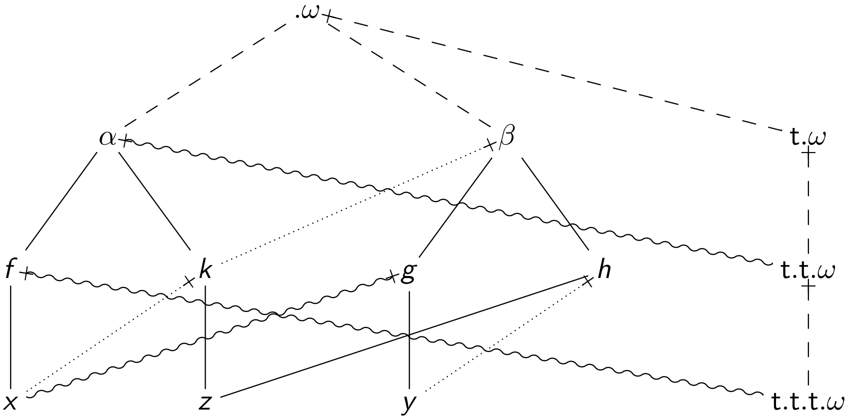




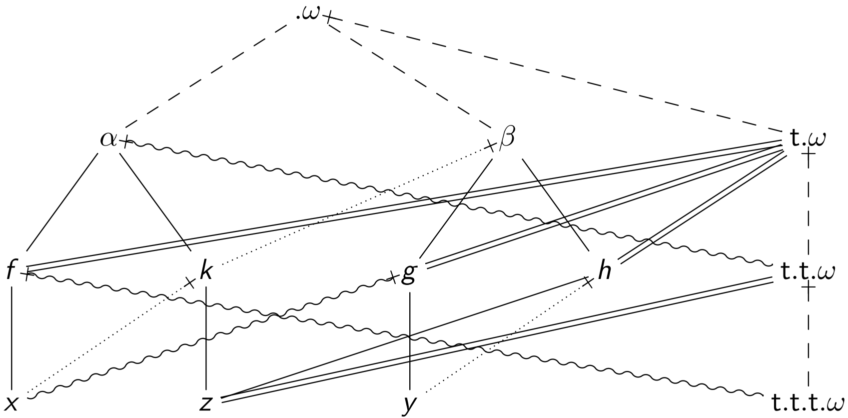
Example (step 4 – Innner)



Example (step 4 – Glob↓)



Example (step 4 – Glob↑)



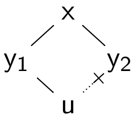
Positive opetopes in this framework

This gives a bijection between positive opetopes and oriented graded posets P s.t.

- P has a maximum element
- for every x in P of dimension ≥ 1 there exists exactly one element u such that $\mathrm{u} <^{+} \mathrm{x}$. We write $\mathrm{u} = \mathrm{t.x}$; we also use the following notation ($\mathrm{cl}(\_) =$ downward closure):

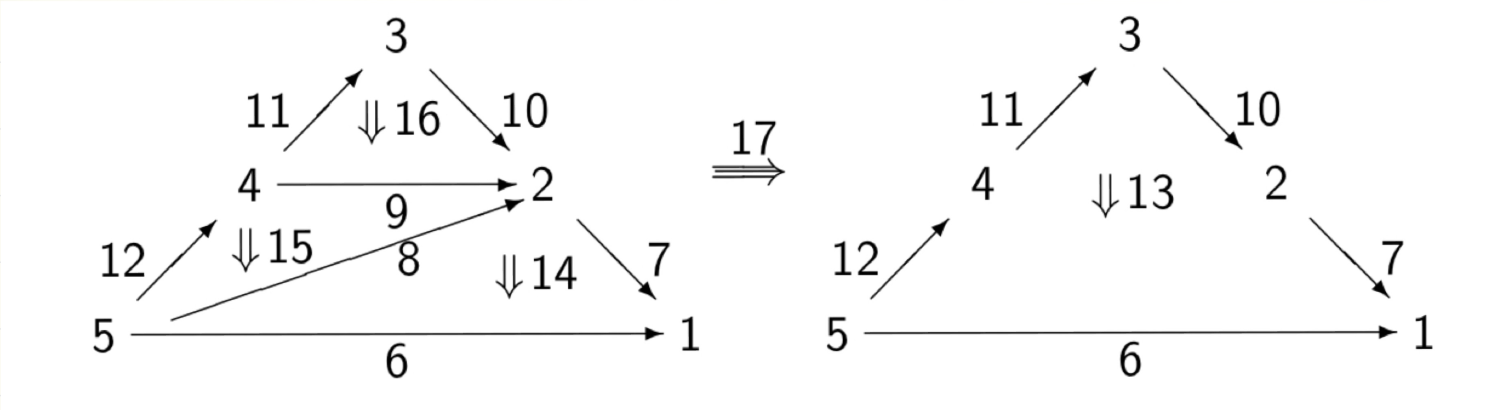
$$\Delta^{+}(\mathrm{x}) = \{\mathrm{t.x}\} \qquad \Delta^{-}(\mathrm{x}) = \{\mathrm{y} \mid \mathrm{y} <^{-} \mathrm{x}\} \qquad \partial^{\alpha}(\mathrm{x}) = \mathrm{cl}(\Delta^{\alpha}(\mathrm{x}))$$

- P satisfies oriented thinness
- for each x in P, $\{\mathrm{y} \mid \mathrm{y} <^{-} \mathrm{x}\}$, can be organised as a tree with leaf edges, as follows: $\mathtt{A}(\mathrm{y}) = \{\mathrm{u} \mid \mathrm{u} <^{-} \mathrm{y}\}$, and $\mathrm{y}_1 \prec_{\mathrm{u}} \mathrm{y}_2$ whenever

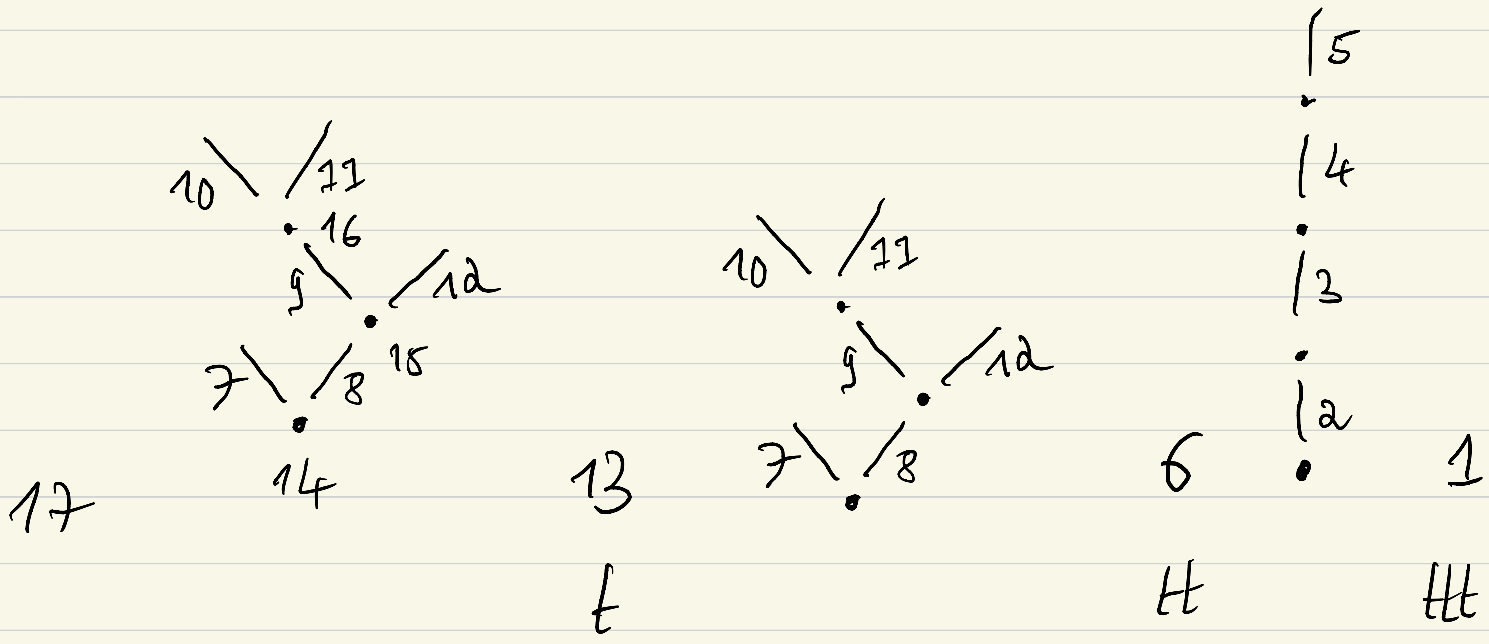


- For all such $\mathrm{y}_1 \prec_{\mathrm{u}} \mathrm{y}_2$, we have $\mathrm{cl}(\mathrm{y}_1) \cap \mathrm{cl}(\mathrm{y}_2) = \mathrm{cl}(\mathrm{u})$.

Enumerating the faces of opetopes



← copied from Marek's talk



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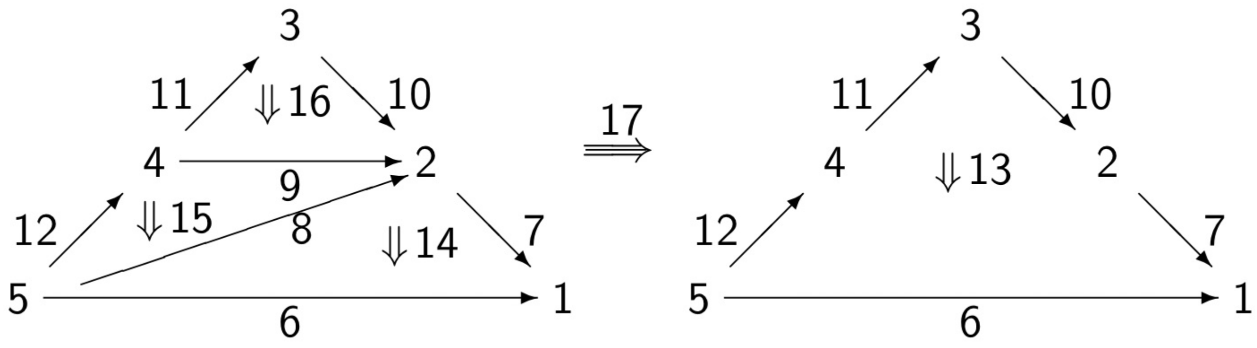
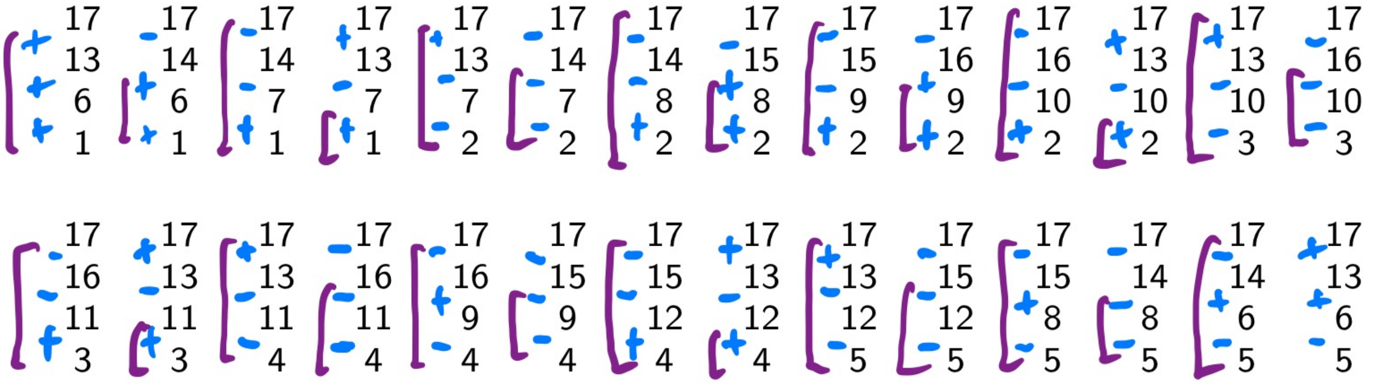
Palm's flags and hamiltonian path





17	17	17	17	17	17	17	17	17	17	17	17	17	17
+	−	−	+	+	−	−	−	−	−	−	+	+	−
13	14	14	13	13	14	14	15	15	16	16	13	13	16
+	+	−	−	−	−	−	+	−	+	−	−	−	−
6	6	7	7	7	7	8	8	9	9	10	10	10	10
+	+	+	+	−	−	+	+	+	+	+	+	−	−
1	1	1	1	2	2	2	2	2	2	2	2	3	3

17	17	17	17	17	17	17	17	17	17	17	17	17	17
−	+	+	−	−	−	−	+	+	−	−	−	−	+
16	13	13	16	16	15	15	13	13	15	15	14	14	13
−	−	−	−	+	−	−	−	−	−	+	−	+	+
11	11	11	11	9	9	12	12	12	12	8	8	6	6
+	+	−	−	−	−	+	+	−	−	−	−	−	−
3	3	4	4	4	4	4	4	5	5	5	5	5	5

From positive opetopes to opetopes (a snapshot)

Each $(n-2)$-opetope ν gives rise to a degenerate n-opetope $\{\{\nu$.

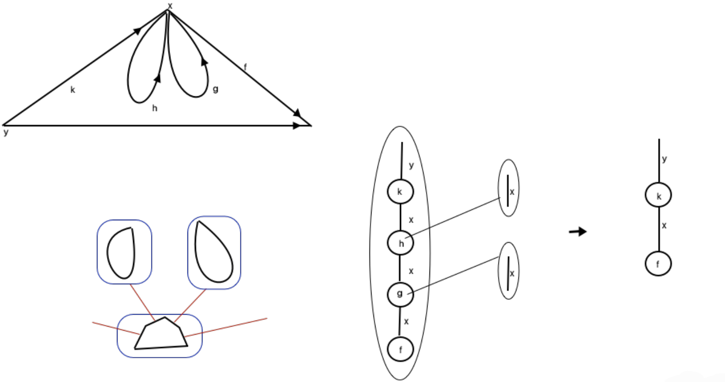
We set

$$\mathsf{t}\ \{\{\nu\ = \mathsf{shift}(\nu)$$

drawn as $|\nu$, where $|$ is the "exceptional tree reduced to one leaf)

(the $(n-2)$ tree reduced to one node $*$, such that $\mathsf{s}_*\,(\mathsf{shift}(\nu)) = \nu$).

Source opetopes are allowed to be degenerate:



Target computation upgraded

Fore general opetopes, the definition of target above in terms of triplets needs to be adjusted as follows. The description is exactly the same as given above for positive opetopes, except for the **specification of the triplets** inducing the glueing triplets in the target (the definition of those being also unchanged). Instead of all triplets $\mathsf{z}_1 \mathrel{-\!\!\prec}_{\mathsf{u}} \mathsf{z}_2$ in $\mathsf{s}_{\mathsf{x}}\,\omega$, we now consider all sequences ($p \geq 0$)

$$\mathsf{z}_1 \mathrel{-\!\!\prec}_{\mathsf{u}} \mathsf{z}_1' \mathrel{-\!\!\prec}_{\mathsf{u}_1} \ldots z_p' \mathrel{-\!\!\prec}_{\mathsf{u}_{p-1}} \mathrel{-\!\!\prec}_{\mathsf{u}_p} \mathsf{z}_2$$

such that

$$\mathsf{s}_{z_1'}\,\mathsf{s}_x\,\omega = \cdots = \mathsf{s}_{z_1'}\,\mathsf{s}_x\,\omega = \{\{\mathsf{s}_u\,\mathsf{s}_{z_1}\,\mathsf{s}_x\,\omega$$

An abstract machine for target computation

The machine produces complete branches of $\mathsf{t}\,\omega$.
Machine states are of four forms:

$$[\langle P\rangle \ \| \ S]{\uparrow} \qquad , \qquad [\langle P\rangle \ \| \ S]? \qquad , \qquad [\langle Q\rangle \ \| \ S]! \qquad \text{and} \qquad \langle P\rangle!,$$

where

- $P ::= \epsilon \,\|\, (P\, f \prec_x)$
- S is a stack of pairs (f, α), with $\alpha \in \omega^\bullet$ and $f \in (\mathsf{s}_{[\alpha]}\,\omega)^\bullet$.

The initial state is

$$[\langle \epsilon\rangle \ \| \ (\rho(\mathsf{s}_{\rho(\omega)}\,\omega), \rho(\omega))]{\uparrow}$$

The respective kinds of state have the following meaning:

$[\langle P\rangle \ \| \ S]?$	going up the tree of ω searching for the next opetope in the explored branch of $\mathsf{t}\,\omega$
$[\langle P\, g\rangle \ \| \ S]!$	the machine has just found one
$[\langle P\rangle \ \| \ S]?$	going down the tree of ω to find the next branch of ω in which to go up again (if any)
$\langle P\rangle!$	final state (branch completed)

The rules

$$\frac{\alpha \mathrel{-\!\!\prec}^{\omega}_{f} \{\{x \quad , \quad f = \mathrm{shift}(x)}{[\langle P \rangle \parallel (f, \alpha) \cdot S]{\uparrow} \longrightarrow [\langle P \rangle \parallel (f, \alpha) \cdot S]?} \ \text{degenerate}$$

$$\frac{\alpha \mathrel{-\!\!\prec}^{\omega}_{f} \beta}{[\langle P \rangle \parallel (f, \alpha) \cdot S]{\uparrow} \longrightarrow [\langle P \rangle \parallel (\rho(\mathsf{s}_{\rho(\beta)} \beta), \beta) \cdot (f, \alpha) \cdot S]{\uparrow}} \ \text{up}$$

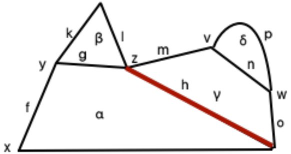
$$\frac{f \in \omega^{|}}{[\langle P \rangle \parallel (f, \alpha) \cdot S]{\uparrow} \longrightarrow [\langle P \, f \rangle \parallel (f, \alpha) \cdot S]!} \ \text{leaf}$$

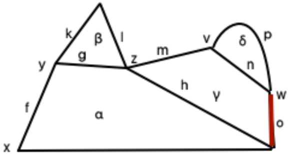
$$\frac{x \in (\mathsf{s}_f \, \mathsf{s}_\alpha \, \omega)^{\bullet}}{[\langle Q \rangle \parallel (f, \alpha) \cdot S]! \longrightarrow [\langle Q \mathrel{-\!\!\prec}_x \rangle \parallel (f, \alpha) \cdot S]?} \ \text{explore}$$

$$\frac{x \in (\mathsf{s}_\alpha \, \omega)^{|}}{[\langle Q \mathrel{-\!\!\prec}_x \rangle \parallel (f, \alpha) \cdot S]? \longrightarrow [\langle Q \mathrel{-\!\!\prec}_x \rangle \parallel S]?} \ \text{down}$$

$$\frac{f \mathrel{-\!\!\prec}^{\mathsf{s}_\alpha \omega}_{x} g}{[\langle Q \mathrel{-\!\!\prec}_x \rangle \parallel (f, \alpha) \cdot S]? \longrightarrow [\langle Q \mathrel{-\!\!\prec}_x \rangle \parallel (g, \alpha) \cdot S]{\uparrow}} \ \text{next}$$

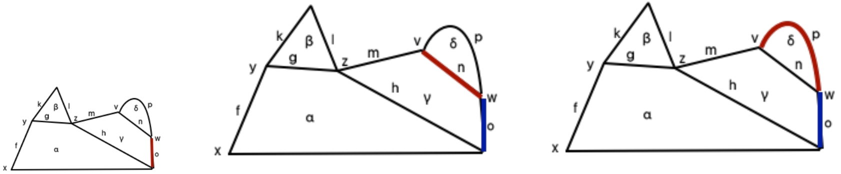
Example execution (steps 1-2)





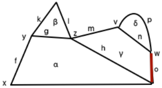
$$\begin{array}{lcll} [\langle\epsilon\rangle \parallel (h,\alpha)]\uparrow & \longrightarrow & [\langle\epsilon\rangle \parallel (o,\gamma)\cdot(h,\alpha)]\uparrow & \text{(up)}\\ & \longrightarrow & [\langle o\rangle \parallel (o,\gamma)\cdot(h,\alpha)]! & \text{(leaf)} \end{array}$$

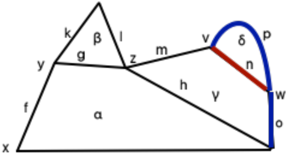
Example execution (steps 3-6)

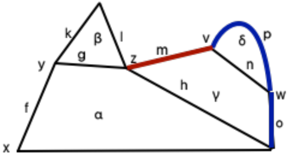


$$
\begin{array}{lll}
\longrightarrow & [\langle o \prec_w \rangle \parallel (o,\gamma)\cdot(h,\alpha)]? & \text{(explore)} \\
\longrightarrow & [\langle o \prec_w \rangle \parallel (n,\gamma)\cdot(h,\alpha)]\uparrow & \text{(next)} \\
\longrightarrow & [\langle o \prec_w \rangle \parallel (p,\delta)\cdot(n,\gamma)\cdot(h,\alpha)]\uparrow & \text{(up)} \\
\longrightarrow & [\langle o \prec_w p \rangle \parallel (p,\delta)\cdot(n,\gamma)\cdot(h,\alpha)]! & \text{(leaf)}
\end{array}
$$

Example execution (steps 7-10)

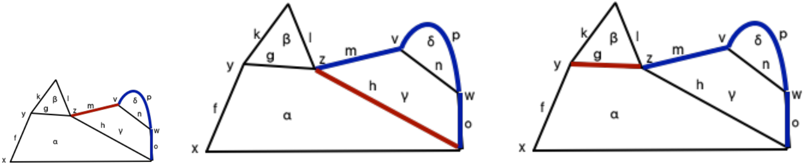






$$\begin{array}{lll}
\longrightarrow & [\langle o \prec_w p \prec_v \rangle \parallel (p,\delta)\cdot(n,\gamma)\cdot(h,\alpha)]? & \text{(explore)} \\
\longrightarrow & [\langle o \prec_w p \prec_v \rangle \parallel (n,\gamma)\cdot(h,\alpha)]? & \text{(down)} \\
\longrightarrow & [\langle o \prec_w p \prec_v \rangle \parallel (m,\gamma)\cdot(h,\alpha)]\uparrow & \text{(next)} \\
\longrightarrow & [\langle o \prec_w p \prec_v m \rangle \parallel (m,\gamma)\cdot(h,\alpha)]! & \text{(leaf)}
\end{array}$$

Example execution (steps 11-13)

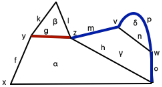


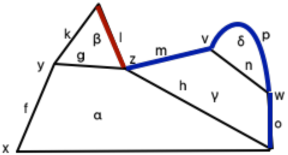
$$\longrightarrow \quad [\langle o \mathrel{-\!\!\!\prec}_w p \mathrel{-\!\!\!\prec}_v m \mathrel{-\!\!\!\prec}_z \rangle \parallel (m,\gamma)\cdot(h,\alpha)]? \qquad \text{(explore)}$$

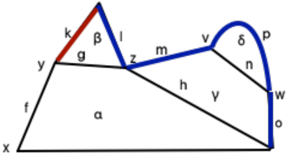
$$\longrightarrow \quad [\langle o \mathrel{-\!\!\!\prec}_w p \mathrel{-\!\!\!\prec}_v m \mathrel{-\!\!\!\prec}_z \rangle \parallel (h,\alpha)]? \qquad \text{(down)}$$

$$\longrightarrow \quad [\langle o \mathrel{-\!\!\!\prec}_w p \mathrel{-\!\!\!\prec}_v m \mathrel{-\!\!\!\prec}_z \rangle \parallel (g,\alpha)]\uparrow \qquad \text{(next)}$$

Example execution (steps 14-18)

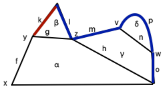


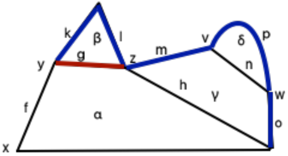


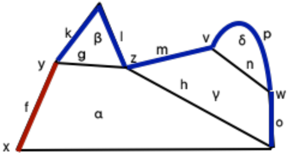


$$
\begin{array}{lll}
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z \rangle \parallel (l,\beta)\cdot(g,\alpha)]\uparrow & \text{(up)} \\
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z l \rangle \parallel (l,\beta)\cdot(g,\alpha)]! & \text{(leaf)} \\
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z l \prec_u \rangle \parallel (l,\beta)\cdot(g,\alpha)]? & \text{(explore)} \\
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z l \prec_u \rangle \parallel (k,\beta)\cdot(g,\alpha)]\uparrow & \text{(next)} \\
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \rangle \parallel (k,\beta)\cdot(g,\alpha)]! & \text{(leaf)}
\end{array}
$$

Example execution (steps 19-22)







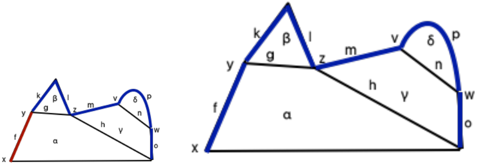
$$\longrightarrow \quad [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y \rangle \parallel (k, \beta) \cdot (g, \alpha)]? \qquad \text{(explore)}$$

$$\longrightarrow \quad [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y \rangle \parallel (g, \alpha)]? \qquad \text{(down)}$$

$$\longrightarrow \quad [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y \rangle \parallel (f, \alpha)]\uparrow \qquad \text{(next)}$$

$$\longrightarrow \quad [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y f \rangle \parallel (f, \alpha)]! \qquad \text{(leaf)}$$

Example execution (steps 23-25)



$$
\begin{array}{lll}
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y f \prec_x \rangle \parallel (f, \alpha)]? & \text{(explore)} \\
\longrightarrow & [\langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y f \prec_x \rangle \parallel \emptyset]? & \text{(down)} \\
\longrightarrow & \langle o \prec_w p \prec_v m \prec_z l \prec_u k \prec_y f \prec_x \rangle ! & \text{(final)}
\end{array}
$$

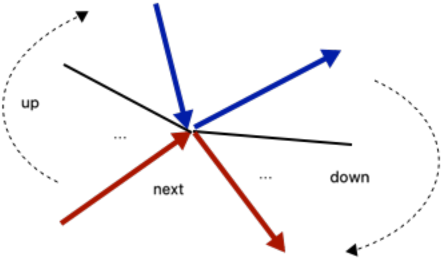
46

Invariants of the machine

- Two successives items (f, α), (g, β) in the stack are always such that g is the target of α.
- The sequence of states of the machine between two successive (leaf) moves is always of the form

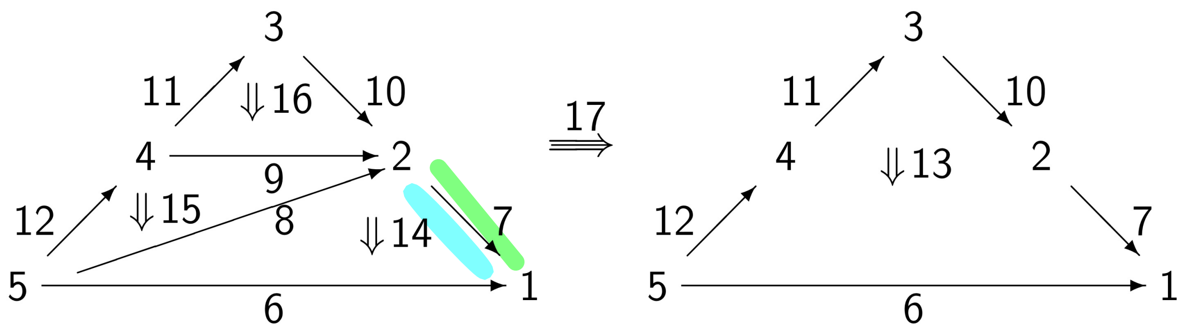
$$\text{(explore) (down)}^{m-1} \text{ (next) (up)}^{n-1}$$

with $m, n \geq 1$. Graphically:

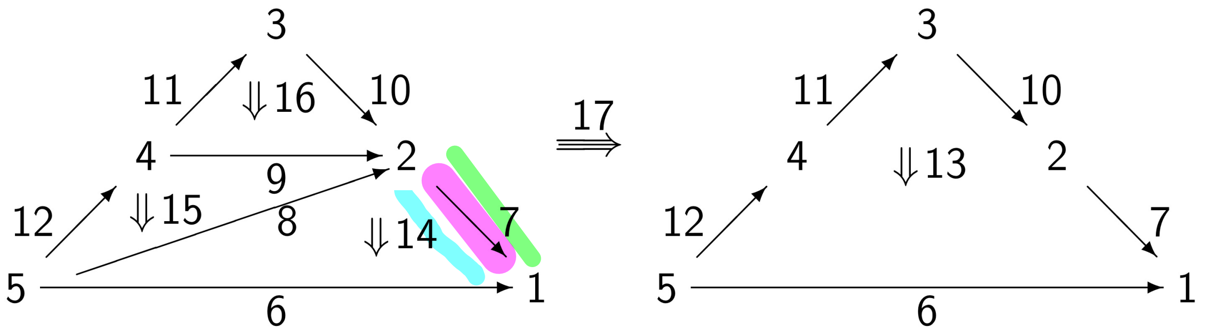


(More on this picture at the end of the talk)

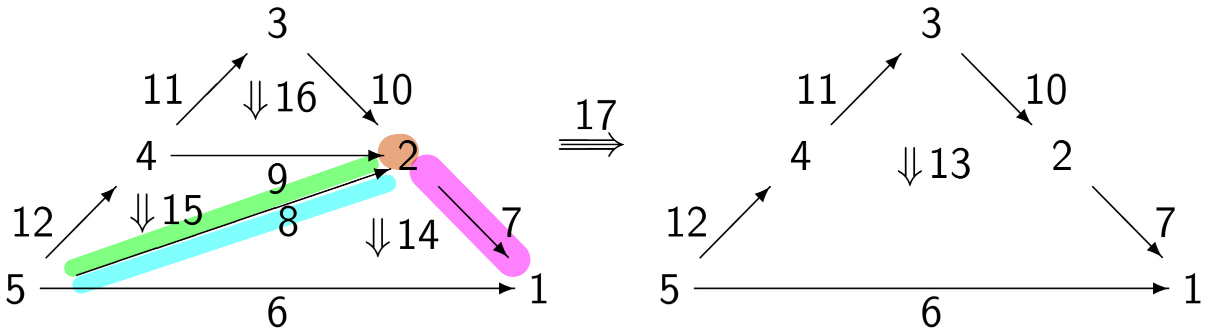
↑



LEAF

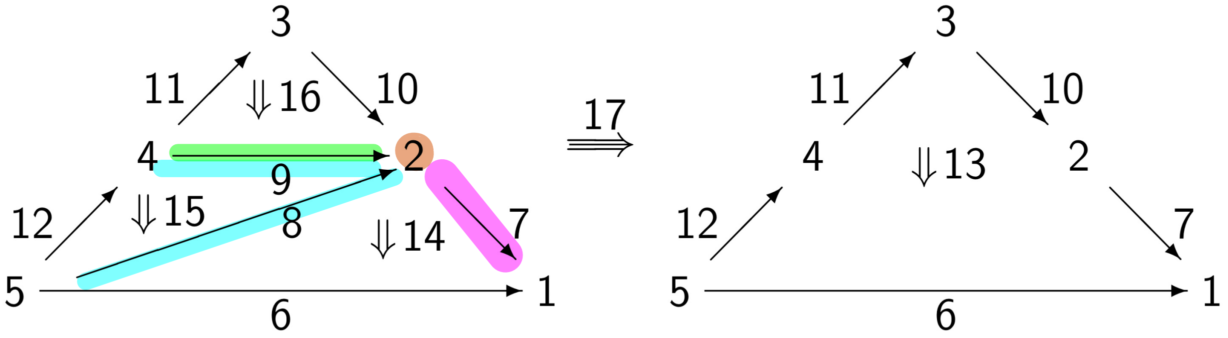


EXPLORE + NEXT

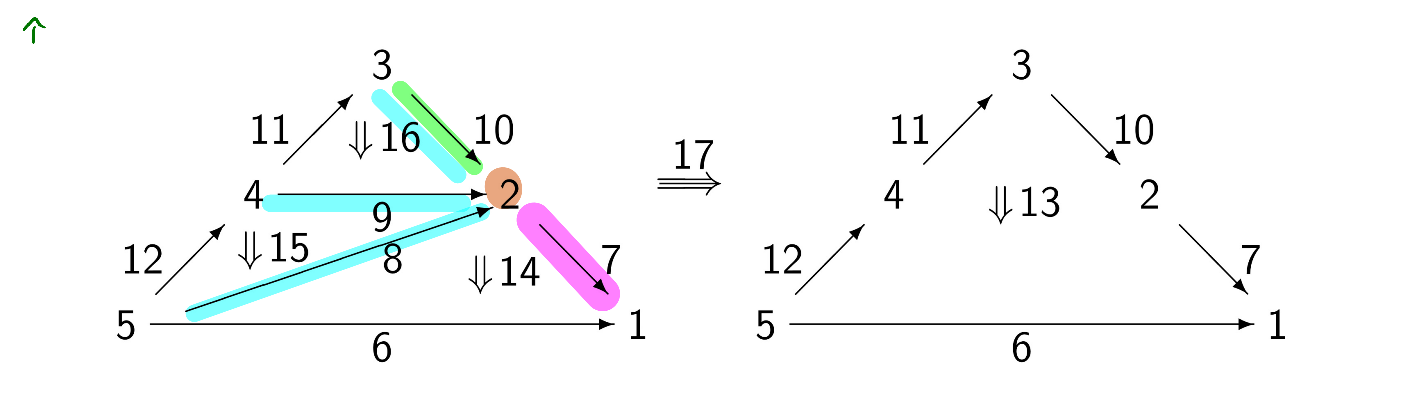


UP

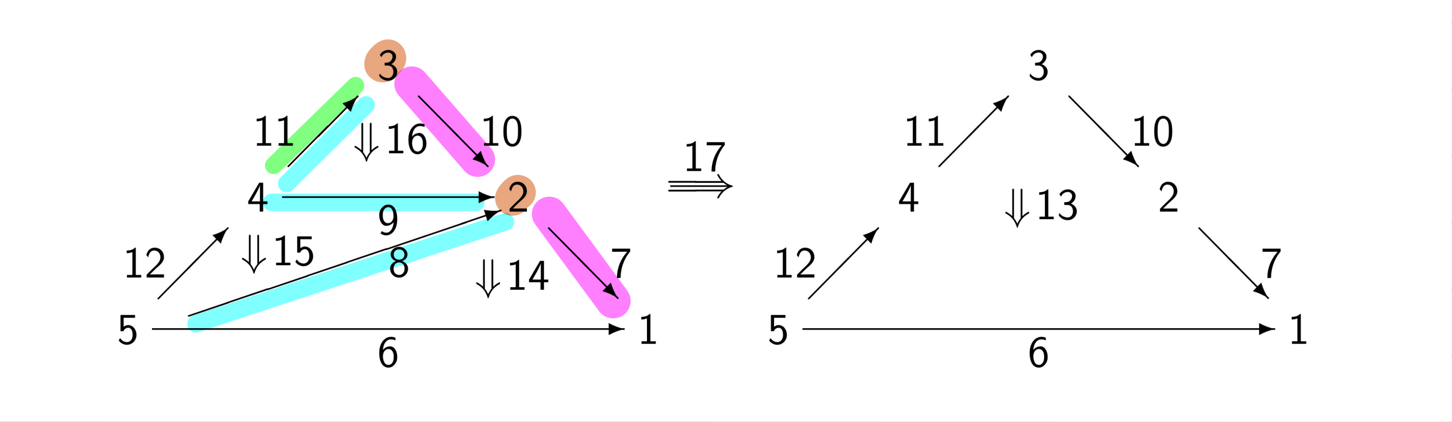
↑



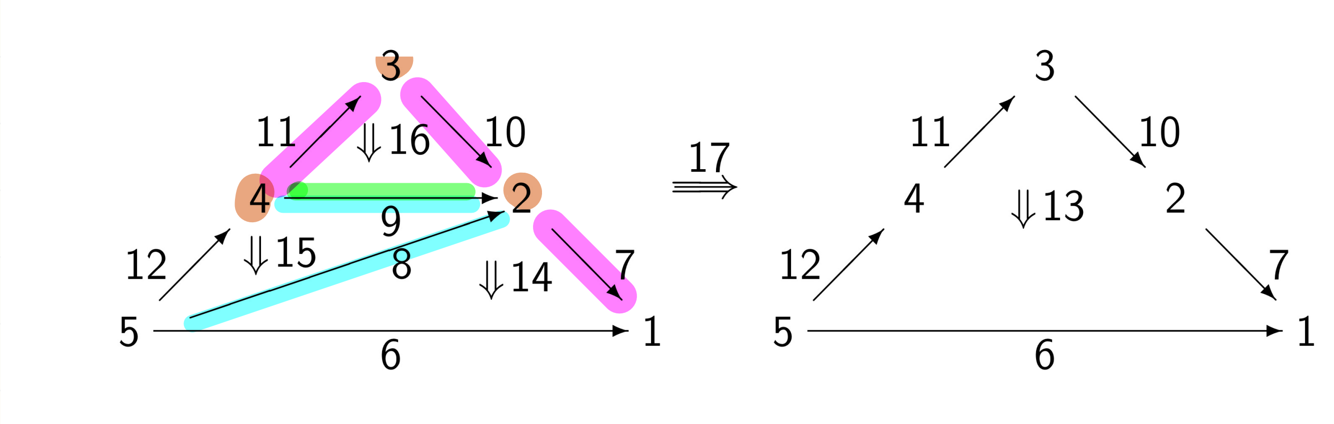
UP



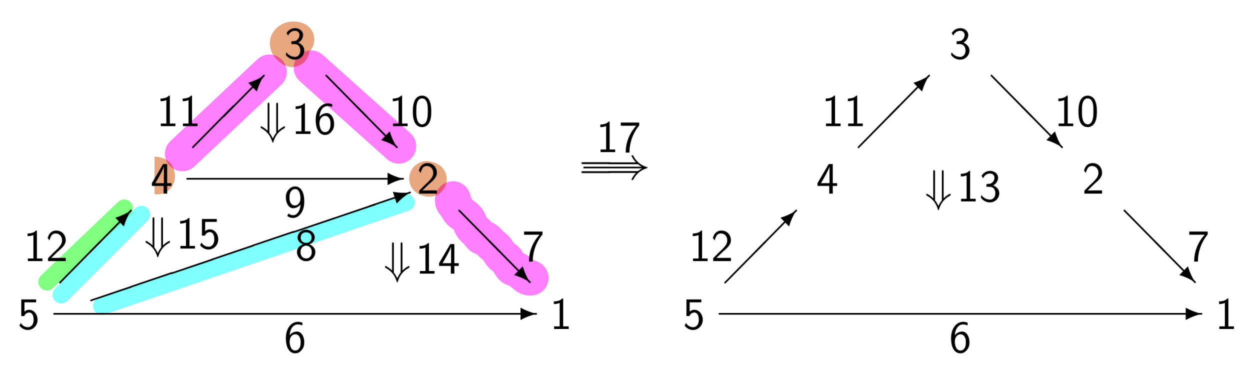
LEAF + EXPLORE + NEXT



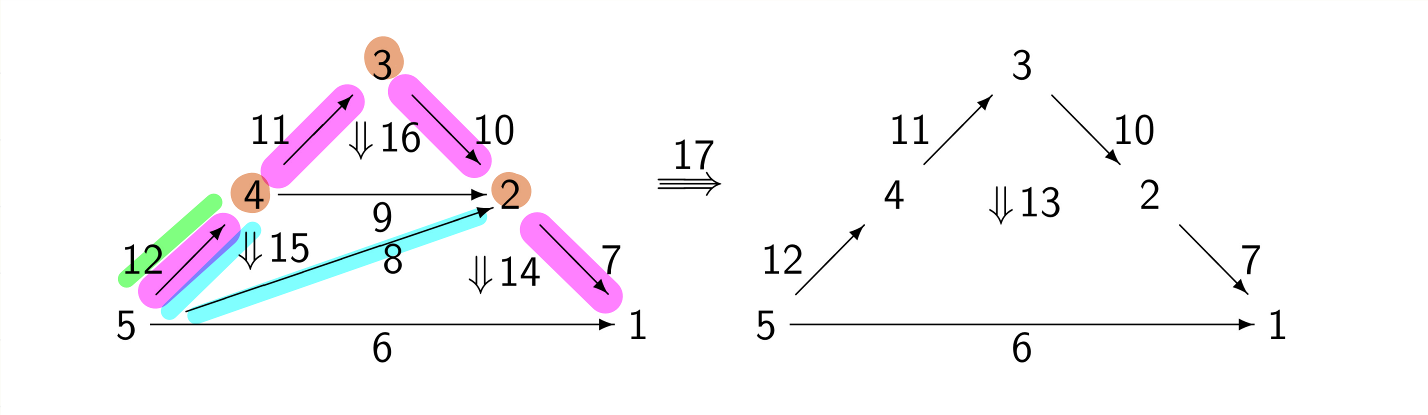
LEAF + EXPLORE + DOWN



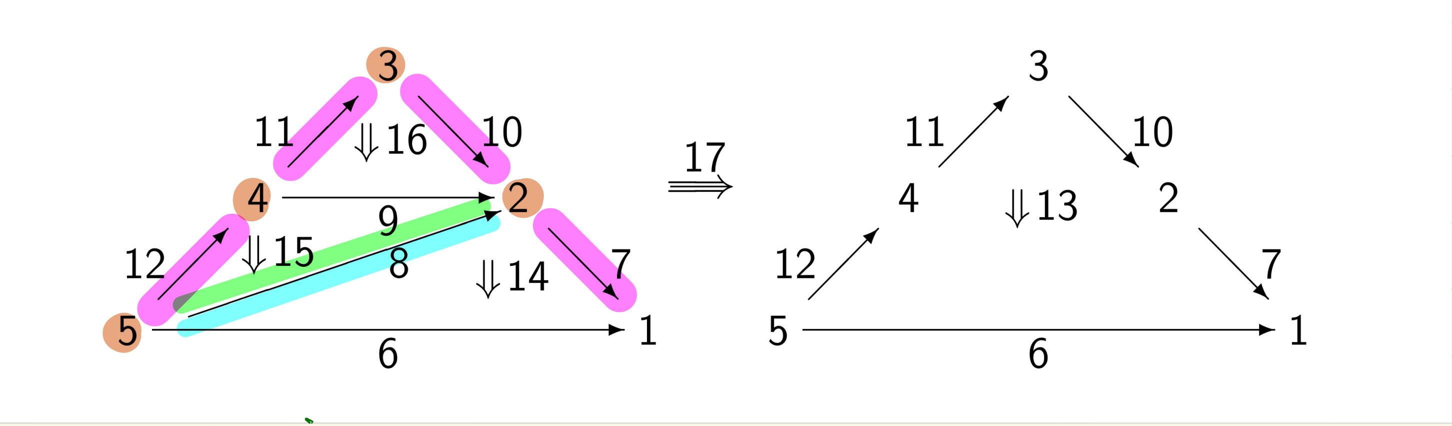
NEXT



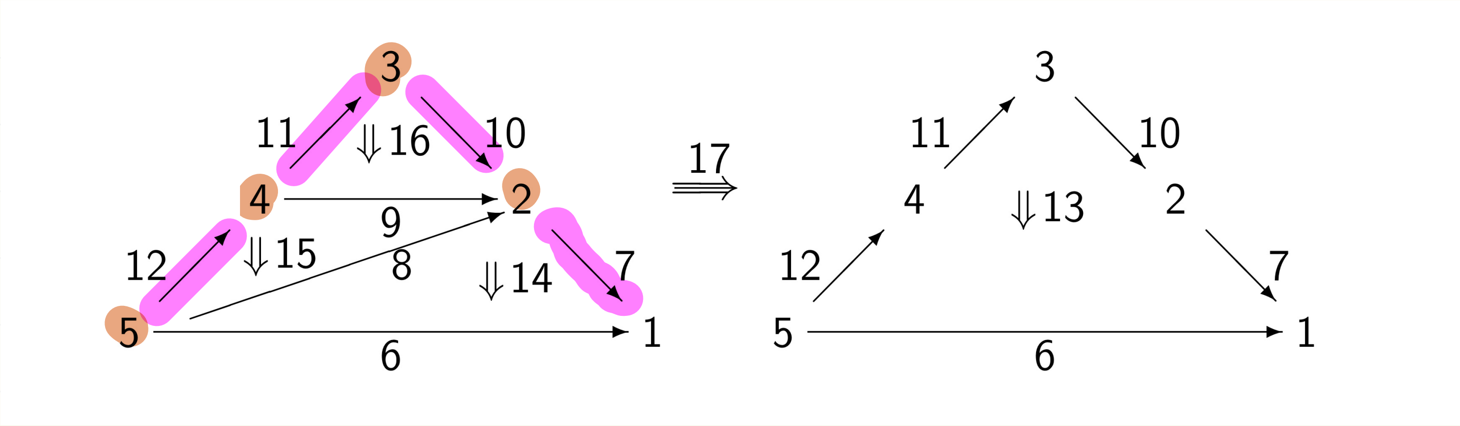
LEAF

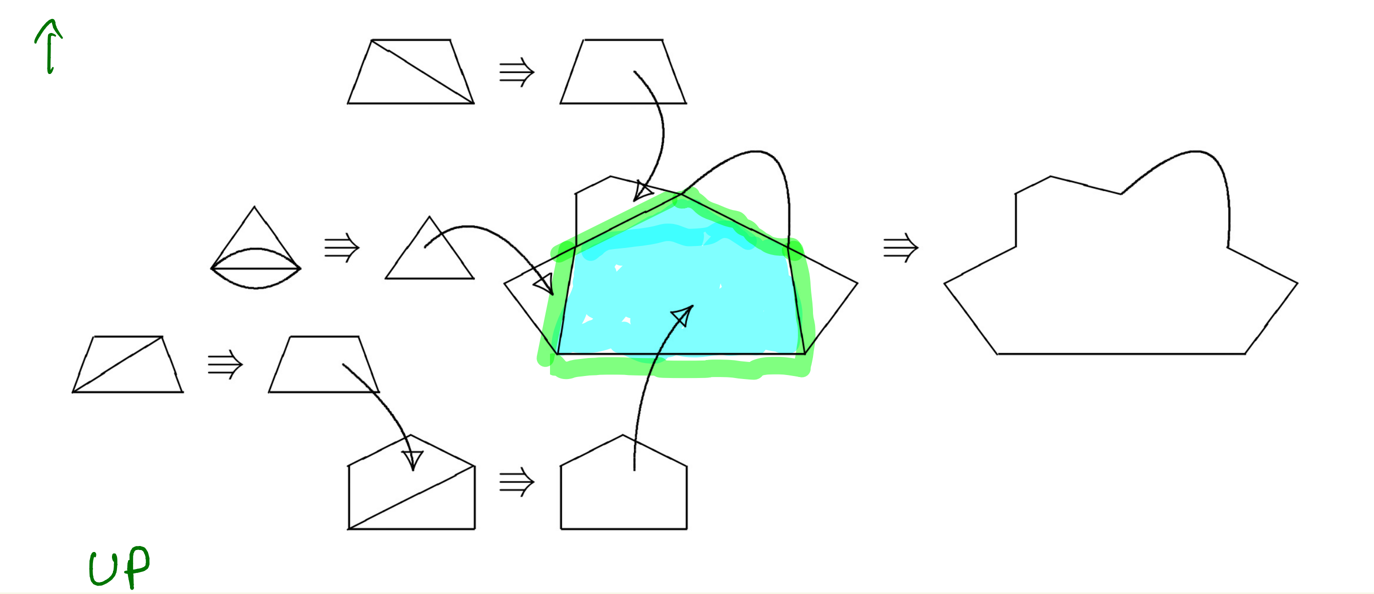


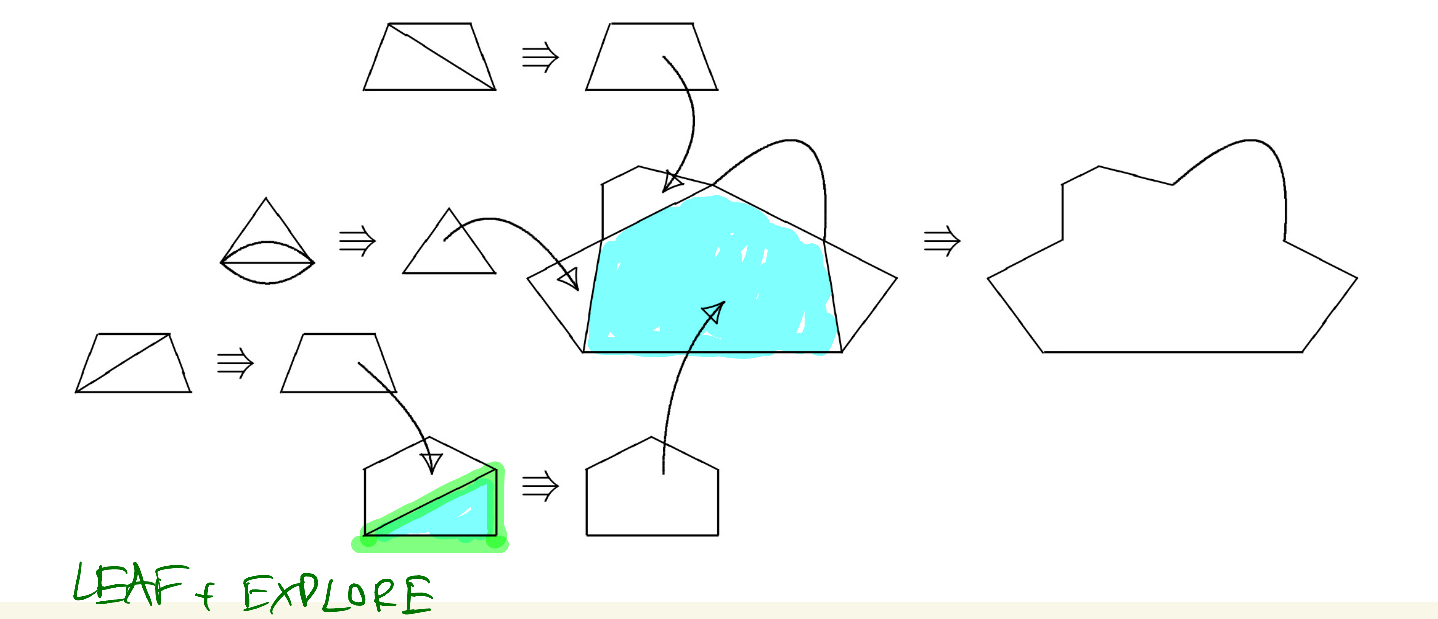
EXPLORE + DOWN

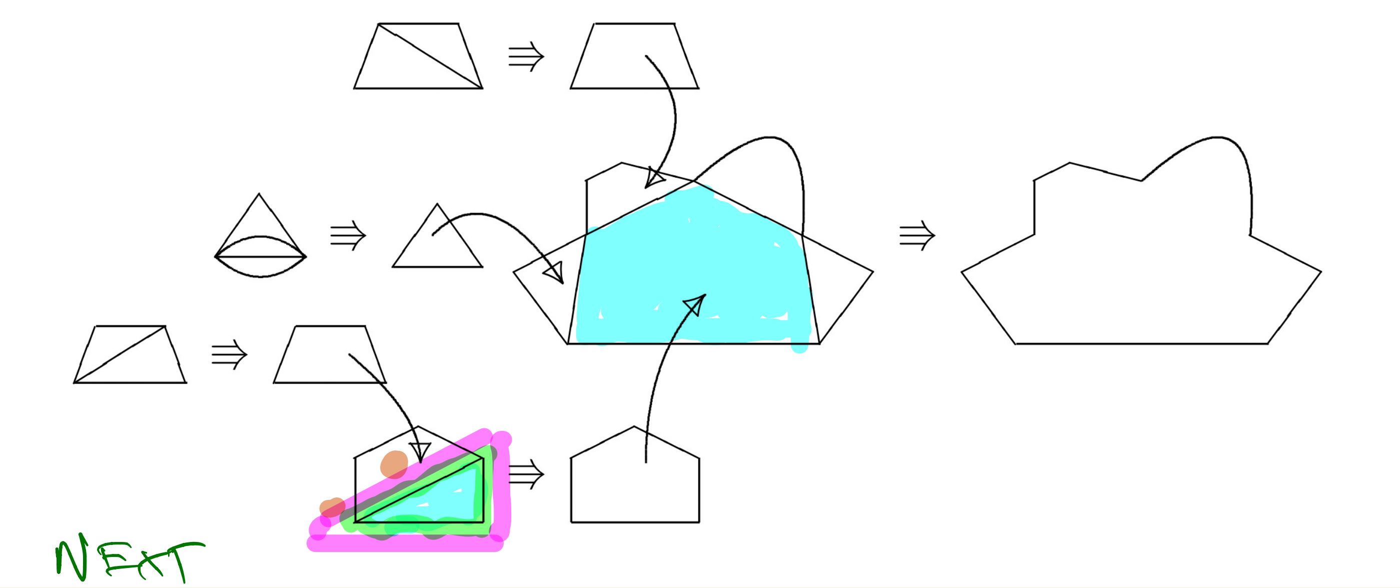


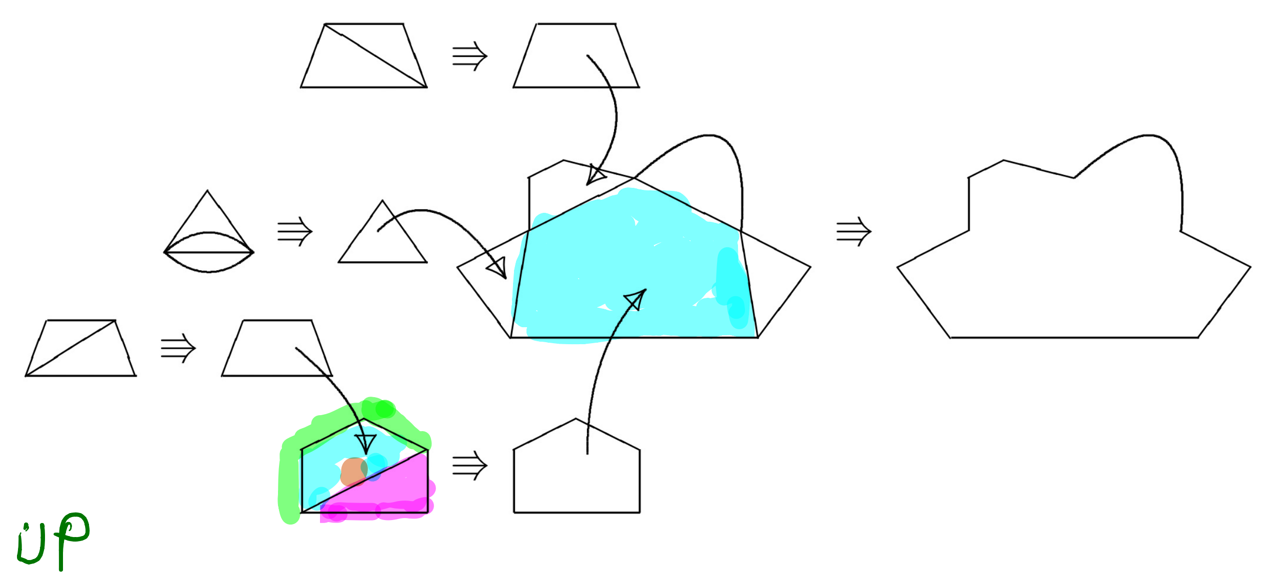
DOWN + FINAL

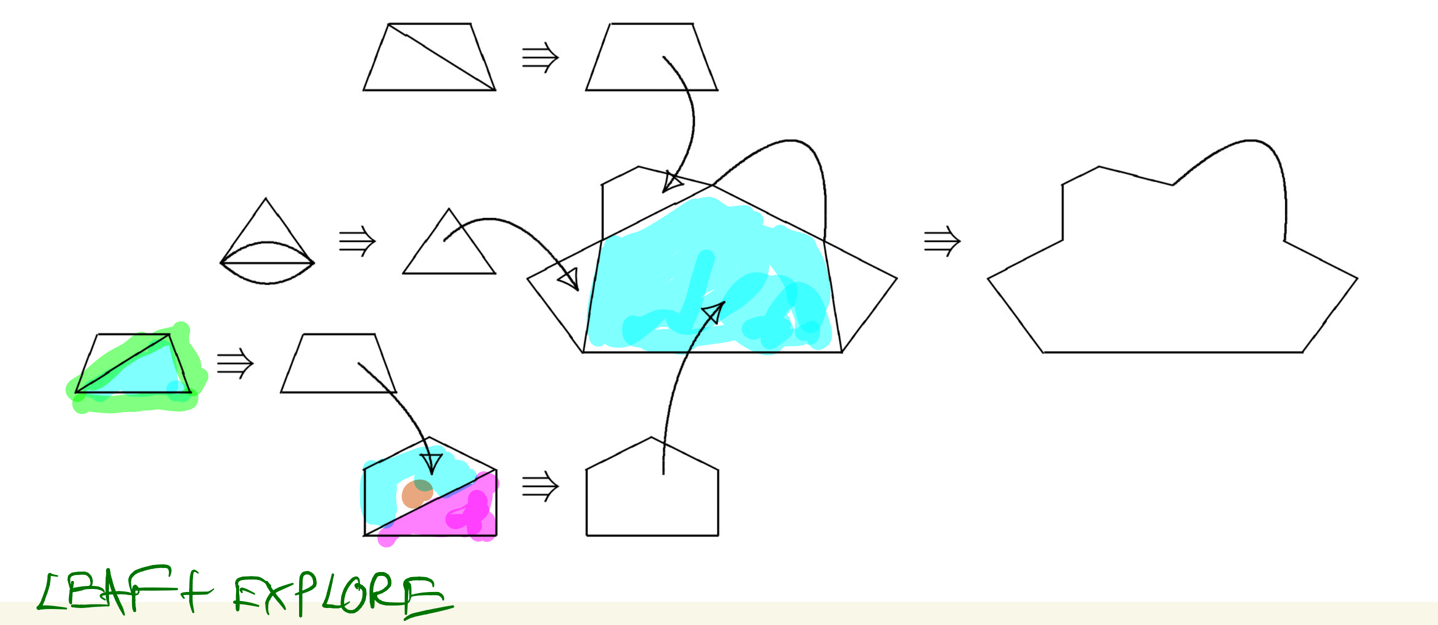


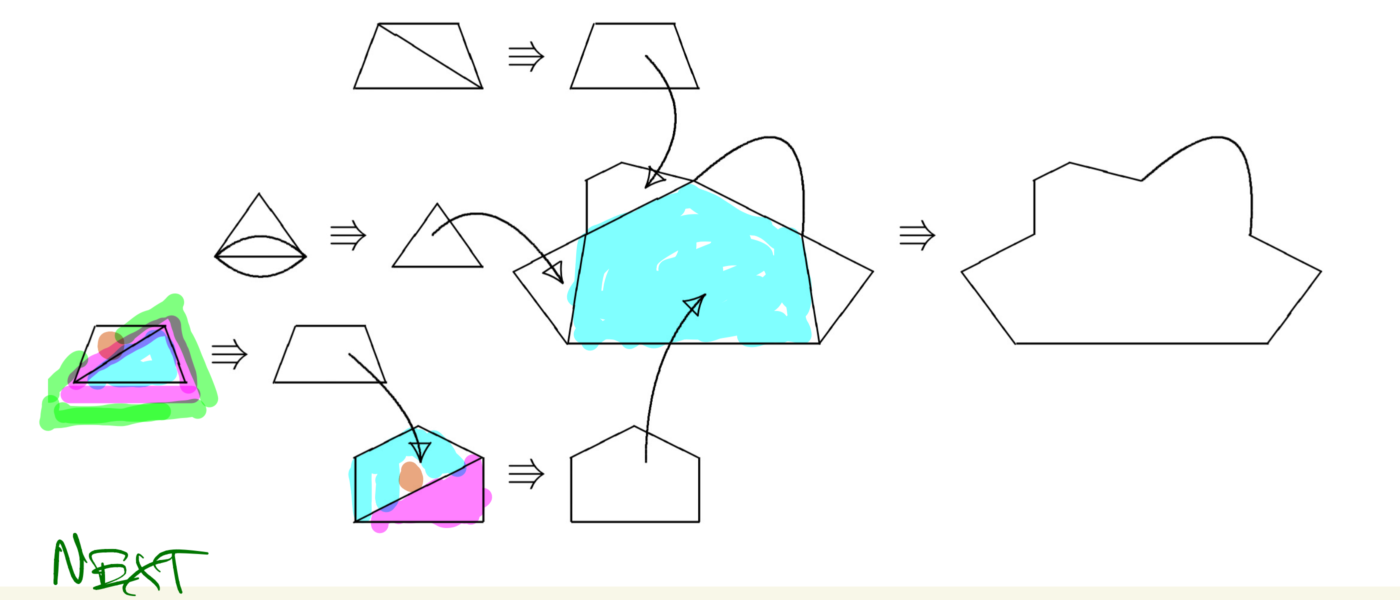

↑
UP


LEAF + EXPLORE


NEXT


UP


LEAF + EXPLORE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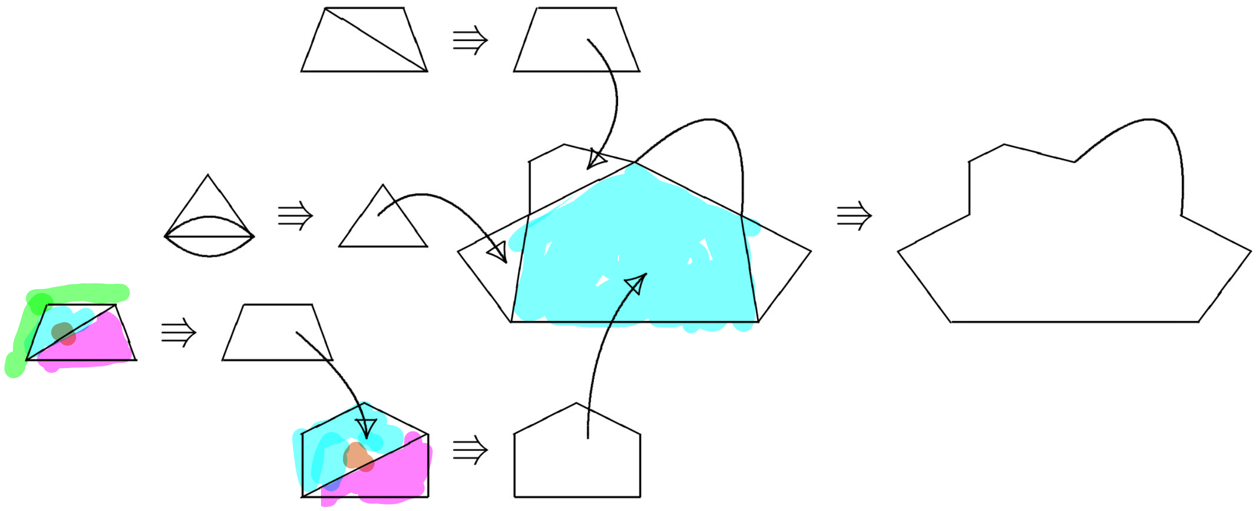
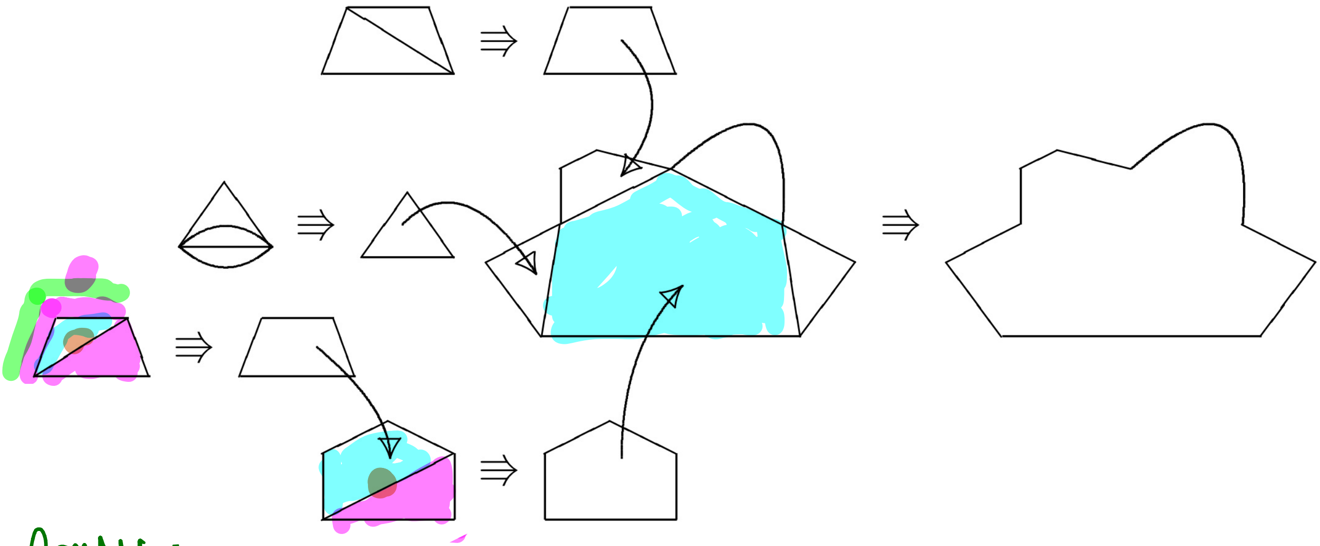
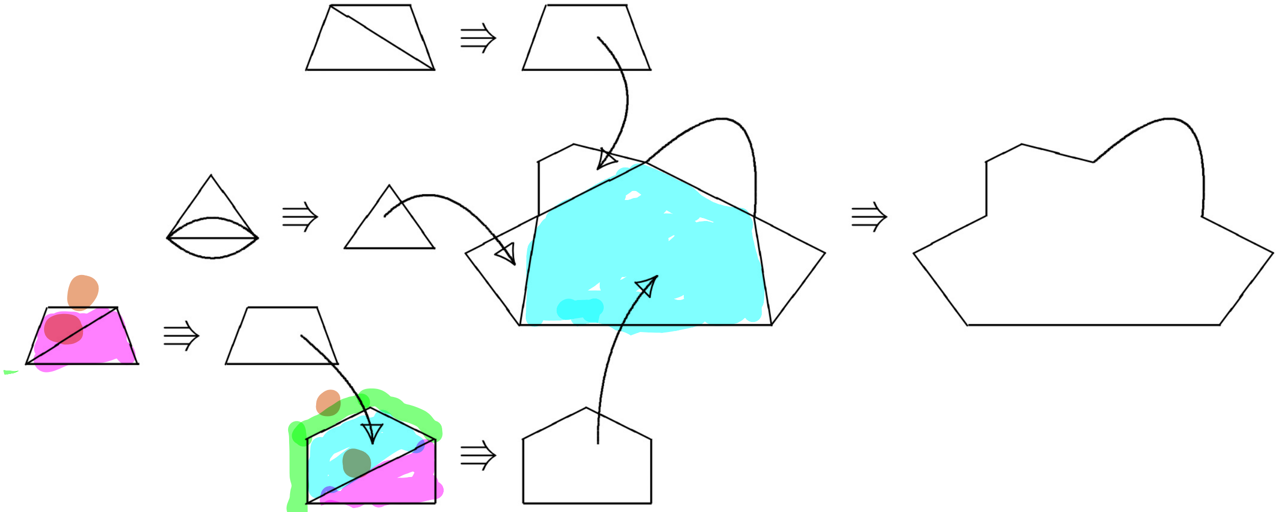


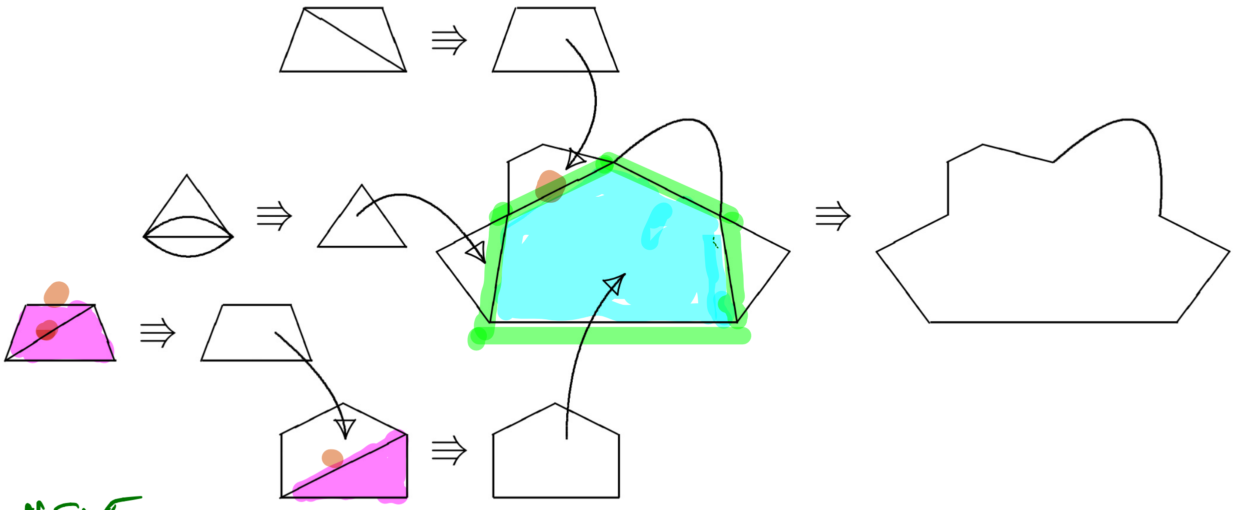
LEAF + EXPLORE



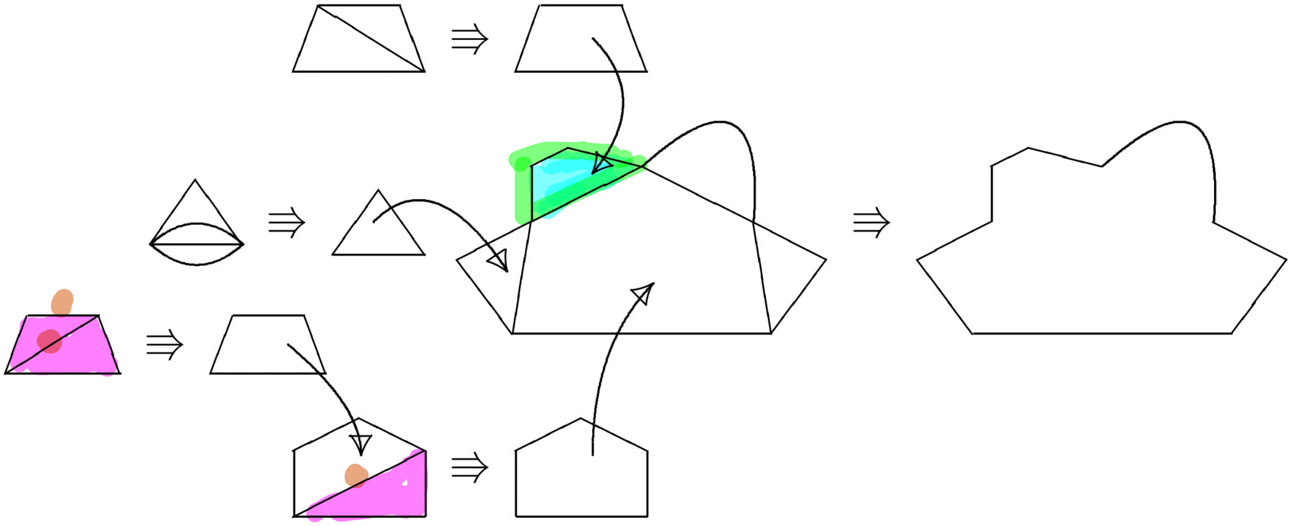
DOWN



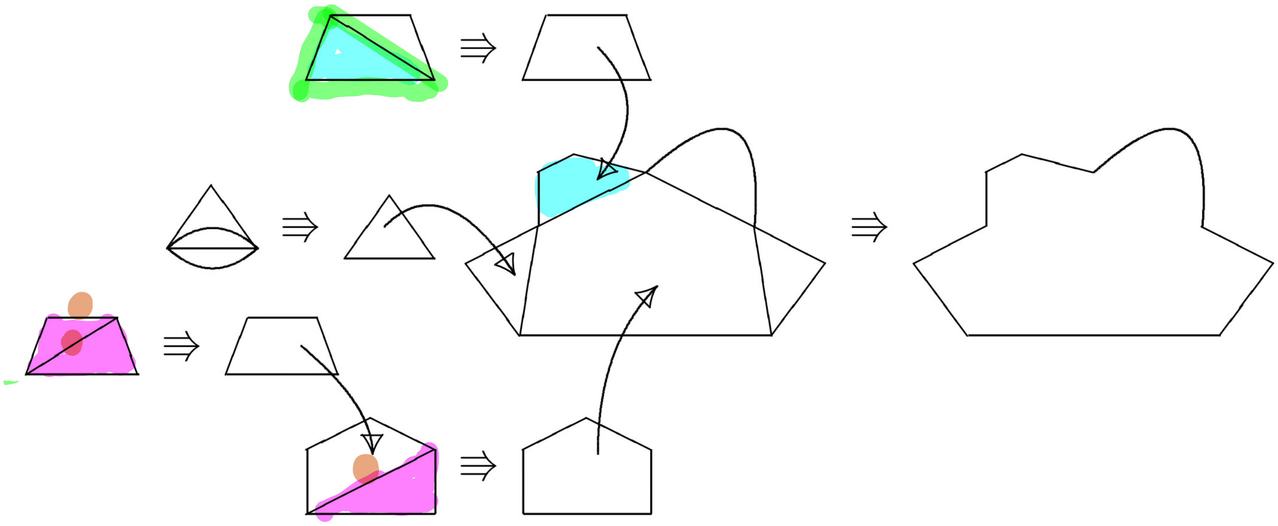
DOWN



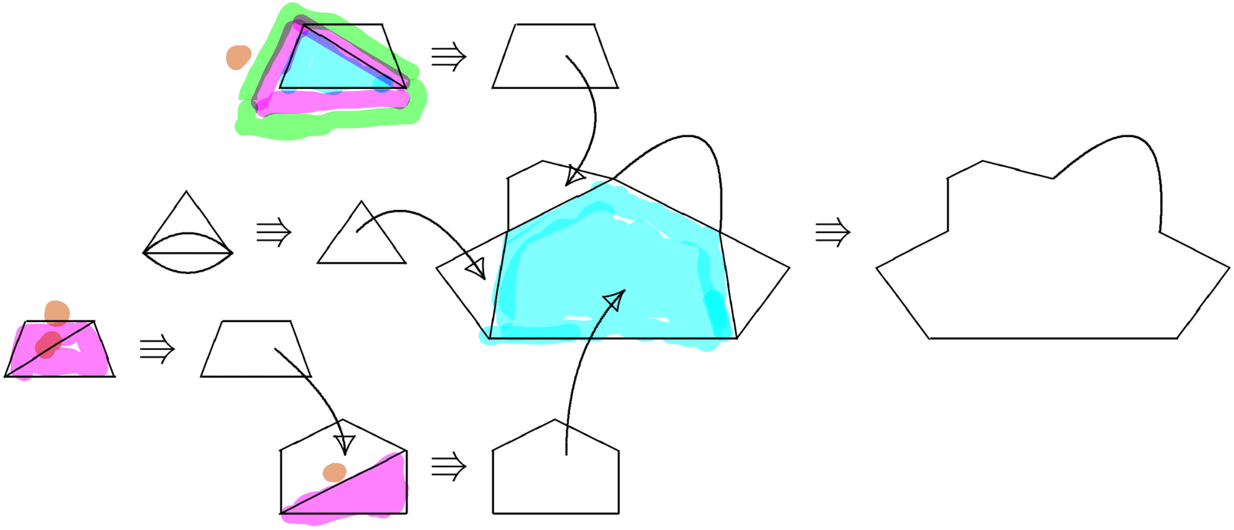
NEXT



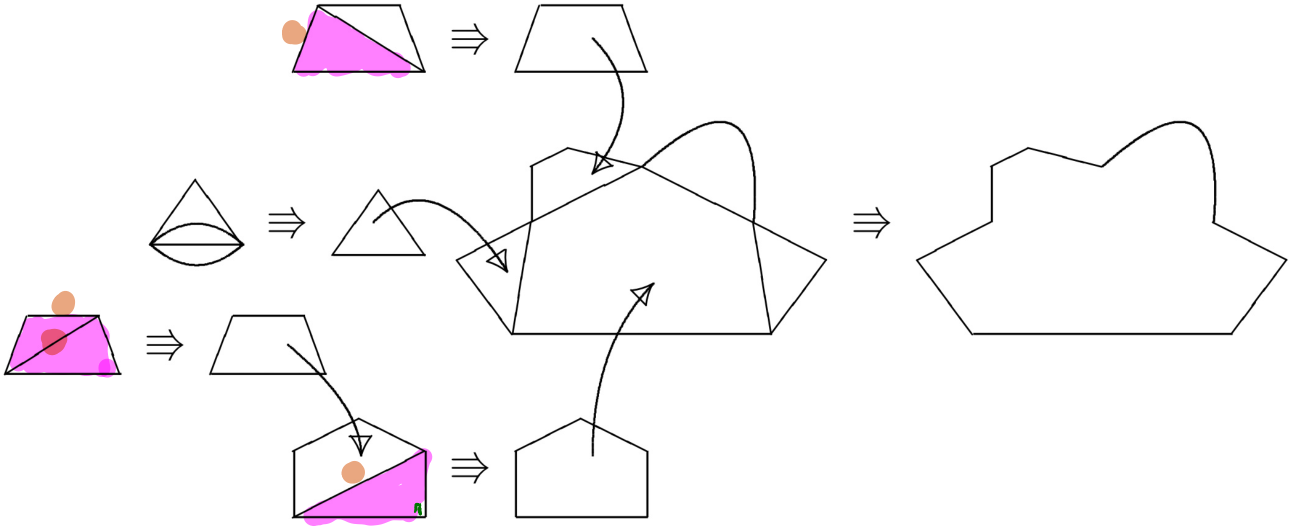
UP



LEAF + EXPLORE

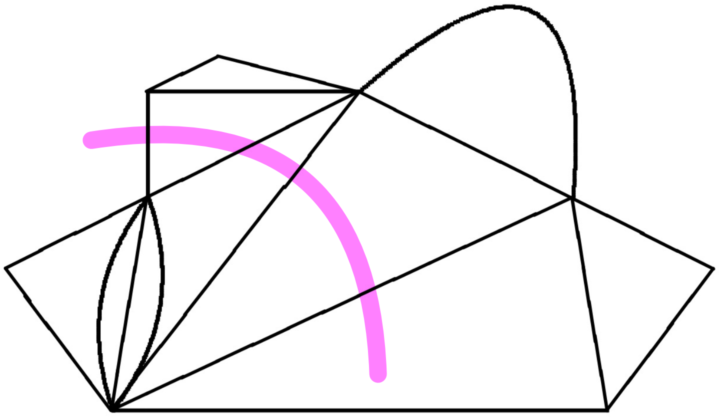


DOWN + DOWN + FINAL

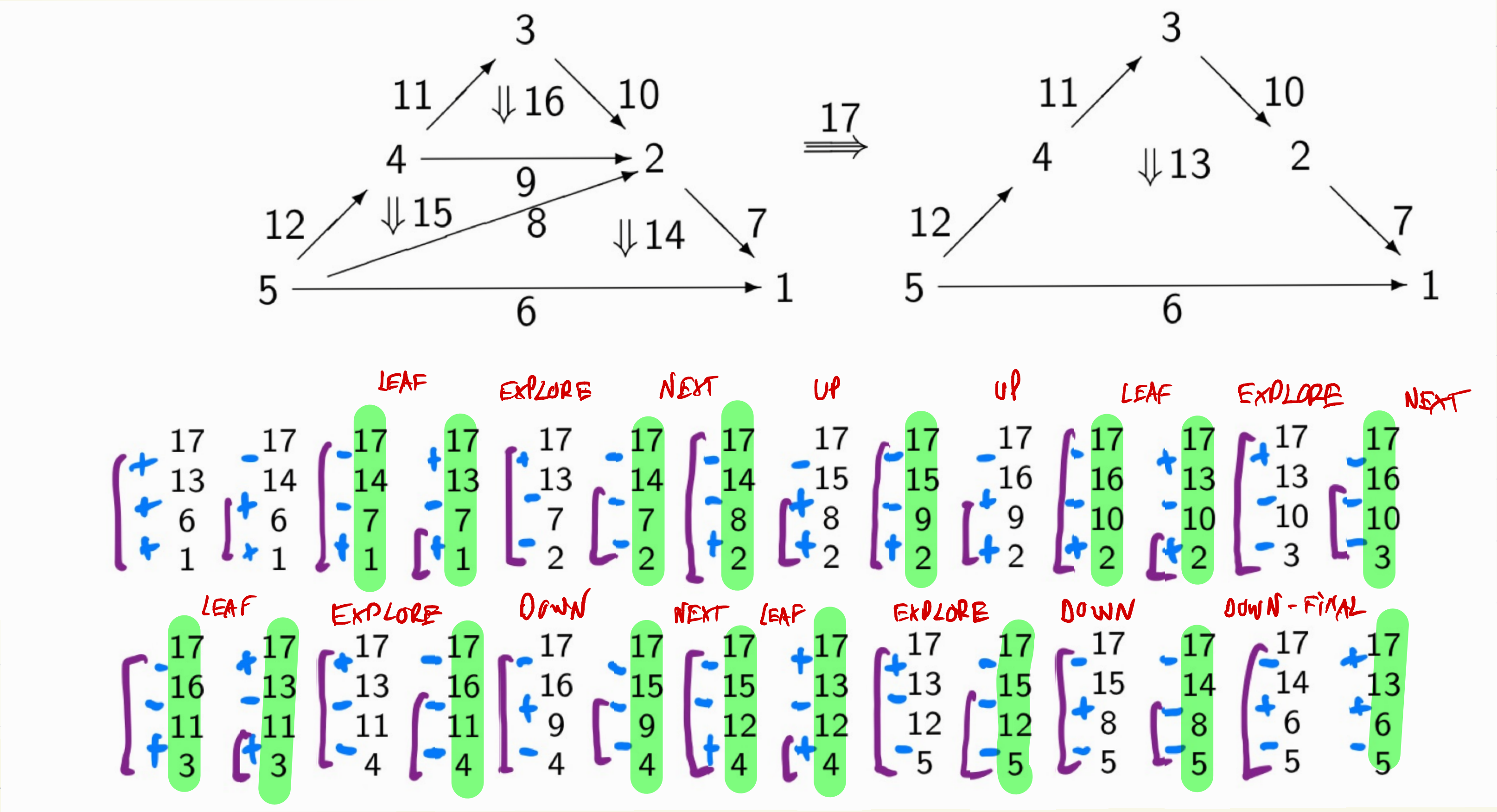


BRANCH

EXPLORED



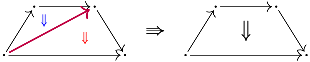
The target abstract machine travels through the Hamiltonian path!



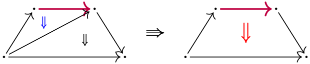
The category Ope

It has as objects all opetopes, and morphisms by generators s_x (for each node of the tree) and t, and relations

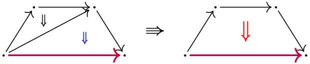
(Inner) $s_x s_u = s_y t$ (all edges)



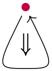
(Glob↑) $t s_u = s_x s_u$ (all leaves, ω non degenerate)



(Glob↓) $s_x t = t t$ ($x =$ root, ω non degenerate)



(Degen) $t s_* = t t$ (ω degenerate)



Opetopic sets are presheaves over Ope.

PART II

Opetopic sets = many-to-one polygraphs

Polygraphs (a.k.a. computads)

A polygraph is (a presentation of) a strict ω-category (i.e. all truncations are strict *n*-categories). It is given by the following data:

- a set $\mathcal{P}_0$ of generating 0-cells,
- a set $\mathcal{P}_1$ of generating 1-cells, each coming with specified source and target in $\mathcal{P}_0$. This gives rise to a free strit 1-category $\mathcal{P}_1^*$ over these generators.

  $\vdots$

- a set $\mathcal{P}_{n+1}$ of $(n+1)$-generating cells, each coming with a specified source and target in $\mathcal{P}_n^*$. This gives rise to a free strict $(n+1)$-category $\mathcal{P}_{n+1}^*$ over these generators.

  $\vdots$

Polygraphic syntax

The n cells (or n-morphims) of $\mathcal{P}^*_n$ are equivalence classes of n-terms built via the following rules:

for $n=2$ $\circ_1$ = vertical comp.
$\circ_0$ = horizontal

- If $x \in \mathcal{P}_n$, then x is an $n-$term.
- If t is an $(n-1)$-term, then $\mathrm{id}(t)$ is an n-term.
- If t_1, t_2 are n-terms and $i < n$, then $t_1 \circ_i t_2$ is an n-term, provided $\mathfrak{s}^{n-i}s$ and $\mathfrak{t}^{n-i}t$ are provably equal as $(n-1)$-terms.

Sources and targets are derived information:

$$\begin{array}{lll}
\mathfrak{s}\,(\mathrm{id}(t)) = t & \mathfrak{t}\,(\mathrm{id}(t)) = t & \\
\mathfrak{s}\,(t_1 \circ_i t_2) = \mathfrak{s}\,t_1 \circ_i \mathfrak{s}\,t_2 & \mathfrak{t}\,(t_1 \circ_i t_2) = \mathfrak{t}\,t_1 \circ_i \mathfrak{t}\,t_2 & (i < n-1) \\
\mathfrak{s}\,(t_1 \circ_{n-1} t_2) = \mathfrak{s}\,t_2 & \mathfrak{t}\,(t_1 \circ_{n-1} t_2) = \mathfrak{t}\,t_1 & (i < n-1).
\end{array}$$

Equational theory (for n-terms)

$$\begin{array}{ll}
(t_1 \circ_i t_2) \circ_i t_3 = t_1 \circ_i (t_2 \ \circ_i t_3) & \text{(category)} \\
\mathrm{id}^{n-i}(\mathfrak{s}^{n-i}t) \circ_i t = t \qquad t \circ_i \mathrm{id}^{n-i}(\mathfrak{t}^{n-i}t) = t & \text{(category)} \\
(s_1 \circ_i s_2) \circ_j (t_1 \circ_i t_2) = (s_1 \circ_j t_1) \circ_i (s_2 \circ_j t_2) \quad (i \neq j) & \text{(exchange law)} \\
\mathrm{id}(t_1) \circ_i \mathrm{id}(t_2) = \mathrm{id}(t_1 \circ_i t_2) \quad (i < n-1) & \text{(2-category)}
\end{array}$$

Occurrences of generating n-cells in an n-cell

Let t be an n-term. We say that

- if $x \in \mathcal{P}_n$, then x has one occurrence (of a generating n-cell) (labelled by x),
- $\mathrm{id}(t)$ has no occurrence,
- the set of occurrences of $t_1 \circ_i t_2$ is the (disjoint) union of the sets of occurrences of t_1 and t_2.

This notion is invariant under the choice of representatives of t.

It can be formulated using the language of contexts.

An *n-context* is a term with one occurrence of a special n-term $[\,]$, with specified source and target, called the hole.
We use the notation $C[\,]$ for the context, and $C[s]$ for the result of replacing $[\,]$ with some actual n-term s with the same source and target as the hole. This is called filling the hole.

Then occurrences (with their labelling generating n-cell) of t are in bijection with the pairs $(C[\,], x)$ such that $x \in \mathcal{P}_n$ and $t = C[x]$.

Illustrating occurences

$$\begin{tikzcd}
{} \arrow[r, "f"'] & {} \arrow[r, bend left, "g"] \arrow[rr, bend right=50, "\ell"'] \arrow[r, phantom, shift right=3, "\Downarrow\alpha"] & {} \arrow[r, "k"'] \arrow[r, bend left=70, "h"] \arrow[r, phantom, shift left=3, "\beta\Downarrow"] & {} \arrow[r, "m"'] & {} \arrow[r, bend left=60, "n"] \arrow[r, bend right=50, "o"'] \arrow[r, phantom, "\Downarrow\gamma"] & {}
\end{tikzcd}$$

$$(\gamma \circ_0 id(m)) \circ_0 [\alpha \circ_1 (\beta \circ_0 id(g)) \circ_0 id(f)]$$

Occurences α, β, γ

Many-to-one polygraphs

A polygraph is called many-to-one if for all n and $x \in \mathcal{P}_n$, we have $\mathfrak{t}x \in \mathcal{P}_{n-1}$ (all generating cells have as target a generating cell).

Theorem. Many-to-one polygraphs are the same thing as opetopic sets (giving rise to an equivalence of categories).

- The theorem has been proved by Victor Harnik, Michael Makkai and Marek Zawadowski (HMZ), replacing "opetopic" with "multitopic". (On the other hand, Eugenia Cheng has proved the equivalence between multitopic sets and opetopic sets.)
- Cédric Ho Thanh has a more direct proof, relying in part on notions and results of Simon Henry.
- Here, we offer a "plug-in" in Cédric's proof, making it entirely self-contained.

Remark. It follows from Henry's work that many-to-one polygraphs form a presheaf category $\mathrm{Set}^{(??)^{op}}$(without an explicit description of ??).

The key lemma

A many-to-one polygraph gives rise naturally to a family of polynomial endofunctors $\nabla_n \mathcal{P}$ (for $n \geq 1$):

$$\mathcal{P}_{n-1} \xleftarrow{\mathfrak{s}} \mathtt{A}_n \xrightarrow{\mathtt{p}} \mathcal{P}_n \xrightarrow{\mathfrak{t}} \mathcal{P}_{n-1}$$

where $\mathtt{A}_n(t)$ is the set of occurrences of $(n-1)$-generating cells of $\mathfrak{s}\, t$, and where $\mathtt{s}$ is the corresponding labelling (or filling).

Let $\mathcal{P}$ be a many-to-one polygraph. We write $\mathcal{P}_n^{mto}$ for the set of many-to-one n-cells, i.e. the cells whose target is a generating cell.

Lemma. For all n, there exists a bijective correspondence

between $\mathcal{P}_n^{mto}$ and the set $\mathrm{Tr}\nabla_n \mathcal{P}$ of $(\nabla_n \mathcal{P})$-trees.

- There exists a composition map $(\_)^\circ : \mathrm{Tr}\nabla_n \mathcal{P} \to \mathcal{P}_n^{mto}$ based on a notion of placed composition defined by HMZ.
- Ho Thanh proves that $(\_)^\circ$ is bijective using some machinery developped by Henry.
- We provide here an **explicit inverse** to $(\_)^\circ$.

Down-to-earth reading of the key lemma

A $\nabla_n \mathfrak{P}$-tree has

- nodes decorated by generating n-cells
- edges decorated by generating $(n-1)$-cells

counting the "exceptional tree" with just one leaf edge and no node.

Looks like a (decorated) opetope!

Sketch of the proof of the theorem from the lemma

Here is the skeleton of the rest of Cédric's proof.

- One defines a realisation functor $|\_| : \mathrm{Ope} \to \mathcal{P}ol^{mto}$ (idea: name all sources and targets of an opetope). The goal is then to show that the induced adjunction

  $$((\text{left Kan extension of } |\_|) \dashv \text{nerve})$$

  is actually an equivalence.
- The key lemma
  - allows to define a shape function from $\mathcal{P}_n$ to $\mathbb{O}_n$ (hereditarily use the key lemma, stripping the decorations by generating cells, and retaining only the underlying opetope),
  - and to establish a bijection

    $$\text{between } \mathcal{P}_n \text{ and } \Sigma_{\omega \in \mathbb{O}_n} \mathcal{P}ol^{mto}(|\omega|, \mathcal{P}) = \Sigma_{\omega \in \mathbb{O}_n} (N\mathcal{P}_\omega)$$

    over $\mathbb{O}_n$ (restoring the decorations!).
  - This allows to prove that the unit and counit of the adjunction are isos.

Rest of the talk: proof of key lemma

(1) Recall the placed composition of HMZ and define the composition map $(\_)^{\circ} : \mathrm{Tr}\nabla_n\mathcal{P} \to \mathcal{P}_n^{mto}$. For this we need a tool/notation that we call context lifting.

(2) Define an invariant associated to evey cell (not only the many-to-one ones) = a forest, i.e. a (possibly empty) set of non-degenerate trees.

(3) Look more closely at this invariant when the cell is many-to-one: it provides the inverse of $(\_)^{\circ}$.

present contribution!

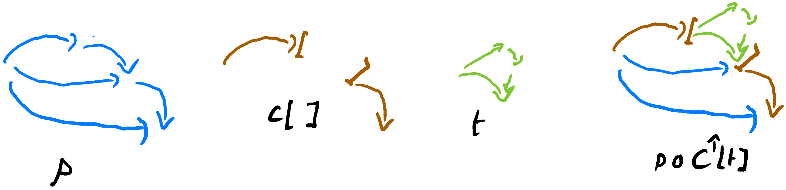
Placed composition (back to many-to-one polygraphs)

Consider two many-to-one n-cells s and t such that $\mathfrak{s}\, s = C[\mathfrak{t}\, t]$ for some context $C[\,]$. Then the term

$$\underline{s} \circ \underline{C}^{\uparrow}[\underline{t}]$$

is well-defined and called the placed composition of s, t at $C[\,]$.

In dim. 2



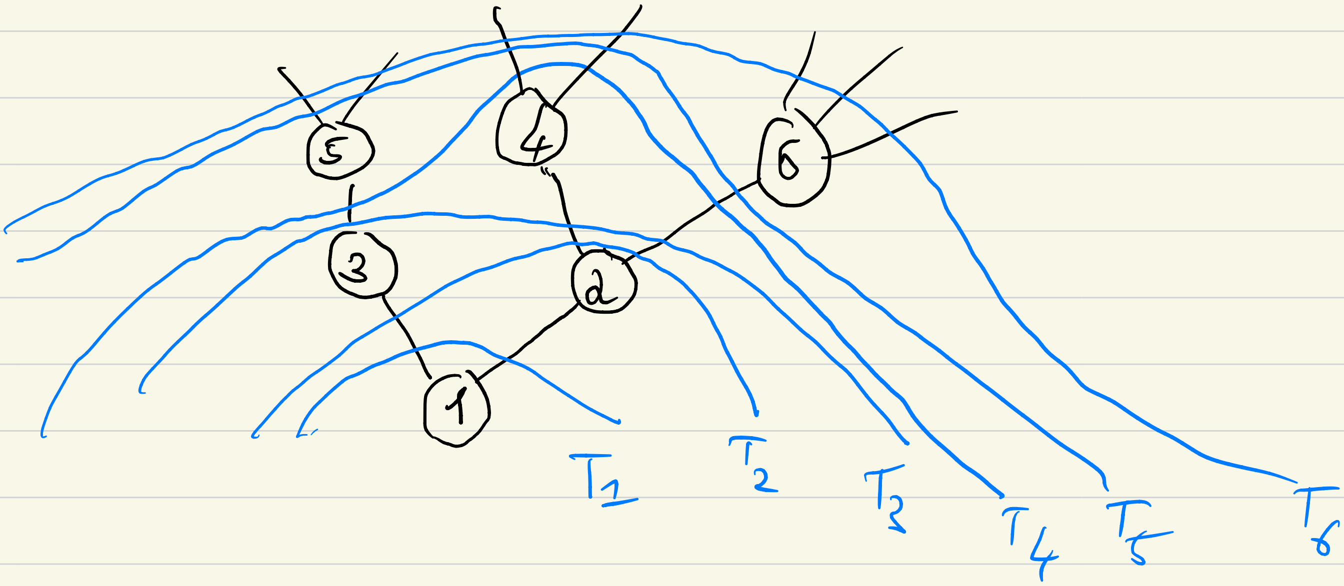
Composition of $(\nabla_n\mathcal{P})$-trees

- It T is degenerate, i.e., reduced to a leaf decorated with a $(n-1)$-generating cell y, then we set $T^\circ = \mathrm{id}(y)$.
- If T is non degenerate, i.e. has at least one node, we fix an admissible (i.e. ancestor respecting) enumeration of the nodes of T. This induces a sequence of trees: the i-th tree has the first i nodes of T, and the first one is just a single node tree decorated with generating cell x_1 (the root of T). We set $x_1^\circ = x_1$ and define T_{i+1}° as a placed composition of T° and x_{i+1} (the decoration of the $(i+1)$-th node) guided by the edge connecting the $(i+1)$-th node to T_i, which itself reads as a context by the definition of $(\nabla_n\mathcal{P})$.

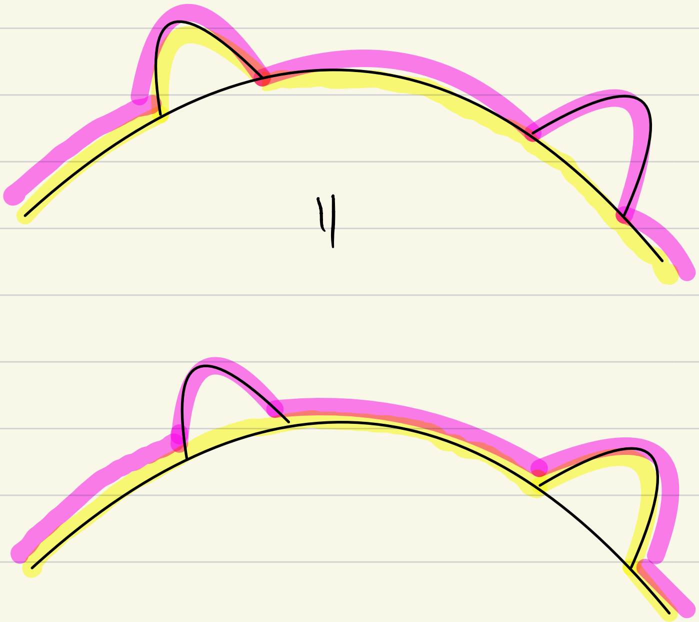
That this definition is independent of the choice of admissible enumeration is a consequence of the following property, for $(n-1)$-contexts with two holes $C[\,]_1[\,]_2$ and generating n-cells x_1, x_2 such that $\mathfrak{t}\,x_i$ can fit in $[\,]_i$ $(i = 1, 2)$ (Godement rule!) :

$$\boxed{C^{\uparrow 1}[x_1]_1[\mathfrak{t}\,x_2]_2 \circ_{n-1} C^{\uparrow 2}[\mathfrak{s}\,x_1]_1[x_2]_2 = C^{\uparrow 2}[\mathfrak{t}\,x_1]_1[x_2]_2 \circ_{n-1} C^{\uparrow 1}[x_1]_1[\mathfrak{s}\,x_2]_2}$$

Illustrating admissible enumerations



Illustrating the "Godement" equation, a.k.a. Mickey mouse (as I learned from Marek)



The other way around: the forest of a cell (setting the scene)

We shall associate with every representative t of a cell in $\mathcal{P}_n^*$ a forest $\#(t)$ whose nodes are decorated by elements of $\mathcal{P}_n$ and whose edges are decorated by elements of $\mathcal{P}_{n-1}$, in such a way that the following properties hold.

- The set of leaf edges (resp. of root edges) of $\#(t)$ is in bijective correspondence with a subset L (resp. R) of nodes of the forest recursively associated with the source of t (resp. the target of t), and the bijection preserves the decorations.
- The set of nodes of $\#(\mathfrak{s}\, t)$ that are not in L is in bijective correspondence with the set of nodes of $\#(\mathfrak{t}\, t)$ that are not in R. We abuse notation by writing this as

$$\#(\mathfrak{s}\, t) \setminus \mathrm{leaves}(\#(t)) = \#(\mathfrak{t}\, t) \setminus \mathrm{roots}(\#(t)).$$

The forest of a cell (definition)

(the polygraph is many-to-one, the cell is arbitrary)

- If $t = x$ is a generating n-cell, then $\#(t)$ is forest consisting of one tree reduced to one node, decorated by x. The leaf edges of the forest are in one-to-one correspondence with the nodes of $\#(\mathfrak{s}\, x)$ and receive the corresponding decorations, and the root edge is decorated with $\mathfrak{t}\, x$.
- If $t = \mathrm{id}(t')$, then we set $\#(t)$ to be the empty forest (whatever t' is).
- If $t = t_1 \circ_i t_2$, with $i < n - 1$, then $\#(t)$ is the disjoint union of the forests $\#(t_1)$ and $\#(t_2)$.
- If $t = t_1 \circ_{n-1} t_2$, then $\#(t)$ is obtained by grafting some trees of $\#(t_2)$ above $\#(t_1)$: if a root u of $\#(t_2)$ is such that $u \in L$ (L relative to t_1), we graft the tree of root u of $\#(t_2)$ on the corresponding tree of $\#(t_1)$.

This definition does not depend on the choice of a representative of an n-cell.

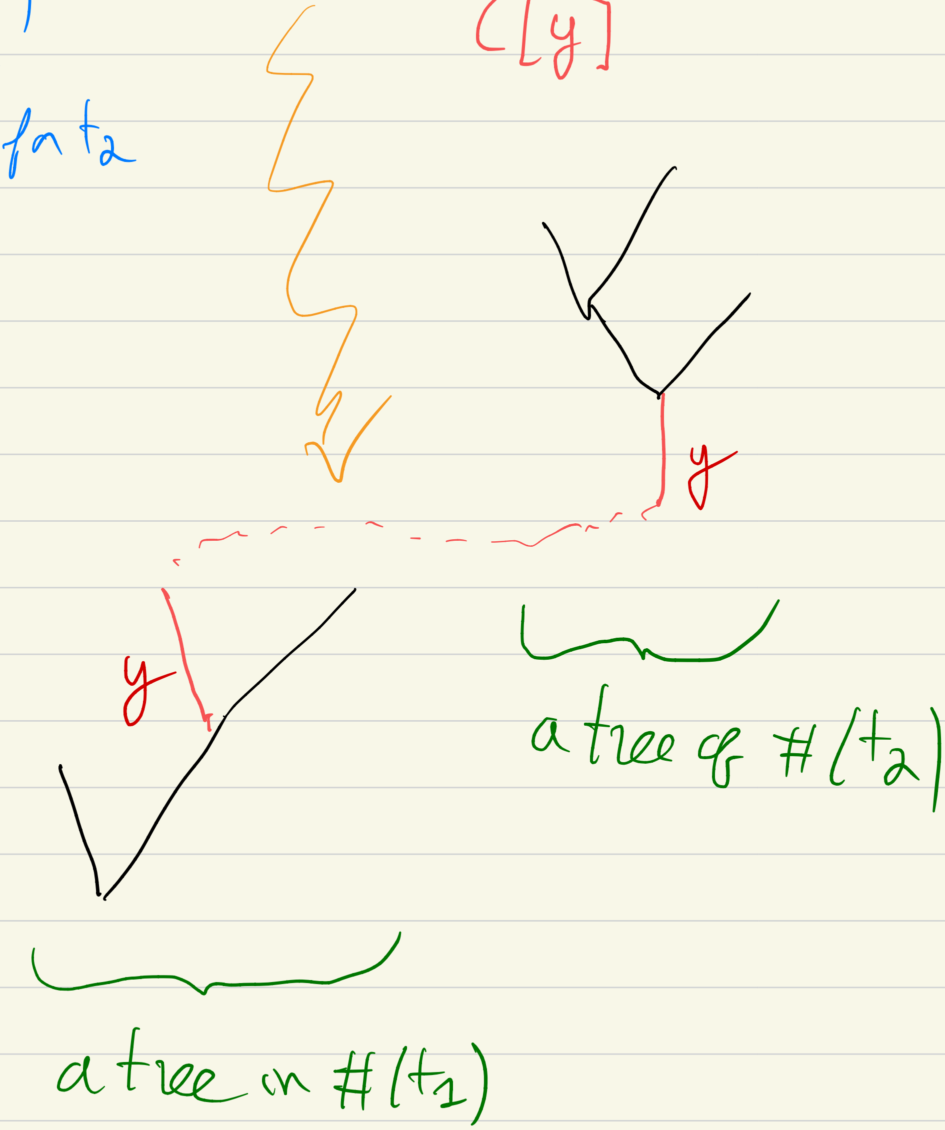
Generic illustration for $\#(t_1 \circ_{n-2} t_2)$

add an edge if $\text{source}(t_1) = \text{target}(t_2)$

$$\underbrace{\text{source}(t_1)}_{C[y]} = \underbrace{\text{target}(t_2)}_{C[y]}$$

$\{y \in L \cap R\}$

for t_1 for t_2



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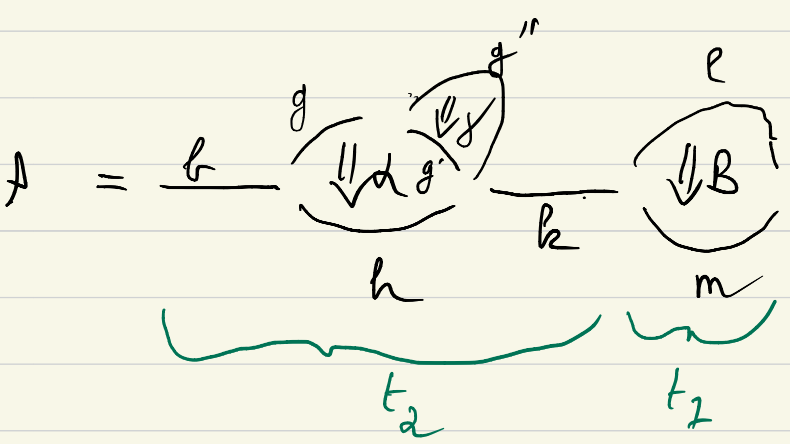
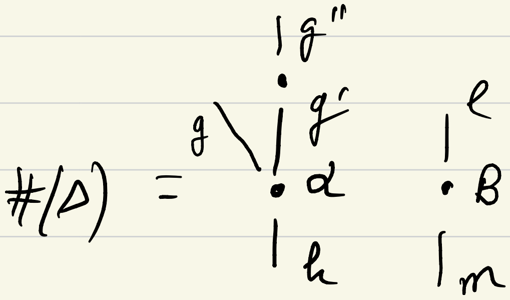
Generic illustration for $\#(t_1 \circ_i t_2)$ $(i<n-1)$





$\#(\mathrm{Dance}(A))^\bullet = \{b, g, g'', k, l\}$ $\quad L = \{g, g'', l\}$

$\#(\mathrm{target}(A))^\bullet = \{b, h, k, m\}$ $\quad R = \{h, m\}$

$\#(\mathrm{Dance}(A))^\bullet \setminus L = \{b, k\} = \#(\mathrm{target}(A))^\bullet \setminus R$

Remark The discrepancy between leaves$(\#(p))$ and nodes of $\#(\mathrm{Dance}(p))$ arises only from the compositions $-\circ_i-$ $\quad i<n-1$ [for p in P_n^*].

Canonical forms for many-to-one cells

Proposition. Any many-to-one n-cell has a representative t that has one of the following shapes:

- $t = x$ for $x \in \mathcal{P}_n$,
- $t = \mathrm{id}(y)$ for $y \in \mathcal{P}_{n-1}$,
- $t = t_1 \circ_{n-1} t_2 \ldots \circ_{n-1} t_n$ $(n \geq 2)$, where
  - $t_1 = x_1 \in \mathcal{P}_n$, and
  - each t_i $(i > 1)$ is of the form $C_i^{\uparrow}[x_i]$, where $x_i \in \mathcal{P}_n$ and $\mathfrak{s}\, t_{i-1} = C[\mathfrak{t}\, x_i]$.

In plain words, t is a placed composition of the generating n-cells occurring in it.

The tree associated with a many to one cell

Corollary. If t is many-to-one, then we have

- $\#(t)$ is empty $\Leftrightarrow$ $t = \mathrm{id}(y)$ for some $y \in \mathcal{P}_{n-1}$.
- $\#(t)$ is not-empty $\Leftrightarrow$ $\#(t)$ consists of a single (non-degenerate) tree. Moreover the set of leaves of $\#(t)$ is in one-to-one correspondence with the set of nodes of $\#(\mathfrak{s}\, t)$ (i.e. $\#(\mathfrak{s}\, t) \setminus L = \emptyset$).

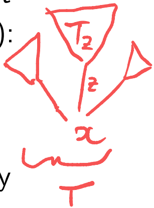
This allows us to define $\underline{\#} : \mathcal{P}_n^{mto} \to \mathrm{Tr}\nabla_n\mathcal{P}$ by

- $\underline{\#}(\mathrm{id}(y))$ is the degenerate tree whose unique leaf is decorated with y,
- $\underline{\#}(t) = \#(t)$ otherwise.

Morale. Even in the canonical forms of many-to-one cells, t_i for $(i > 1)$ is not many-to-one in general. This is why we had to define a wider invariant (forests) working for all cells, and only then narrow it down to the many-to-one cells.

$(\_)^\circ$ and $\underline{\#}$ are inverse

- $(\_)^\circ \circ \underline{\#} = \mathrm{id}$. Clear for $t = x$.
  - If $t = \mathrm{id}(y)$, then $\underline{\#}(x)$ is the degenerate tree decorated with y, hence $(\underline{\#}(x)) = \mathrm{id}(y) = t$.
  - If $t = x_1 \circ_{n-1} t_2 \ldots \circ_{n-1} t_n$, the inductive definition of $\#(t)$ provides an admissible enumeration for $\#(t)$, composing along which yields exactly the same representative t we started from.
- $\underline{\#} \circ (\_)^\circ = \mathrm{id}$. Clear for degenerate T. If $T = x\{z \leftarrow T_z \mid z \in Z\}$, then we fix an order $Z = \{z_1 < \cdots < z_p\}$, take adm. enum. on each T_{z_i}: this determines an adm. enum. of T. One shows that composing along it gives (for suitable lifted contexts $C_i^\uparrow[]$):



$$T^\circ = x \circ_{n-1} C_1^\uparrow[T_{z_1}^\circ] \circ_{n-1} \ldots \circ_{n-1} C_p^\uparrow[T_{z_p}^\circ]$$
$$\#(T^\circ) = x\{z_i \leftarrow \#(C_i^\uparrow[T_{z_i}^\circ]) \mid i = 1, \ldots, p\}$$

and we conclude by induction, thanks to the following easy

Lemma. If $C^\uparrow[]$ is a lifted context, then

$$\#(C^\uparrow[t]) = \#(t) \quad \text{(for all } t \text{ fitting in the hole)}$$

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