

Condorcet meets Gustave (revisited)

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for
Thomas Ehrhard's 60th birthday

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A TALK ABOUT (NON) SEQUENTIALITY
what else ?

Largely inspired from G. Huet's
talk at GGJ 2011

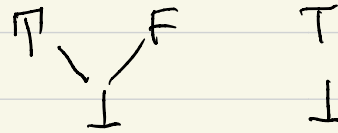
with some additions of my own

Thanks to Antonio Bucciarelli for discussions

Once upon a time in the late 1960's - early 1970's

- Scott domains and Scott continuity.

For our purpose $\text{Bool}_\perp = \{\perp, \top, F\}$ $\perp_\perp = \{\perp, \top\}$



Our playground: monotone (= continuous here) functions

$$f: (\text{Bool}_\perp)^n \rightarrow \perp_\perp$$

Such functions can be described by their trace: the set of minimal points $(x_1 \dots x_n)$ s.t. $f(x_1 \dots x_n) = \top$ and hence can be described by an $m \times n$ -matrix (m = number of minimal points)

- Scott noticed the problematic behaviour of "po2": $(\text{Bool}_\perp)^2 \rightarrow \perp_\perp$ given by

$$\begin{pmatrix} \perp & \top \\ \top & \perp \end{pmatrix}$$

This launched the quest for a (genuine) fully abstract (FA) model of PCF,

Once upon a time in the late 1970's

- Berry (= Gustave, gallium.inria.fr/~huet/PUBLIC/GGJJ.pdf)
"killed" por by requiring all pairs of minimal points to be incompatible = stable functions
(rediscovered by Girard in early 1980's $\rightarrow$ linear logic)
Indeed " por " is not stable

$$\left\{ \begin{pmatrix} \perp & \top \\ \top & \perp \end{pmatrix}, \begin{pmatrix} \perp & \top \\ \top & \perp \end{pmatrix} \right\} \leq (\top, \top)$$

- But Berry noticed the problematic character of the function

$$\text{GUSTAVE} = \begin{pmatrix} \perp & \top & \text{F} \\ \text{F} & \perp & \top \\ \top & \text{F} & \perp \end{pmatrix}$$

In both cases (" por " and GUSTAVE), problematic means not sequential (at $(\perp, \perp, \perp)$)

- A function $f: (\text{Bool}_{\perp})^n \rightarrow \perp_{\perp}$ is sequential if

$$\exists i \left(f(x_1 \dots x_n) = \top \Rightarrow x_i \neq \perp \right)$$

So that we can start to program f by writing
if $x_i = \top$ then ~

- Sequential $\Rightarrow$ stable

Summary so far

- for is continuous but not stable

- GUSTAVE is stable but not sequential

- Then all full abstraction hunters searched ways to "kill" GUSTAVE, and Thomas is one of them, who did it (with Antonio Bucciarelli) via strong stability.

- But in this talk, we love GUSTAVE!

Definition (Huet 2011) A Gustave function is

a function $f : (\text{Bool}_\perp)^n \rightarrow \mathbb{I}_\perp$ that is

- stable, and
- not sequential at $(\perp, \perp, \perp)$

Equivalent definition: each column of the matrix contains a $\perp$:

$$\underbrace{\begin{pmatrix} \perp & \dots & \dots \\ \dots & \perp & \dots \\ \dots & \dots & \perp \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}}_n \left. \vphantom{\begin{pmatrix} \perp & \dots & \dots \\ \dots & \perp & \dots \\ \dots & \dots & \perp \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}} \right\} m$$

(up to permutation of lines)

Once upon a time in 1785 in the kingdom of France

ESSAI
SUR L'APPLICATION
DE L'ANALYSE
À LA
PROBABILITÉ
DES DÉCISIONS

Rendues à la pluralité des voix.

*Par M. LE MARQUIS DE CONDORCET, Secrétaire perpétuel
de l'Académie des Sciences, de l'Académie Française, de
l'Institut de Bologne, des Académies de Pétersbourg, de
Turin, de Philadelphie & de Padoue.*

Quòd si deficiant vires audacia certè
Laus erit, in magnis & voluisse sat est.



A PARIS,
DE L'IMPRIMERIE ROYALE.

M. DCCLXXXV.

Condorcet's (1743-1794) voting procedure

- We have
- a set $C = \{c_1, \dots, c_n\}$ of **candidates**
 - a set $V = \{v_1, \dots, v_{N+1}\}$ of **voters**

Each voter gives a total order of preference of all candidates

Thus the data of a Condorcet ballot consists in a function

$$\varphi : V \rightarrow \text{Bij}(C, \{1, \dots, n\}) \quad (\text{the set of bijections from } C \text{ to } \{1, \dots, n\})$$

We write $c \leq_v c'$ if $\varphi(v)(c) \leq \varphi(v)(c')$
"v ranks c' above c "

We write $c \leq_v c'$ if $\text{card}\{v \mid c \leq_v c'\} \geq N$
"a majority of voters ranks c' above c "

 c' beats c
(Strictly speaking, we should write $c <_v^p c'$, $c <_v^o c'$)

Observe that for all c, c' we have (exclusively)

$$c = c' \quad \text{or} \quad c \leq_v c' \quad \text{or} \quad c' \leq_v c$$

$\downarrow$
also written
 $c >_v c'$

i.e., $\leq_v$ is a total strict relation.

Condorcet paradox

Warning. Unlike $\leq_v$, $\leq_v$ may not be transitive. The basic

example is $C = \{A, B, C\}$, $V = \{1, 2, 3\}$

$$\varphi(1) = A \succ B \succ C$$

$$\varphi(2) = B \succ C \succ A$$

$$\varphi(3) = C \succ A \succ B$$

We have $A \succ_v B$, $B \succ_v C$, $C \succ_v A$

(supported each by two voters over three)

Hence $A \succ_v C$ does not hold!

If $\leq_v$ admits a (necessarily unique) maximum, i.e. c s.t. $\forall c' \neq c$ $c \succ_v c'$, then c is called Condorcet winner.

There is no winner in this example. The possible absence of a winner is called Condorcet paradox.

Note. Lack of transitivity does not prevent the existence of a winner. Just add a candidate D and put

$$\varphi(1) = D \succ A \succ B \succ C$$

$$\varphi(2) = D \succ B \succ C \succ A$$

$$\varphi(3) = D \succ C \succ A \succ B$$

Then $\leq_v$ is not transitive, but D wins.

We have ● $\leq_v$ transitive $\Rightarrow \exists$ winner

● for $\text{card}(C) = 3$, the converse holds

● for $\text{card}(C) < 3$ there is a winner

Abstracting from the set of voters

Without loss of generality (as we shall see), we abstract from V and consider a total strict relation $<$ on a set $C = \{c_1, \dots, c_n\}$ of candidates. The relation $<$ can be represented as a $n \times n$ -matrix $M_<$ with entries in $\text{Bool}_\perp$, setting

$$M_{ij} = \begin{cases} \top & \text{if } c_j < c_i \\ \perp & \text{if } i = j \\ \text{F} & \text{if } c_i < c_j \end{cases}$$

For example, the matrix associated to the counterexample above is

$$\begin{pmatrix} \perp & \top & \text{F} \\ \text{F} & \perp & \top \\ \top & \text{F} & \perp \end{pmatrix}$$

We've met
before...

The matrices $M_<$ have the following features:

- They are square matrices ($n \times n$)
- The diagonal is filled with $\perp$, and these are the only $\perp$ entries
- Setting $\bar{\top} = \text{F}$, $\bar{\text{F}} = \top$ and $\bar{\perp} = \perp$, we have $\forall i, j \quad c_j < c_i \iff \bar{c}_i = \bar{c}_j$

Definition (PLC0022). Such a matrix is called semi-Condorcet

To win or not to win

- As the *peigneur de la Police* could ^{ban in 1/470} have paid, ^{not} it is supereasy to detect the winner if any on a semi-Condorcet matrix: find a (necessarily unique) line i of the form

$$T \dots T \perp T \dots T \quad (C_{ij} = T \text{ for all } j \neq i)$$

- But there is another characterization that gives more information in the case of the absence of a winner: there is a uniquely determined core subset X of candidates that cause (via $\prec$) the paradox.

(In terms of voters, X is minimum such that if we erased the candidates in $C \setminus X$ from the voter, keeping the rest of the expressed preferences, then there is no winner.)

Characterising the absence of a winner

Definition (Huet 2011). We define the relation $\prec^T$ as follows:

$c \prec^T c'$ if

- $c \prec c'$ and
- for all $c'' \neq c$, if $c'' \prec c$ then $c'' \prec c'$

c' defeats c "c' beats c, and all candidates that c beats"

Lemma. The relation $\prec^T$ is transitive.

Algorithm • Initialisation $X := C$

• while $\exists c, c' \in X$ such that $c' \prec_X^T c$, let $X := X - \{c\}$
wt X ! slowly, take $\prec_X =$ restriction of $\prec$ to X , and then $\prec_X^T = (\prec_X)^T$

Proposition (Huet 2011) • Given C and $\prec$ total strict on C ,
there is a winner (i.e. a maximum for $\prec$)



the algorithm terminates with $X = \{c_i\}$ for some i the winner!

• The final state X reached by the algorithm is independent of the choices made by the algorithm (choice of c at each iteration)

• If there is no winner, the machine stops at some X with $\text{card}(X) \geq 3$, and $M_{\prec_X}$ displays a Gustave function.

We call $M_{\prec_X}$ the Candidate witness of $\prec$.

(and $\text{card}(X) \neq 4$ (IC 2012), see below)

Applying the algorithm

Algorithm • Initialisation $X := C$

• while $\exists c, c' \in X$ such that $c' \prec_X^T c$, let $X := X - \{c'\}$

• With a winner

$c_1 \perp \top \top$	$c_1 \perp \top$	$c_1 \perp$
$c_2 \perp \perp \perp$	$c_2 \perp \perp$	
$c_3 \perp \perp \perp$	$c_3 \perp \perp$	

• Without a winner (obtained by reverse engineering!)

$c_1 \perp \top \top \top$
$c_2 \perp \perp \top \top$
$c_3 \top \perp \top \top$
$c_4 \perp \top \perp \perp$
$c_5 \top \perp \perp \perp$

$c_1 \perp \top \top \top$
$c_2 \perp \perp \top \perp$
$c_3 \top \perp \perp \top$
$c_4 \perp \top \perp \perp$

$c_1 \perp \top \perp$
$c_2 \perp \perp \top$
$c_3 \top \perp \perp$

Final: no winner

Question. All the semi-Condorcet matrices above are non-stable unless they are Condorcet matrices.

Is this always true?

Proof of the soundness and completeness of Huet's algorithm

Algorithm • Initialisation $X := C$

• while $\exists c, c' \in X$ such that $c \not\geq_X^T c'$, let $X := X - \{c'\}$

Proof of the proposition.

• If there is a winner c_i , it a priori defeats all candidates, hence taking $C \setminus \{c_i\} = \{c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n\}$ in any order we can let $X := C \setminus \{c_{i_1}\}$, $X := C \setminus \{c_{i_2}, c_{i_1}\}$ etc. until $X := C \setminus \{c_{i_1}, \dots, c_{i_{n-2}}\} = \{c_i\}$

• The converse is an immediate consequence of the following claim:

If $\bullet^1 X \not\subseteq C$, $c' \in C \setminus X$, $c_1, c_2 \in X$

$\bullet^2 c_1 \in X$ wins relatively to X

$\bullet^3 c_2$ defeats c' relatively to $X \cup \{c'\}$

then c wins relatively to $X \cup \{c'\}$.

Proof of the claim. We have

• $c_1 \geq_X^T c_2$ by $\bullet^2$, or equivalently $c_1 \geq_{X \cup \{c'\}}^T c_2$

• Hence $c_1 \geq_{X \cup \{c'\}}^T c_2 \geq_{X \cup \{c'\}}^T c' \Rightarrow c_1 \geq_{X \cup \{c'\}}^T c' \Rightarrow c_1 \geq_{X \cup \{c'\}} c'$
 $\underbrace{\hspace{10em}}_{\bullet^3}$ $\uparrow$ transitivity lemma

Which Gustave functions arise as Condorcet witnesses?

By construction, a Condorcet witness is a (sub) set X of candidates such that $\nexists c_i, c_j \in X, c_i >_X c_j$, or, equivalently

$$\forall c_i, c_j (c_i >_X c_j \Rightarrow \exists k \neq i, j (c_j >_X c_k \text{ and } c_i <_X c_k))$$

Translating this in terms of matrices, we have:

Fact (PLC 2022) A semi-Condorcet matrix M is a Condorcet witness iff the following additional condition holds:

- $\forall i, j$, if $M_{ij} = T$ (and hence $M_{ji} = F$), then there exists $k \neq i, j$ s.t. $M_{ik} = F$ and $M_{jk} = T$.
Such a k is called a Condorcet witness of stability (of the associated function) for i, j .

$$\begin{matrix} i \\ j \end{matrix} \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & T & \dots & F \\ \dots & \dots & \dots & \dots \\ \dots & F & T & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

A function is called a Condorcet function if it is semi-Condorcet and if any pair of minimal points has a Condorcet witness.

Thus, by definition, Condorcet $\subseteq$ Gustave

The eight 3x3 Gustave functions

$\begin{array}{ccc} \perp & F & F \\ T & \perp & T \\ F & T & \perp \end{array}$	$\begin{array}{ccc} \perp & F & F \\ F & \perp & T \\ T & T & \perp \end{array}$	$\begin{array}{ccc} \perp & F & T \\ T & \perp & F \\ F & T & \perp \end{array}$	$\begin{array}{ccc} \perp & F & T \\ F & \perp & F \\ T & T & \perp \end{array}$
$\begin{array}{ccc} \perp & T & F \\ T & \perp & T \\ F & F & \perp \end{array}$	$\begin{array}{ccc} \perp & T & F \\ F & \perp & T \\ T & F & \perp \end{array}$	$\begin{array}{ccc} \perp & T & T \\ F & \perp & F \\ T & F & \perp \end{array}$	$\begin{array}{ccc} \perp & T & T \\ T & \perp & F \\ F & F & \perp \end{array}$

- Each of the six excluded functions is refuted for two reasons:

- not semi-Condorcet
- no Condorcet witness of stability

This is special to $n=3$!

- The remaining ones are GUSTAVE and its mirror

Réje (= Gustave - GB) could not
have been closer to
the marquis de Condorcet!

A curiosity: There is no 4x4 Condorcet function

$\begin{bmatrix} \perp & a & b & c \\ \bar{a} & \perp & d & e \\ \bar{b} & \bar{d} & \perp & f \\ \bar{c} & \bar{e} & \bar{f} & \perp \end{bmatrix}$

1	$a=T$	$b=F$	$d=T$	$\begin{bmatrix} \perp & T & F & c \\ F & \perp & T & e \\ T & F & \perp & f \\ \bar{c} & \bar{e} & \bar{f} & \perp \end{bmatrix}$	1.1	$\perp$	T	F	c	$\perp$	T	F	c	compatible! No
				$e=F$	F	$\perp$	T	F	$\perp$	T	F	$\perp$	T	
				$\begin{bmatrix} T & F & \perp & f \\ \bar{c} & T & \bar{f} & \perp \end{bmatrix}$	forces $\bar{b}=F$	T	F	$\perp$	f	T	F	$\perp$	T	
				$\begin{bmatrix} \bar{c} & T & \bar{f} & \perp \end{bmatrix}$		$\bar{c}$	T	$\bar{f}$	$\perp$	$\bar{c}$	T	F	$\perp$	
					1.2	$\perp$	T	F	c	$\perp$	T	F	F	compatible! No
					$e=T$	F	$\perp$	T	T	F	$\perp$	T	T	
						T	F	$\perp$	f	forces $\bar{c}=T$	T	F	$\perp$	f
						$\bar{c}$	F	$\bar{f}$	$\perp$		T	F	$\bar{f}$	$\perp$
2	$a=T$	$c=F$	$e=T$	$\begin{bmatrix} \perp & T & b & F \\ F & \perp & d & T \\ \bar{b} & \bar{d} & \perp & f \\ T & F & \bar{f} & \perp \end{bmatrix}$	2.1	$\perp$	T	b	F	$\perp$	T	T	F	compatible! No
				$f=T$	F	$\perp$	d	T	$\bar{b}$	$\bar{d}$	$\perp$	T	F	
				$\begin{bmatrix} \bar{b} & \bar{d} & \perp & T \\ T & F & F & \perp \end{bmatrix}$	forces $\bar{b}=F$	$\bar{b}$	$\bar{d}$	$\perp$	T	F	$\bar{d}$	$\perp$	T	
						T	F	F	$\perp$		T	F	F	$\perp$
					2.2	$\perp$	T	b	F	$\perp$	T	b	F	compatible! No
					$f=F$	F	$\perp$	d	T	F	$\perp$	F	T	
						$\bar{b}$	$\bar{d}$	$\perp$	F	forces $\bar{d}=T$	$\bar{b}$	T	$\perp$	F
						T	F	T	$\perp$		T	F	T	$\perp$
3	$a=F$	$b=T$	$d=F$	$\begin{bmatrix} \perp & F & T & c \\ T & \perp & F & e \\ F & T & \perp & f \\ \bar{c} & \bar{e} & \bar{f} & \perp \end{bmatrix}$	3.1	$\perp$	F	T	c	$\perp$	F	T	F	compatible! No
				$f=T$	T	$\perp$	F	e	F	T	$\perp$	T	T	
				$\begin{bmatrix} \bar{c} & \bar{e} & F & \perp \end{bmatrix}$	forces $\bar{c}=T$	$\bar{c}$	$\bar{e}$	F	$\perp$	$\bar{c}$	$\bar{e}$	F	$\perp$	
						T	$\perp$	F	e		T	$\perp$	F	e
					3.2	$\perp$	F	T	c	$\perp$	F	T	c	compatible! No
					$f=F$	T	$\perp$	F	e	T	$\perp$	F	T	
						F	T	$\perp$	F	forces $\bar{c}=F$	F	T	$\perp$	F
						$\bar{c}$	$\bar{e}$	T	$\perp$		$\bar{c}$	F	T	$\perp$
4	$a=F$	$c=T$	$e=F$	$\begin{bmatrix} \perp & F & b & T \\ T & \perp & d & F \\ \bar{b} & \bar{d} & \perp & f \\ F & T & \bar{f} & \perp \end{bmatrix}$	4.1	$\perp$	F	b	T	$\perp$	F	b	T	compatible! No
				$f=T$	T	$\perp$	d	F	$\bar{b}$	$\bar{d}$	$\perp$	T	T	
				$\begin{bmatrix} \bar{b} & \bar{d} & \perp & T \\ F & T & F & \perp \end{bmatrix}$	forces $\bar{d}=F$	$\bar{b}$	$\bar{d}$	$\perp$	T	F	$\bar{b}$	F	$\perp$	T
						F	T	F	$\perp$		F	T	F	$\perp$
					4.2	$\perp$	F	b	T	$\perp$	F	F	T	compatible! No
					$f=F$	T	$\perp$	d	F	T	$\perp$	d	F	
						$\bar{b}$	$\bar{d}$	$\perp$	F	forces $\bar{b}=T$	T	$\bar{d}$	$\perp$	F
						F	T	T	$\perp$		F	T	T	$\perp$

An example of a 5x5 Condorcet function

<u>L</u>	<u>T</u>	F	T	F
<u>F</u>	L	<u>T</u>	F	F
T	F	L	T	T
F	T	F	L	T
T	T	F	F	L

L	T	<u>F</u>	T	F
F	L	<u>T</u>	F	F
<u>T</u>	F	L	T	T
F	T	F	L	T
T	T	F	F	L

L	T	F	<u>T</u>	F
F	L	<u>T</u>	F	F
T	F	L	T	T
<u>F</u>	T	F	L	T
T	T	F	F	L

L	T	F	T	<u>F</u>
F	L	<u>T</u>	F	F
T	F	L	T	T
F	T	F	L	T
<u>T</u>	T	F	F	L

L	T	F	T	F
F	<u>L</u>	<u>T</u>	F	F
T	<u>F</u>	L	T	T
F	T	F	L	T
T	T	F	F	L

L	T	F	T	F
F	<u>L</u>	<u>T</u>	<u>F</u>	F
T	F	L	T	T
F	<u>T</u>	F	L	T
T	T	F	F	L

L	T	F	T	F
F	<u>L</u>	<u>T</u>	F	<u>F</u>
T	F	L	T	T
F	T	F	L	T
T	<u>T</u>	F	F	L

L	T	F	T	F
F	L	<u>T</u>	F	F
T	F	<u>L</u>	<u>T</u>	T
F	T	<u>F</u>	L	T
T	T	F	F	L

L	T	F	T	F
F	L	<u>T</u>	F	F
T	F	<u>L</u>	<u>T</u>	<u>T</u>
F	<u>T</u>	F	L	T
T	<u>T</u>	<u>F</u>	F	L

L	T	F	T	F
F	L	<u>T</u>	F	F
T	F	L	T	T
<u>F</u>	T	F	<u>L</u>	<u>T</u>
T	T	F	<u>F</u>	L

Synthesising a 6x6 Condorcet function

⊥	T	F	T	F	T
F	⊥	T	F	F	.
T	F	⊥	T	T	.
F	T	F	⊥	T	
T	T	F	F	⊥	F
F				T	⊥

⊥	T	F	T	F	T
F	⊥	T	F	F	T
T	F	⊥	T	T	
F	T	F	⊥	T	
T	T	F	F	⊥	F
F	F			T	⊥

⊥	T	F	T	F	T
F	⊥	T	F	F	T
T	F	⊥	T	T	F
F	T	F	⊥	T	
T	T	F	F	⊥	F
F	F	T		T	⊥

⊥	T	F	T	F	T
F	⊥	T	F	F	T
T	F	⊥	T	T	F
F	T	F	⊥	T	F
T	T	F	F	⊥	F
F	F	T	T	T	⊥

Question: Are there Condorcet functions for all $n \neq 1, 2, 4$?

Back to voting

Proposition (PLC 2022) For any set $C = \{c_1, \dots, c_n\}$ of candidates, for any total strict relation $<$ on C , there exists a set of voters (of odd cardinality) and a Condorcet ballot such that $<_V = <$.

Lemma For fixed C , if (V, φ) is a Condorcet ballot, then, taking $W = (V.1) \cup (V.2) \cup \{*\}$ and ψ defined as follows:

● $\psi(V.1) = \varphi(V)$ ● $\psi(V.2) = \varphi(W)$ ● $\psi(*) = \text{whatever choice}$
we have $<_V = <_W$.

Proof. We have, for all $c, c' \in C$:

$$|\text{card } \{w \mid w \in W \setminus \{*\} \text{ and } c <_w c'\}| = |\text{card } \{w \mid w \in W \setminus \{*\} \text{ and } c' <_w c\}| \geq 2$$

And hence the single vote of $*$ cannot make a difference.

Proof of Proposition. We proceed by induction on the number of candidates.

Assume we have found V such that $<_V = \text{restriction of } < \text{ to } X \subseteq C$, and pick $c \in C \setminus X$. Then take W, ψ as in the Lemma (relatively to X),

- extending ψ on $W \setminus \{*\}$ by setting
 - every $v.1$ places c in first position
 - every $v.2$ places c in last position
- constraining the choice of the vote of $*$ so as to have
 - $*$ prefers c to all candidates c' such that $c' < c$
 - $*$ prefers all candidates c' such that $c' < c$ to c

CHAPEAU

BAS À

THOMAS!