MODEL STRUCTURE ON ∞-PREOPERADS

Contents

Définition

1. Tree decompositions

Definition 1.1 ([HM22, ~~p.~~ 1.20])**.** A *non planar, rooted tree* T is the data of a finite set $V(T)$ of *vertices*, a nonempty finite set $E(T)$ of *edges*, a distinguished element $r \in E(T)$ called the *root*, together with:

(1) a function $t\colon E(T) \setminus \{r\} \to V(T)$, which we think of as assigning to an edge e the vertex $t(e)$ of which it is an *input*;
(2) a function $O\colon V(T) \to E(T)$, assigning to each vertex v its output edge $O(v)$. (injective, voir mail)

such that for any edge $e \in E(T)$, $e \neq r$, there exists a number k such that $(O \circ t)^k(e) = r$.

The edges in the complement of the image of O are called the *leaves* of the tree T. The vertices in the complement of the image of I are called *stumps*, or *nullary vertices*. An *outer edge* is an edge that is either the root or a leaf. An *inner edge* is any other edge of T , i.e., an edge in the image of O that is not the root. Such an edge is both an output edge and an input edge of some other vertex.

t

Harmoniser

Add some pictures.

For a tree T, we denote by $\mathrm{E}^{\mathrm{int}}(T)$ the set of its internal edges. Since ~~for~~ any internal edge e writes as $e = O(x)$ for some other edge x, and we define $s(e) \in V(T)$ as the vertex obtained as $s(e) = t(x)$. We will refer to the two vertices $\{s(e), t(e)\}$ as the two external vertices of the inner edge e.

injectivité de O!

Add some pictures.

On a pas [illegible] [illegible] de $O(T)$

Definition 1.2. For a given tree T, the partial order on its set of edges $E(T)$ is defined as follows: for any two edges $e, f \in E(T)$, we say that $e \leq f$ if and only if there exists a number $k \in \mathbb{N}$ such that $(O \circ t)^k(f) = e$.

In particular, the root is the minimal element of $E(T)$, while the leaves are the maximal elements.

Definition 1.3 (The dendroidal category $\mathbf{\Omega}$.)**.** add definition

We have four subcategories of $\mathbf{\Omega}$ which are of interest:

$$\mathbb{C} \subseteq \mathbb{A} \subseteq \mathbb{B} \subseteq \mathbf{\Omega}_r^o \subseteq \mathbf{\Omega}$$

expliquer que composition of external face = partial défini

at least one leaf ?

no unary node ?

Donner 1 déf. précise.

- $\mathbf{\Omega}_r^o$ is the full subcategory of $\mathbf{\Omega}$ whose objects are open and reduced trees;
- $\mathrm{Ob}(\mathbb{B}) = \mathrm{Ob}(\mathbb{A}) = \mathrm{Ob}(\mathbb{C}) = \mathrm{Ob}(\mathbf{\Omega}_r^o)$, and
  - morphisms in $\mathbb{B}$ are root preserving;
  - morphisms in $\mathbb{A}$ preserve root and induce a bijection on the set of leaves;
  - morphisms in $\mathbb{C}$ preserve the root and induce the identitity on the set of leaves.

Observe that in $\mathbf{\Omega}_r^o$ there are no degeneracy morphisms, since they would add a~~n~~ etc. unary vertex, and starting from $\mathbb{A}$ we no longer have outer faces, as they don't preserve the set of leaves.

Pas si évident à voir

deg

$\in \Omega_2^o$ $\in \Omega_2^o$

Remark 1.1. For any open and reduced tree T, the slice categories

$$\mathbb{C}/T, \quad \mathbb{A}/T, \quad \mathbb{B}/T$$

are finite, and the coslice categories

$$T/\mathbb{C}, \quad T/\mathbb{A}$$

are finite as well. However, the coslice category $T/\mathbb{B}$ is not finite.

Definition 1.4. An *orthogonal factorization system* on a category $\mathscr{C}$ is the data of two wide subcategories $\mathscr{L}, \mathscr{R} \subseteq \mathscr{C}$, identified with the two classes of morphisms they determine,

(1) $\mathrm{Iso}(\mathscr{C}) \subset \mathscr{L} \cap \mathscr{R}$;
(2) every morphism $f : x \to y$ in $\mathscr{C}$ factors as $f = l \circ r$, where $l \in \mathscr{L}$ and $r \in \mathscr{R}$;
(3) the factorization is functorial: any commutative diagram such as

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "\mathscr{L}\ni l"'] & \bullet \arrow[d, "l'\in\mathscr{L}"] \\
\bullet \arrow[r, dashed, "\exists!"] \arrow[d, "\mathscr{R}\ni r"'] & \bullet \arrow[d, "r'\in\mathscr{R}"] \\
\bullet \arrow[r, "v"'] & \bullet
\end{tikzcd}$$

admits a unique morphism (dotted) making the two squares commute.

Remark 1.2. Point (3) implies that the factorization of a morphism as $f = r \circ l$ is unique up to a~~n~~ unique isomorphism.

Remark 1.3. By [HM22, Proposition 3.10], each of the four subcategories of trees are equipped with a factorization system $(\mathscr{L}, \mathscr{R})$. Indeed:

- In $\mathbb{C}$ there are no isomorphisms ~~but~~ other than identities. The only nontrivial class of morphisms in the factorization system is the left one: we have $(\mathscr{L}, \mathscr{R}) = (\text{Inner Faces}, \{\mathrm{id}_T\}_T)$.
- Any map in $\mathbb{A}$ factors as an inner face map followed by an isomorphism, i.e. we have $(\mathscr{L}, \mathscr{R}) = (\mathbb{C}, \mathrm{Iso}(\mathbb{A}))$. In particular, for any tree T in $\mathbb{C}$, we have $\pi_0(\mathbb{A}/T) = \mathrm{Ob}(\mathbb{C}/T)$.
- Any morphism in $\mathbb{B}$ factors as a morphism in $\mathbb{A}$ followed by a root-preserving external face map:
$$(\mathscr{L}, \mathscr{R}) = (\mathbb{A}, \text{Ext Faces}).$$
- Any morphism in $\mathbf{\Omega}_r^o$ factors as a morphism in $\mathbb{A}$ followed by an external face map (non-necessarily root preserving):
$$(\mathscr{L}, \mathscr{R}) = (\mathbb{A}, \text{Ext Faces}).$$

Notation 1.4. For any vertex $v \in T$, we call C_v the subcorolla of T determined by v. We denote the corresponding subtree inclusion by $\hat{v} : C_v \to T$. The map $\hat{v}$ is an external face: it is hence a morphism in $\mathbf{\Omega}_r^o$, and it is a morphism in $\mathbb{B}$ if an only if v is the root vertex.

We denote by v_r, or sometimes by r when no ambiguity arise, the root vertex.

Definition 1.5. Consider a tree $T \in \mathbf{\Omega}_r^o$. A *decomposition of T of degree at most* 2 is a pushout square

Il faut préciser

leaf

définir

root

$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & S \arrow[d, "g"] \\ R \arrow[r, "f"'] & T \end{tikzcd}$$

inside $\mathbf{\Omega}_r^o$. We loosely write

$$T = S \bigcup R$$

for the pushout $T = S \cup_e R$, where e is the inner edge of T defined as $e = f(r_R)$, where r_R is the root of R. Observe that necessarily $e = g(l)$, where l is a leaf of S.

The decomposition is *trivial* whenever one between f or g is an isomorphism.

Remark 1.5. If the above diagram is cocartesian in $\mathbf{\Omega}_r^o$, then the two maps $S \to T, R \to T$ have to be inclusions of subtrees, namely external face maps, so in particular they are not morphisms in $\mathbb{A}$. This means that T can no-longer be expressed as a colimit of $S \to T$ and $R \to T$ when we move to the category $\mathbb{A}$. For this reason, from now on we will say that T is *the grafting of R onto the leaf l of S*.

maladroit il faut dire comment tu le définis

Definition 1.6. A *decomposition* of a tree T *of degree at most k* is given by:

(1) a decomposition of degree at most 2 of T, for which we write $T = S^b \bigcup S^t$, and
(2) decompositions of S^b and S^t of degree at most $k-1$,

such that overall there is at most one trivial decomposition of a tree appearing in the sequence.

Remark 1.6. An *isomorphism* of decomposition is an isomorphism of pushout diagrams. Unravelling the definition, we see that classes of isomorphisms of decompositions of a tree T are in bijection with the subsets of the set of internal edges of T $E \subseteq \mathrm{E}^{\mathrm{int}}(T)$.

From now on, whenever we talk about uniqueness of objects and arrows in $\mathbb{A}$ we will always mean uniqueness up to isomorphism.

définir

Construction 1.7. Let $\alpha\colon S \to T$ be a morphism in $\mathbb{A}$. For any vertex v of S, there exists a subtree $T(\alpha)_v \hookrightarrow T$, which we call *the blow-up of v by α*. The tree $T(\alpha)_v$ is uniquely determined by the $(\mathbb{A}, \mathrm{Ext\ Faces})$-factorization system in $\mathbf{\Omega}_r^o$: it is the essentially unique tree making the following diagram commute:

Définir

exact

$$\begin{tikzcd} C_v \arrow[r, "\hat{v}"] \arrow[d, dashed, "\mathbb{A}"'] & S \arrow[d, "\alpha"] \\ T(\alpha)_v \arrow[r, dashed, "\mathrm{ext}"'] & T \end{tikzcd}$$

charac

Equivalently, the subtrees $T(\alpha)_v$'s can be carachterized as the connected components of the planar graph that we obtain if we embed the tree T in the plane and we cut it along the edges in $\alpha(\mathrm{E}^{\mathrm{int}}(\operatorname{dom}\alpha))$. expliquer en quoi 1 morphisme α induit 1 fonction $E^{\mathrm{int}}(\mathrm{dom}\,\alpha) \to E^{\mathrm{int}}(\mathrm{cod}\,\alpha)$

Remark 1.8. Although decompositions of a tree cannot be expressed as colimits inside the category $\mathbb{A}$, we can parametrize them via morphisms into the tree.

Indeed, any morphism $\alpha\colon S \to T$ induces a decomposition of T into the grafting of all the subtrees $T(\alpha)_v$'s, and we loosely write

$$T = \bigcup_{v \in V(\alpha)} T(\alpha)_v.$$

We call the trees $T(\alpha)_v$ the *blocks* of the decomposition induced by α. We can go the other way round as well:

- If we are given a binary decomposition of T as $T = S \cup_e R$, then we can construct an inner face map inducing that decomposition. Indeed, we can construct a tree $T_{\{e\}}$ by contracting all inner edges of T *except* the edge e. Then there exists an essentially unique inner face map

$$\alpha_e \colon T_{\{e\}} \to T,$$

inducing a decomposition $T = T(\alpha_e)_{t(e)} \bigcup T(\alpha_e)_{s(e)}$, and by direct inspection we see that we recover the original subtrees S and R precisely as $S = T(\alpha_e)_{t(e)}$ and $R = T(\alpha_e)_{s(e)}$.

When there is no ambiguity, we write

$$T_{s(e)} \coloneqq T(\alpha_e)_{s(e)} \qquad \text{and} \qquad T_r \coloneqq T_{t(e)} = T(\alpha_e)_{t(e)},$$

where we identify the vertices $t(e)$ and $s(e)$ with the unique two vertices of $T_{\{e\}} = \operatorname{dom} \alpha$.

- More generally, for any set of inner edges $E \subseteq \mathrm{E}^{\mathrm{int}}(T)$, we call T_E the tree obtained from T by contracting all the inner edges in $\mathrm{E}^{\mathrm{int}}(T) \setminus E$; there exists a face map $\alpha_E \colon T_E \to T$, and α_E determines a decomposition of T into the subtrees obtained by cutting along the edges in E. Observe that in the category $\mathbb{C}$ there is a~~n~~ unique such α_E, while in $\mathbb{A}$ uniqueness is up to isomorphism.

Example 1.9.

Pas très heureux d'utiliser la même lettre / fonte pour un objet et un morphisme.

- For any natural number n, there is an essentially unique n-corolla C_n. There exists a morphism $C_n \to T$ in $\mathbb{A}$ if and only if n is the number of leaves of T, in which case the morphism is essentially unique. For this reason we write $C_T \to T$ for such a morphism and corolla. The corolla C_T corresponds to the tree $T_\emptyset$ obtained by contracting all inner edges in T, and the map

$$C_T = T_\emptyset \longrightarrow T$$

donner une intuition pour les directions de β et φ.

represents the trivial decomposition of T, where the only block is T itself.
- The identity map $\mathrm{id}_T \colon T \to T$ corresponds to the most fine decomposition of T, where for any vertex v of T there is the block $T(\mathrm{id}_T)_v = C_v$, so we can write

$$T = \bigcup_{v \in V(T)} C_v.$$

finest

Dire que $C_v = C_{|t^{-1}(v)|}$

Definition 1.7. The category of *dendroidal necklaces* **dNec** has:

- as objects *dendroidal necklaces*, i.e. the data of (T, α), where T is a tree in $\mathbb{A}$ and $\alpha \colon S \to T$ is a morphism in $\mathbb{A}$;
- as morphisms, maps $F = (\phi, f) \colon (T, \alpha) \to (Q, \beta)$ where ϕ, f are morphisms in $\mathbb{A}$ ~~and the following square~~ fitting in a above square commutes:

$$\begin{array}{ccc} \operatorname{dom}\alpha & \xrightarrow{\alpha} & T \\ {\scriptstyle\phi}\uparrow & & \downarrow{\scriptstyle f} \\ \operatorname{dom}\beta & \xrightarrow[\beta]{} & Q \end{array}$$

S plus naturel comme symbole ici.

• composition . . .

Definition 1.8. We say that a morphism $F = (\phi, f) \colon (T, \alpha) \to (Q, \beta)$ is *inert* if $f \colon T \xrightarrow{\sim} Q$ is an isomorphism, while a morphism is called *active* if $\phi \colon \operatorname{dom}\beta \xrightarrow{\sim} \operatorname{dom}\alpha$ is an isomorphism. We say that the morphism is *strictly inert* (resp. *strictly active*) when the isomorphism is the identity.

Remark 1.10. There is a forgetful functor

$$U \colon \mathbf{dNec} \to \mathbb{A}.$$

It sends a dendroidal necklace (T, α) to its *underlying tree* T, and a morphism $(f, \phi) \colon (T, \alpha) \to (S, \beta)$ to its *active part* $\phi \colon T \to S$. On the other hand, $\mathbb{A}$ can be embedded into **dNec** via a functor

$$j \colon \mathbb{A} \longrightarrow \mathbf{dNec}$$

which sends a tree T to the trivial decomposition $(T, \cancel{\alpha_T}\, T_\emptyset \to T)$ and a morphism $f \colon T \to S$ to the active morphism $(\mathrm{id}, f) \colon (T, \cancel{\alpha_T}\, T_\emptyset \to T) \to (S, \cancel{\alpha_S}\, S_\emptyset \to S)$. We have that $U \circ j = \mathrm{id}$.

$$\begin{array}{ccc} T_\emptyset & \longrightarrow & T \\ \| & & \downarrow \\ S_\emptyset & & S \end{array}$$

iso, induit par la bijection entre $T_\emptyset$ et $S_\emptyset$ induite par α.

T→S page précédente, i.e. S→T : éviter de trop changer

1.9. **Tree decompositions and active-inert morphisms.** We study some propert~~y~~ies of active and inert morphisms with respect to tree decompositions.

$$\begin{tikzcd} \mathrm{dom}\,\alpha \arrow[r, "\alpha"] \arrow[d, equal] & S \arrow[d] \\ \mathrm{dom}(f) \arrow[ur, phantom] \arrow[r, "\gamma"] & {} \end{tikzcd}$$

Construction 1.11. Consider a strictly active morphism between two dendroidal necklaces

$$(\mathrm{id}, f)\colon (S,\alpha) \to (T,\gamma).$$

In other words, it is a factorization $\gamma = f\circ\alpha$. In particular, T and S can be written as the graftings

$$S = \bigcup_{v\in \mathrm{dom}\,\alpha} S(\alpha)_v \qquad\qquad T = \bigcup_{v\in\mathrm{dom}\,\gamma} T(\gamma)_v$$

For any vertex v in $\mathrm{dom}\,\alpha = \mathrm{dom}\,\gamma$, the map f induces a morphism

$$f_v\colon S(\alpha)_v \longrightarrow T(\gamma)_v.$$

The morphism f_v is determined by the $(\mathbb{A}, \mathrm{Ext\ Faces})$-factorization system in $\mathbf{\Omega}^o_r$. Indeed, the map $\beta\circ\alpha\circ\hat{v}\colon C_v \to T$ can be factored in two ways:

f

$$\begin{tikzcd} C_v \arrow[r, "\hat{v}"] \arrow[d, dashed, "\mathbb{A}"] & R \arrow[r, "\alpha"] & S \arrow[d, "\beta"] \\ T(\beta\alpha)_v \arrow[rr, dashed, "\mathrm{ext}"] & & T \end{tikzcd} \qquad \begin{tikzcd} C_v \arrow[r, "\hat{v}"] \arrow[dr, dashed, "\mathbb{A}"] & R \arrow[r, "\alpha"] & S \arrow[r, "\beta"] & T \\ & S(\alpha)_v \arrow[ur, dashed, "\delta"] \arrow[rr, dashed, "\mathbb{A}"] & & \tilde{T} \arrow[u, dashed, "\mathrm{ext}"] \end{tikzcd}$$

By uniqueness we have $\tilde{T} \simeq T(\gamma)_v$, hence we get a map $f_v\colon S(\alpha)_v \to T(\gamma)_v$.

The original map $f\colon S\to T$ can be recovered as the grafting of the $f_v's$, and we write $f = \bigcup_v f_v$.

Proposition 1.12. *For any two composable strictly active morphisms*

$$(S,\alpha) \xrightarrow{(id,f)} (T, f\alpha) \xrightarrow{(id,g)} (T',\beta)$$

and any vertex $v\in V(dom\,\alpha)$, it holds that $(g\circ f)_v = g_v\circ f_v$.

Proof. Both morphisms $(g\circ f)_v$ and $g_v\circ f_v$ can be obtained as left maps in the factorization of the same map, and therefore they need to coincide.

Indeed, recall that $\beta = g\circ f\circ\alpha$; the morphism $(g\circ f)_v$ is uniquely determined by the diagram:

$$\begin{tikzcd} C_v \arrow[r, "\hat{v}"] \arrow[dr, "\mathbb{A}"] & R \arrow[r, "\alpha"] & S \arrow[r, "f"] & T \arrow[r, "g"] & T' \\ & S(\alpha)_v \arrow[ur, dashed] \arrow[rr, dashed, "(g\circ f)_v"] & & T'(\beta)_b \arrow[ur, dashed, "\mathrm{ext}"] & \end{tikzcd}$$

On the other hand, the composition $\gamma_v\circ\beta_v$ is given by:

$$\begin{tikzcd} C_v \arrow[r, "\hat{v}"] \arrow[dr, "\mathbb{A}"] & R \arrow[r, "\alpha"] & S \arrow[r, "\beta"] & T \arrow[r, "g"] & T' \\ & S(\alpha)_v \arrow[ur, dashed] \arrow[r, dashed, "\beta_v"] & T(f\alpha)_v \arrow[ur, dashed, "\mathrm{ext}"] \arrow[r, dashed, "\gamma_v"] & T'(\beta)_v \arrow[ur, dashed, "\mathrm{ext}"] & \end{tikzcd}$$

$\square$

Definition 1.10. Consider a tree $T\in\mathbb{A}$ and two decompositions $\alpha,\beta\in\mathbb{A}/T$. We say that (α,β) is a *nested decomposition* of T if every block determined by β is a subtree of a block determined by α. We say that α is the *first level* of the decomposition, while β is the second level.

In other words, (α,β) is a nested decomposition if for any $w\in\mathrm{dom}\,\beta$ there exists a unique vertex $v_w\in\mathrm{dom}\,\alpha$ such that $T(\beta)_w \subseteq T(\alpha)_{v_w}$.

If we call V_v the subset of $V(\operatorname{dom}\beta)$ formed by those vertices w for which $v = v_w$, then $V(\operatorname{dom}\beta) = \bigsqcup_{v\in\operatorname{dom}\alpha} V_v$ and for any $v \in \operatorname{dom}\alpha$ we have

$$T(\alpha)_v = \bigcup_{w\in V_v} T(\beta)_w.$$

Proposition 1.13. *For any tree T, a pair $(\alpha,\beta) \in (\mathbb{A}/T)^2$ is a nested decomposition if and only if there exists an inert morphism of necklaces $(f, id)\colon (T,\beta) \to (T,\alpha)$. Moreover, such inert morphism is unique up to isomorphism.*

Proof. Suppose it exists an inert morphism $(f,\mathrm{id})\colon (T,\beta) \to (T,\alpha)$ and let S denote the domain of β. For any vertex $v \in \alpha$, there is an induced decomposition morphism

$\alpha \to T \leftarrow \beta$

$$\beta_v\colon S(f)_v \longrightarrow T(\alpha)_v,$$

and we have that for any vertex $w \in S(f)_v$ there is a natural isomorphism

$$(T(\alpha)_v(\beta_v))_w \simeq T(\beta)_w.$$

Thus (α, β) is a nested decomposition

For the reverse implication, suppose that (α,β) is a nested decomposition, and select a vertex $v \in \operatorname{dom}\alpha$. The decomposition of $T(\alpha)_v$ as the grafting of the $T(\beta)_w$, for $w \in V_v$, corresponds to a morphism $S_v \to T(\alpha)_v$, and these assemble to a morphism of trees

$$S \coloneqq \bigcup_{v\in\operatorname{dom}\alpha} S_v \longrightarrow \bigcup_{v\in\operatorname{dom}\alpha} T(\alpha)_v = T \quad (*).$$

? ce symbole ne semble pas avoir été introduit.

By the correspondence between decomposition of trees and morphisms in the slice category $\mathbb{A}/T$, the map $(*)$ has to coincide (up to isomorphism) with β, and it is immediate to check that the assignment

$$V(\operatorname{dom}\alpha) \ni v \longrightarrow S_v \in \mathsf{Subtrees}(S)$$

uniquely extends to a leaves-preserving morphism of trees $f\colon \operatorname{dom}\alpha \to S$ which is injective on edges. This means that f is an inner face map, and by construction we have (up to isomorphism) the factorization $\alpha = \beta\circ f$, as wanted. $\square$

From uniqueness of left maps in the factorization system for $\mathbb{A}$ we also deduce the following.

Proposition 1.14. *Consider a sequence of two inert morphisms*

$$(T,\gamma) \xrightarrow{(\cancel{f},id)} (T,\beta) \xrightarrow{(\cancel{g},id)} (T,\chi).$$

φ, ψ ...

Denote by R the domain of β and by S the domain of γ. For any vertex $v \in \operatorname{dom}\chi$, there are induced morphisms

$$\beta_v\colon R(g)_v \longrightarrow T(\chi)_v, \qquad \gamma_v\colon S(gf)_v \longrightarrow T(\chi)_v, \qquad f_v\colon R(g)_v \longrightarrow S(gf)_v,$$

and we have that $\beta_v = \gamma_v \circ f_v$.

2. The tree comonad

We denote by $\mathrm{Ch}(\mathbb{A})$ the category of functors $\mathrm{Fun}(\mathbb{A}^{\mathrm{op}},\mathrm{Ch})$. In this section we construct a comonad on $\mathrm{Ch}(\mathbb{A})$ which describes linear ∞-preoperads, as we shall see in section 3.

Notation 2.1. To a tree $T \in \mathbf{\Omega}$ and a morphism $\alpha\colon S \to T$ we can associate the following numbers:

- $l_T = \#\mathrm{Leaves}(T)$;
- $\dim(T) = \#\mathrm{E}^{\mathrm{int}}(T)$;
- $\dim(\alpha) = \#\alpha(\mathrm{E}^{\mathrm{int}}(S)))$;
- $\mathrm{codim}(\alpha) = \dim(T) - \dim(\alpha)$.

Observe that, if $\alpha : S \to T$ is a morphism in $\mathbb{A}$, then $\dim(\alpha) = \dim(S)$.

utiliser \mathit{KM} pour rapprocher le M du K.

Construction 2.2. Given $M \in \mathrm{Ch}(\mathbb{A})$ and a tree $T \in \mathbb{A}$, define the chain complex $KM(T)$ as:

$$KM(T) := \left(\prod_{\alpha \in \mathbb{A}/T} \bigotimes_{v \in \operatorname{dom} \alpha} M(T(\alpha)_v) \right)^{\mathrm{inv}},$$

where invariants are taken under the action of isomorphisms of $\mathbb{A}/T$. Observe that the product is finite, so the above expression may as well be written by using direct sums since $\mathsf{Ch}(R)$ is abelian. We can get rid of invariants by taking representatives of classes of morphisms, and in this case we write $KM(T)$ as the finite product

$$KM(T) \simeq \prod_{\alpha \in \mathbb{C}/T} \bigotimes_{v \in \operatorname{dom} \alpha} M(T(\alpha)_v).$$

Observe that we can equivalently describe $KM(T)$ via dendroidal necklaces: the first product ranges over dendroidal necklaces X with underlying tree T, i.e. of the form (T, α), while the second product can be seen as parametrized by those $X \in \pi_0(\mathbf{dNec})$ having T as underlying tree.

In particular, for any $\alpha \in \mathbb{A}/T$, the α-*component* of $KM(T)$ is, by definition, the chain complex given by given by

$$(\mathrm{proj}_\alpha \circ KM)(T) = \bigotimes_{v \in \operatorname{dom} \alpha} M(T(\alpha)_v).$$ X

For a tree morphism $\beta \colon T \to T'$ in $\mathbb{A}$, we define the map $KM(\beta) \colon KM(T') \to KM(T)$ via its components $\mathrm{proj}_\alpha \circ KM(\beta)$, for any $\alpha \colon S \to T$ in $\mathbb{A}$: not introduced.

$$\mathrm{proj}_\alpha \circ KM(\beta) := \left(\bigotimes_{v \in \operatorname{dom} \alpha} M(\beta_v) \right) \circ \mathrm{proj}_{\alpha\beta}.$$

For $\psi \colon M \to N$ morphism in $\mathrm{Ch}(\mathbb{A})$, the map $K(\psi) \colon KM \to KN$ is defined, for any tree $T \in \mathbb{A}$, as:

$$(K\psi)_T := \left(\prod_{\alpha \in \mathbb{A}/T} \bigotimes_{v \in \operatorname{dom} \alpha} \psi_{T(\alpha)_v} \right)^{\mathrm{inv}}.$$

In particular, the map $(K\psi)_T$ sends the α-component of $KM(T)$ to the α-component of $KN(T)$.

Proposition 2.3. *The assignments just defined determine a functor*

$$K \colon \mathit{Ch}(\mathbb{A}) \longrightarrow \mathit{Ch}(\mathbb{A}).$$

Proof. First of all, we need to check that for any $M \in \mathrm{Ch}(\mathbb{A})$, KM is functorial on tree morphisms. It is clear that $KM(\mathrm{id}_T) = \mathrm{id}_{KM(T)}$ for any tree T, so we need to check that, for any two morphisms $R \xrightarrow{\gamma} T \xrightarrow{\beta} T'$, we have $KM(\beta \circ \gamma) = KM(\gamma) \circ KM(\beta)$. Fix $\alpha \in \mathbb{A}/R$, then we have the equalities

$$\begin{aligned}
&\mathrm{proj}_\alpha \circ KM(\beta\gamma) = \\
&= \left(\bigotimes_{v \in \operatorname{dom} \alpha} M((\beta\gamma)_v) \circ \mathrm{proj}_{\beta\gamma\alpha} \right) && \text{naturality of projection} \\
&= \left(\bigotimes_{v \in \operatorname{dom} \alpha} M((\gamma)_v) \circ \bigotimes_{v \in \operatorname{dom} \alpha} M(\beta_v) \circ \mathrm{proj}_{\beta\gamma\alpha} \right) && \text{Proposition 1.12} \\
&= \bigotimes_v M(\gamma_v) \circ \mathrm{proj}_{\gamma\alpha} \circ KM(\beta) && \text{naturality of projection} \\
&= \mathrm{proj}_\alpha \circ KM(\gamma) \circ KM(\beta). && \text{naturality of projection}
\end{aligned}$$

Since this holds for any $\alpha \in \mathbb{A}/R$, the desired equality holds.

Secondly, we need to verify that K is functorial with respect to morphisms in $\mathrm{Ch}(\mathbb{A})$. Consider two composable maps $M \xrightarrow{\psi} N \xrightarrow{\phi} L$: since $K\psi$ sends the α-component of $KM(T)$ to the α-component of $KN(T)$ for any $S \in \mathbb{A}$, one needs to check that, for any $v \in \operatorname{dom}\alpha$, we have

$$\bigotimes_{v \in \operatorname{dom}\alpha} (\psi \circ \phi)_{T(\alpha)_v} = \bigotimes_{v \in \operatorname{dom}\alpha} \psi_{T(\alpha_v)} \circ \phi_{T(\alpha)_v},$$

which is true since ψ and ϕ are natural transformations.

□

2.1. **The comonad structure.** We now extend K to a comonad by defining comultiplication and counit.

Recall that, for any $M \in \mathrm{Ch}(\mathbb{A})$ and any tree T, we can write $KM(T)$ by using dendroidal necklaces as

$$KM(T) = \left(\prod_{(T,\alpha)} \bigotimes_{v \in \operatorname{dom}\alpha} M(T(\alpha)_v) \right)^{\mathrm{inv}}.$$

As pointed out in Proposition 1.13, isomorphism classes of inert morphisms into a fixed dendroidal necklace (T, α) represent refinements of the partition of T given by α. Indeed, a couple of partitions of T (α, β) forms a nested partition if and only if the class ~~of~~ isomorphism of inert maps $(T, \beta) \rightarrowtail (T, \alpha)$ is non-empty, and this happens if and only if there exists a factorization $\alpha = \beta \circ f$. In particular, f is (essentially) uniquely determined by α and β.

We use this formalism to express the iterated functor K^2.

Indeed, for any $M \in \mathrm{Ch}(\mathbb{A})$ and any tree T, the object $K^2M(T)$ is, by definition, the chain complex

$$K^2M(T) = \left(\prod_{(T,\alpha)} \bigotimes_{v \in \operatorname{dom}\alpha} \left(\prod_{(T(\alpha)_v, \beta_v)} \bigotimes_{w \in \operatorname{dom}\beta_v} M((T(\alpha)_v(\beta_v))_w) \right)^{\mathrm{inv}} \right)^{\mathrm{inv}};$$

As observed in Proposition 1.13, $(T(\alpha)_v(\beta_v)))_w = T(\beta)_w$, and since finite product commutes with tensor product we can rewrite the above as

$$K^2M(T) = \left(\prod_{(T,\alpha)} \prod_{(T,\beta) \rightarrowtail (T,\alpha)} M(T(\beta)_w) \right)^{\mathrm{inv}} = \left(\prod_{(T,\alpha)} \prod_{(\alpha,\beta) \text{nested}} M(T(\beta)_w) \right)^{\mathrm{inv}},$$

where invariants are taken with respect to isomorphism classes of dendroidal necklaces (T, α) with underlying tree T and of inert morphisms into (T, α) (or, equivalently, of dendroidal necklaces (T, α) with underlying tree T and of nested decompositions (α, β)).

In particular, we can get rid of invariants if we consider $(T, \alpha) \in \pi_0(\mathbf{dNec})$ and strictly inert maps $(f, \mathrm{id}) \colon (T, \beta) \rightarrowtail (T, \alpha)$.

Example 2.4. For any $(T, \alpha) \in \pi_0(\mathbf{dNec})$, the pair (α, α) corresponds to the trivial inert morphism $(\mathrm{id}, \mathrm{id}) \colon (T, \alpha) \hookrightarrow (T, \alpha)$. In particular, for any vertex v of $\operatorname{dom}\alpha$, there is an isomorphism $(\operatorname{dom}\alpha)(\mathrm{id})_v \simeq C_v$ (the subcorolla of T determined by v), and the induced map α_v is just the trivial decomposition of $T(\alpha)_v$, i.e.

$$\alpha_v \colon C_v = C_{T(\alpha)_v} \longrightarrow T(\alpha)_v.$$

In conclusion, we have

$$\mathrm{proj}_{(\alpha,\alpha)} \circ K^2M(T) = \bigotimes_{v \in \operatorname{dom}\alpha} M(T(\alpha)_v).$$

Construction 2.5. We define the comultiplication $\Delta_{M,T} \colon KM(T) \to K^2M(T)$ by defining its (α, β)-components as

$$\mathrm{proj}_{(\alpha,\beta)} \circ \Delta_{M,T} \coloneqq \mathrm{proj}_\beta.$$

In other words, $\Delta_{M,T}$ is the unique map making the following triangles commute, for any nested decomposition (α,β):

$$\begin{tikzcd}
\prod\limits_{(T,\alpha)} \bigotimes\limits_{v\in \mathrm{dom}\,\alpha} M(T(\alpha)_v) \arrow[rr, "\Delta_{M,T}"] \arrow[dr, "\mathrm{proj}_\beta"'] & & \prod\limits_{(T,\alpha)} \prod\limits_{\cancel{(T,\beta)\to(T,\alpha)}} \bigotimes\limits_{w\in \mathrm{dom}\,\beta} M(T(\beta)_w) \arrow[dl, "\mathrm{proj}_{\alpha,\beta}"] \\
& \bigotimes\limits_{w\in V(R)} M(T(\beta)_w)) &
\end{tikzcd}$$

(α,β) nested

Define $\epsilon_{M,T}\colon KM(T)\to M(T)$ as the projection on the component relative to the trivial decomposition $C_T\to T$.

It is clear that $\epsilon\colon K\to 1_{\mathrm{Ch}(\mathbb{A})}$ is a natural transformation. The same holds for Δ.

Proposition 2.6. *The map $\Delta\colon K\to K^2$ is a natural transformation.*

Proof. Fix $M\in \mathrm{Ch}(\mathbb{A})$ and a tree morphism $\alpha\colon S\to T$, we want to check that

$$\mathrm{proj}_{(\chi,\beta)}\circ K^2M(\alpha)\circ\Delta_{M,T} = \mathrm{proj}_{(\chi,\beta)}\circ\Delta_{M,S}\circ KM(\alpha) \quad (*)$$

for any nested decomposition (χ,β) determining a component in $K^2M(S)$. The left hand side in expression (*) fits into the commutative diagram

$$\begin{tikzcd}
KM(T) \arrow[r, "\Delta_{M,T}"] \arrow[dr, "\mathrm{proj}_{(\alpha,\beta)}"'] & K^2M(T) \arrow[r, "K^2M(\alpha)"] \arrow[d, "\mathrm{proj}_{(\alpha\chi,\alpha\beta)}"] & K^2M(S) \arrow[d, "\mathrm{proj}_{(\chi,\beta)}"] \\
& \bigotimes\limits_{w\in V(R)} M(T(\alpha\beta)_w) \arrow[r] & \bigotimes\limits_{w\in V(R)} M(S(\beta)_w)
\end{tikzcd}$$

while the right hand side fits in the commutative diagram

$$\begin{tikzcd}
KM(T) \arrow[r, "KM(\alpha)"] \arrow[dr, "\mathrm{proj}_{(\alpha,\beta)}"'] & KM(S) \arrow[r, "\Delta_{M,S}"] \arrow[dr, "\mathrm{proj}_\beta"] & K^2M(S) \arrow[d, "\mathrm{proj}_{(\chi,\beta)}"] \\
& \bigotimes\limits_{w\in \mathrm{dom}\,\beta} M(T(\alpha\beta)_w) \arrow[r] & \bigotimes\limits_{w\in \mathrm{dom}\,\beta} M(S(\beta)_w))
\end{tikzcd}$$

where in both diagrams the bottom horizontal arrow is given by $\bigotimes_{w\in\mathrm{dom}\,\beta} M(\alpha_w)$.

In particular, the equality $(*)$ holds, as wanted.

Consider now a morphism $\psi\colon M\to N$ in $\mathrm{Ch}(\mathbb{A})$, we need to check that, for any tree $S\in\mathbb{A}$, we have the equality

$$(K^2\psi)_S\circ\Delta_{M,S} = \Delta_{N,S}\circ(K\psi)_S.$$

Fix an admissible pair (χ,β). The following two diagrams commute:

terminologie non introduite

nested decomposition?

$$\begin{tikzcd}
KM(S) \arrow[r, "\Delta_{M,S}"] \arrow[dr, "\mathrm{proj}_\beta"'] & K^2M(S) \arrow[r, "(K^2\psi)_T"] \arrow[d, "\mathrm{proj}_{(\chi,\beta)}"] & K^2N(S) \arrow[d, "\mathrm{proj}_{(\chi,\beta)}"] \\
& \bigotimes\limits_{w\in \mathrm{dom}\,\beta} M(S(\beta)_w) \arrow[r] & \bigotimes\limits_{w\in \mathrm{dom}\,\beta} N(S(\beta)_w)
\end{tikzcd}$$

$$\begin{tikzcd}
KM(S) \arrow[r, "(K\psi)_S"] \arrow[dr, "\mathrm{proj}_\beta"'] & KN(S) \arrow[r, "\Delta_{N,S}"] \arrow[dr, "\mathrm{proj}_\beta"] & K^2N(S) \arrow[d, "\mathrm{proj}_{(\chi,\beta)}"] \\
& \bigotimes\limits_{w\in \mathrm{dom}\,\beta} M(S(\beta)_w) \arrow[r] & \bigotimes\limits_{w\in \mathrm{dom}\,\beta} N(S(\beta)_w)
\end{tikzcd}$$

where the lower horizontal arrow of both diagrams is given by $\bigotimes_{w\in V(R)} \psi_{T(\beta)_w}$.

This proves that $\mathrm{proj}_{(\chi,\beta)} \circ (K^2\psi)_S \circ \Delta_{M,S} = \mathrm{proj}_{(\chi,\beta)} \circ \Delta_{N,S} \circ (K\psi)_S$, and since this holds for any nested decomposition (χ, β) the statement is proven. $\square$

We need to prove the comonadic identities.

Proposition 2.7. *The following diagram commutes:*

$$\begin{tikzcd}
K \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & K^2 \arrow[d, "K\Delta"] \\
K^2 \arrow[r, "\Delta K"'] & K^3
\end{tikzcd}$$

Proof. Fix $M \in \mathrm{Ch}(\mathbb{A})$ and a tree $T \in \mathbb{A}$. Observe that we can write $K^3M(T)$ as the product

$$K^3M(T) = \prod_{(\chi,\beta,\gamma)} \bigotimes_{v\in \mathrm{dom}\,\gamma} M(T(\gamma)_v),$$

where (χ, β, γ) ranges over the isomorphism classes of composable inert morphisms, represented by strictly iner maps

inner

$$(T, \gamma) \overset{(f,\mathrm{id})}{\rightarrowtail} (T, \beta) \overset{(g,\mathrm{id})}{\rightarrowtail} (T, \chi).$$

We check the coassociativity condition by proving that

$$\mathrm{proj}_{(\chi,\beta,\gamma)}((K\cancel{\Delta}) \circ \Delta)_{M,T} = \mathrm{proj}_{(\chi,\beta,\gamma)}(\Delta K \circ \Delta)_{M,T} \quad (*)$$

for any such triple. Denote by $U = \mathrm{dom}\,\chi$, $R = \mathrm{dom}\,\beta$ and $S = \mathrm{dom}\,\gamma$. Consider the LHS in $(*)$, then we can build the commutative diagram:

$$\begin{tikzcd}
KM(T) \arrow[r, "\Delta_{M,T}"] \arrow[ddddrr, bend right, "\mathrm{proj}_\gamma"'] & K^2M(T) \arrow[r, "(K\Delta_M)_T"] \arrow[d, "\mathrm{proj}_\chi"] \arrow[ddddr, bend right, "\mathrm{proj}_{(\chi,\gamma)}"] & K^3M(T) \arrow[d, "\mathrm{proj}_\chi"] \\
& \bigotimes_{u\in U} KM(T(\chi)_u) \arrow[r, "\bigotimes_u \Delta_{M,T(\chi)_u}"] \arrow[dr, bend right, "\bigotimes_u \mathrm{proj}_{\gamma_u}"'] & \bigotimes_{u\in U} K^2M(T(\chi)_u) \arrow[d, "\bigotimes_u \mathrm{proj}_{(\beta_u,\gamma_u)}"] \\
& & \bigotimes_{u\in U} \bigotimes_{w\in S(fg)_u} M(T(\chi)_u(\gamma_u)_w) \arrow[d, "\simeq"] \\
& & \bigotimes_u \bigotimes_w M(T(\gamma)_w)) \arrow[d, "\simeq"] \\
& & \bigotimes_{w\in S} M(T(\gamma)_w
\end{tikzcd}$$

Interrompe la ligne [je ne sais pas ce que tu utilises pour écrire les diagrammes)

where the composition of the vertical arrows on the right hand side of the diagram gives precisely $\mathrm{proj}_{(\chi,\beta,\gamma)}$.

On the other hand, by considering the RHS in $(*)$, we obtain the following commutative diagram:

ΔK

$$\begin{tikzcd}
KM(T) \arrow[r, "\Delta_{M,T}"] \arrow[dddddrr, bend right, "\mathrm{proj}_\gamma"'] & K^2M(T) \arrow[r, "(K\Delta_M)_T"] \arrow[dr, "\mathrm{proj}_\beta"'] \arrow[dddddr, bend right, "\mathrm{proj}_{(\beta,\gamma)}"'] & K^3M(T) \arrow[d, "\mathrm{proj}_{(\chi,\beta)}"] \\
& & \bigotimes_{r\in R} KM(T(\beta)_r) \arrow[d, "\bigotimes_r \mathrm{proj}_{\gamma_r}"] \\
& & \bigotimes_{r\in R}\bigotimes_{w\in S(f)_r} M(T(\beta)_r(\gamma_r))_w) \arrow[d, "\simeq"] \\
& & \bigotimes_{r\in R}\bigotimes_{w\in S(f)_r} M(T(\gamma)_w) \arrow[d, "\simeq"] \\
& & \bigotimes_{w\in S} M(T(\gamma)_w)
\end{tikzcd}$$

Since here as well the composition of the vertical arrows on the right hand side gives $\mathrm{proj}_{(\chi,\beta,\gamma)}$, the thesis is proved. □

Proposition 2.8. *The following diagram commutes:*

$$\begin{tikzcd}
& K \arrow[dl, equal] \arrow[d, "\Delta"] \arrow[dr, equal] & \\
K & K^2 \arrow[l, "\epsilon K"] \arrow[r, "K\epsilon"'] & K
\end{tikzcd}$$

Proof. Fix $M \in \mathrm{Ch}(\mathbb{A})$ and a tree $T \in \mathbb{A}$. We check commutativity of the above triangles component-wise. Choose $\alpha \in \mathbb{A}/T$. The triangle on the left hand side commutes because of the commutative diagram

$$\begin{tikzcd}
KM(T) \arrow[r, "\Delta_{M,T}"] \arrow[dr, "\mathrm{proj}_\alpha"'] & K^2M(T) \arrow[r, "\epsilon_{KM,T}"] \arrow[d, "\mathrm{proj}_{(C_T\to T,\alpha)}"] & KM(T) \arrow[dl, "\mathrm{proj}_\alpha"] \\
& \bigotimes_{v\in\mathrm{dom}\,\alpha} M(T(\alpha)_v) &
\end{tikzcd}$$

proving that $\mathrm{proj}_\alpha \circ (\epsilon K\Delta)_{M,T} = \mathrm{proj}_\alpha$, as wanted.

For the right hand side, a similar argument applies, in the form of the commutative diagram:

$$\begin{tikzcd}
KM(T) \arrow[r, "\Delta_{M,T}"] \arrow[drr, bend right, "\mathrm{proj}_\alpha"'] & K^2M(T) \arrow[r, "(K\epsilon_M)_T"] \arrow[d, "\mathrm{proj}_\alpha"] \arrow[dr, "\mathrm{proj}_{(\alpha,\alpha)}"] & KM(T) \arrow[d, "\mathrm{proj}_\alpha"] \\
& \bigotimes_{v\in\mathrm{dom}\,\alpha} KM(T(\alpha)_v) \arrow[r, "\bigotimes_v \epsilon_{M,T(\alpha)_v}"] & \bigotimes_{v\in\mathrm{dom}\,\alpha} M(T(\alpha)_v)
\end{tikzcd}$$

□

Je suggère 1 seul compteur par Définition, Remarks etc
C'est beaucoup plus facile pour retrouver par ex "Remark 1.8" qui ne sera plus avant la Déf. 1.7 ...

3. Coalgebras over K are ∞-preoperads

Here we assume to work with reduced presheaves, namely $M \in \mathrm{Ch}(\mathbb{A})$ such that $M(\eta) = R$, where R is the ground ring. In particular, for ∞-preoperads we take $\theta_\eta = \mathrm{id}$. Otherwise, need to understand how to obtain canonically/meaningfully a map $M(\eta) \to R$.

When we drop units from the definitions it means that our operads are equivalent to the augmentation ideal of augmented operads, whereas the cobar or bar constructions are equivalent to the augmentation ideal of the classical unital cobar construction.

In this section, we use notation introduced in Remark 1.8.

Recall that ~~**Definition 3.1.**~~ a K-*coalgebra* is (M, θ), where $M \in \mathrm{Ch}(\mathbb{A})$ and $\theta\colon M \to KM$ is a natural transformation making the following two diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\theta"] & KM \arrow[d, "\Delta_M"] \\ KM \arrow[r, "K\theta"] & K^2M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\theta"] \arrow[dr, equal] & \\ KM \arrow[r, "\epsilon_M"] & M \end{tikzcd}$$

Definition 3.2. A *linear ∞-preoperad* is a functor $M \in \mathrm{Ch}(\mathbb{A})$ together with structure maps

$$\theta_{T,e}\colon M(T) \longrightarrow M(T_{s(e)}) \otimes M(T_{t(e)})$$

for any tree T and any internal edge $e \in \mathrm{E}^{\mathrm{int}}(T)$, which have to satisfy:

- *Coassociativity*: for any two edges $a, b \in \mathrm{E}^{\mathrm{int}}(T)$,
  - if a and b are not comparable in the poset $\mathrm{E}^{\mathrm{int}}(T)$,then
  $$(\theta_{T_{t(a)},b} \otimes 1_{M(T_{s(a)})}) \circ \theta_{T,a} = (1_{M(T_{s(b)})} \otimes \theta_{T_{t(b)},a}) \circ \theta_{T,b}.$$
  - if $a > b$ in $\mathrm{E}^{\mathrm{int}}(T)$, then
  $$(1_{M(T_{s(a)})} \otimes \theta_{T_{t(a)},b}) \circ \theta_{T,a} = (\theta_{T_{s(b)},a} \otimes 1_{M(T_{t(b)})}) \circ \theta_{T,b}.$$
- *Naturality*: whenever a morphism $\gamma\colon T \to T'$ is obtained as the grafting of two morphisms $\chi\colon R \to R'$ and $\beta\colon S \to S'$ along a leaf a of R, namely
  $$\gamma = \chi \cup_a \beta\colon T = R \bigcup_a S \to R' \bigcup_{\chi(a)} S' = T'$$
  then the following diagram has to commute:
  $$\begin{tikzcd} M(T') \arrow[r, "\theta_{T',\chi(a)}"] \arrow[d, "M(\gamma)"'] & M(R') \otimes M(S') \arrow[d, "M(\chi)\otimes M(\beta)"] \\ M(T) \arrow[r, "\theta_{T,a}"'] & M(R) \otimes M(S). \end{tikzcd}$$

Here some pictures

Remark 3.1. In other words, naturality and coassociativity conditions are equivalent to asking that $(M, \{\theta_{T,e}\}_{T,e}$ is a lax monoidal functor with respect to the operadic composition in $\mathbb{A}$ and the tensor product in Ch.

Remark 3.2.

- The coassociativity condition deals with the fact that, for any two different internal edges $a, b \in \mathrm{E}^{\mathrm{int}}(T)$, the decomposition morphism $\alpha_{\{a,b\}}$ can be written as $\alpha_{\{a,b\}} = \gamma \circ \alpha_{\{a\}}$ or as $\alpha_{\{a,b\}} = \gamma' \circ \alpha_{\{b\}}$. Whether a and b are comparable or not in the poset $\mathrm{E}^{\mathrm{int}}(T)$ determines different blocks in the decompositions of T. In particular:
  - For any two non-comparable internal edges $a, b \in \mathrm{E}^{\mathrm{int}}(T)$, we have that
  $$T_{s(a)} = T(\alpha_a)_{s(a)} = T(\alpha_{\{a,b\}})_{s(a)}$$

and

$$T_{s(b)} = T(\alpha_b)_{s(b)} = T(\alpha_{\{a,b\}})_{s(b)}.$$

– If $a > b$ in $\mathrm{E}^{\mathrm{int}}(T)$, then

$$T_{s(a)} = T(\alpha_a)_{s(a)} = T(\alpha_{\{a,b\}})_{s(a)})$$

but

$$T_{s(b)} = T(\alpha_b)_{s(b)} \neq T(\alpha_{\{a,b\}})_{s(b)}.$$

Or here some picture

- Any morphism of trees $\gamma\colon T \to T'$ can be written as a grafting of two morphisms. Indeed, if one considers an internal edge $a \in \mathrm{E}^{\mathrm{int}}(T)$ and defines $\chi = \gamma_{t(a)}$ and $\beta = \gamma_{s(a)}$, we get that $\gamma = \chi \cup_a \beta$.

3.3. K-coalgerbas are ∞-preoperads.

Proposition 3.3. *Let (M, θ) be a K-coalgebra. Then $(M, \theta_{T,e})_{T\in\mathbb{A}, e\in E^{int}(T)}$ is an ∞-preoperad, where $\theta_{T,e} := proj_{\alpha_{\{e\}}} \circ \theta_T$.*

harmoniser partout α_e versus $\alpha_{\{e\}}$

Proof. We need to check naturality and coassociativity.

For the naturality condition, consider a morphism $\gamma = \chi \cup_a \beta$ as in Definition 3.2. The naturality diagram can be factored as:

$$\begin{array}{ccccc}
M(T') & \xrightarrow{\theta_{T'}} & KM(T') & \xrightarrow{\mathrm{proj}_{\alpha_{\chi(a)}}} & M(R') \otimes M(S') \\
{\scriptstyle M(\gamma)}\downarrow & & {\scriptstyle KM(\gamma)}\downarrow & & \downarrow{\scriptstyle M(\gamma_{t(a)})\otimes M(\gamma_{s(a)}) = M(\chi)\otimes M(\beta)} \\
M(T) & \xrightarrow[\theta_T]{} & KM(T) & \xrightarrow[\mathrm{proj}_{\alpha_a}]{} & M(R) \otimes M(S)
\end{array}$$

The square on the left commutes because of naturality of θ, while the right one commutes because of the definition of the action of KM on tree morphisms. As a consequence, the naturality diagram commutes as a whole, as wanted.

We are left to check the coassociativity condition. First of all, we reformulate it as follows.

Pas clair. Pour énoncer 3.4, tu as besoin que M soit 1 coalgèbre, puisque tu utilises θ_T et pas seulement $\theta_{T,e}$

Proposition 3.4. *The coassociativity condition for a presheaf $M \in Ch(\mathbb{A})$ with maps $\{\theta_{T,e}\}_{T,e}$ is equivalent to the following: for any choice of two different internal edges $a, b \in E^{int}(T)$, it holds that*

implied by suffit (et c'est ce que tu fais!)

$$\mathrm{proj}_{(\alpha_a, \alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T = \mathrm{proj}_{(\alpha_b, \alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T \quad (*).$$

Now, if Proposition 3.4 is true we can conclude our proof, since

$$\begin{aligned}
&\mathrm{proj}_{(\alpha_a,\alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T = && \\
&= \mathrm{proj}_{(\alpha_a,\alpha_{\{a,b\}})} \circ (\Delta_M \circ \theta)_T && \text{comonadic identities} \\
&= \mathrm{proj}_{(\alpha_a,\alpha_{\{a,b\}})} \circ \Delta_{M,T} \circ \theta_T && \text{naturality} \\
&= \mathrm{proj}_{\alpha_{\{a,b\}}} \circ \theta_T && \\
&= \mathrm{proj}_{(\alpha_b,\alpha_{\{a,b\}})} \circ \Delta_{M,T} \circ \theta_T && \text{naturality of projection} \\
&= \mathrm{proj}_{(\alpha_b,\alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T. && \text{comonadic identities}
\end{aligned}$$

Therefore we are left with proving the claim.

Proof (of Proposition 3.4) . Suppose that a and b are incomparable internal edges in T and consider the following diagram:

Reformuler Coalg ⇒ (*) ⇒ coassociativity.

$$(\theta_{T_{t(a)},b} \otimes 1_{M(T_{s(a)})}) \circ \theta_{T,a} = (1_{M(T_{s(b)})} \otimes \theta_{T_{t(b)},a}) \circ \theta_{T,b}.$$

$$\begin{tikzcd}
M(T) \arrow[r, "\theta_T"] \arrow[dr, "\theta_{T,a}"'] & KM(T) \arrow[r, "K\theta_T"] \arrow[d, "\mathrm{proj}_{\alpha_a}"] & K^2M(T) \arrow[d, "\mathrm{proj}_{\alpha_a}"] \\
 & M(T_{t(a)}) \otimes M(T_{s(a)}) \arrow[r, "\theta_{T_{t(a)}} \otimes \theta_{T_{s(a)}}"] & KM(T_{t(a)}) \otimes KM(T_{s(a)}) \arrow[d, "\mathrm{proj}_{\alpha_{\{a,b\}_{t(a)}}} \otimes \mathrm{proj}_{\alpha_{\{a,b\}_{s(a)}}}"] \\
 & & M(T_r) \otimes M(T_{s(b)}) \otimes M(T_{s(a)})
\end{tikzcd}$$

where the ~~the~~ composition of the two vertical arrows on the right is precisely $\mathrm{proj}_{(\alpha_a, \alpha_{\{a,b\}})}$. By Remark 3.2, $\alpha_{\{a,b\}_{s(a)}} : C_{T_{s(a)}} \to T_{s(a)}$ is the trivial decomposition morphism, and by the coalgebra identities, we have that

$$\mathrm{proj}_{\alpha_{\{a,b\}_{s(a)}}} \circ \theta_{T_{s(a)}} = \epsilon_{M,T_{s(a)}} \circ \theta_{T_{s(a)}} = 1_{M(T_{s(a)})},$$

so the diagram commutes, proving that

$$\mathrm{proj}_{(\alpha_a,\alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T = (\theta_{T_{t(a)},b} \otimes 1_{M(T_{s(a)})}) \circ \theta_{T,a}.$$

Mutatis mutandis, the same diagram and the same arguments prove that

$$\mathrm{proj}_{(\alpha_b,\alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T = (1_{M(T_{s(b)})} \otimes \theta_{T_{t(b)},a}) \circ \theta_{T,b},$$

so the proposition is proven for a and b independent internal edges.

Dis en effet ce qui est différent.

By similar arguments we conclude also in the case where $a > b$ in $\mathrm{E}^{\mathrm{int}}(T)$. Should I specify how to do the equality in the RHS, given that it is what is different from the non-comparable case? □

$$\mathrm{proj}_{(\alpha_a,\alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T = \mathrm{proj}_{(\alpha_b,\alpha_{\{a,b\}})} \circ (K\theta \circ \theta)_T$$ □

3.4. ∞-preoperads are K-coalgebras.

We prove that the structure maps of an ∞-preoperad M assemble into a K-coalgebra map $M \to KM$.

(ou bien subsets of the set of ...)

Recall that, whenever we fix a tree $T \in \mathbb{A}$, isomorphism classes of dendroidal necklaces with underlying tree T are parametrized by subsets of the inner edges of T: a subset $E \subseteq \mathrm{E}^{\mathrm{int}}(T)$ corresponds to the dendroidal necklace $(T, \alpha_E : T_E \to T)$, and with this notation we can write the comonad as

$$KM(T) = \prod_{E \subseteq \mathrm{E}^{\mathrm{int}}(T)} \bigotimes_{v \in \mathrm{dom}\,\alpha_E} M(T(\alpha_E)_v).$$

Construction 3.5. Consider an ∞-preoperad $(M, \{\theta_{T,e}\})$. For any tree T, we define a map $\theta_T : M(T) \to KM(T)$ by specifying its components

$$\theta_{T,\alpha_E} : M(T) \longrightarrow \bigotimes_{v \in \mathrm{dom}\,\alpha_E} M(T(\alpha_E)_v)$$

for any $E \subseteq \mathrm{E}^{\mathrm{int}}(T)$. We do this by double induction, on the cardinality of $\mathrm{E}^{\mathrm{int}}(T)$ and on that of E. Recall that E inherits a partial order from that on $\mathrm{E}^{\mathrm{int}}(T)$.

Note that

- $\#E = 0$, then $T_\emptyset = C_T \to T$ is the discrete decomposition morphism and we define
$$\theta_{T, C_r \to T} := \mathrm{id}_{M(T)}.$$
- $\#E = 1$, $E = \{e\}$, then
$$\theta_{T,\alpha_e} := \theta_{T,e}.$$
- $\#E \geq 2$. Choose $e \in E$ a maximal element in E; we call such an edge an *E-admissible edge*. In other words, an E-admissible edge is either not comparable with any other edge in E or is a maximal element in a connected component of E. For such an edge e, the subtree $T_{s(e)}$ induced by the binary decomposition $\alpha_{\{e\}}$ is isomorphic to the subtree $T(\alpha_E)_{s(e)}$ in the ~~multi-block~~ decomposition induced by α_E.

  We define
$$\theta_{T,\alpha_E} := \left(1_{M(T_{s(e)})} \otimes \theta_{T_{t(e)},(\alpha_E)_{t(e)}}\right) \circ \theta_{T,e}.$$

Observe that the induction step is legitimate, since $(\alpha_E)_{t(e)} = \alpha_{E\setminus\{e\}} \colon (T_{t(e)})_{E\setminus\{e\}} \to T_{t(e)}$ and $\#\mathrm{E}^{\mathrm{int}}(T_{t(e)}) = \#\mathrm{E}^{\mathrm{int}}(T) - 1$

Proposition 3.6. *The definition of θ_{T,α_E} does not depend on the choice of the admissible edge e.*

Proof. We prove this by induction on $\#E$. Consider another admissible edge $m \in E$, then necessarily m and e are not comparable.

If $\#E = 2$, then the thesis is equivalent to the coassociativity of the structure maps $\{\theta_{T,x}\}_{T,x}$ of the ∞-preoperad M.

Suppose now $\#E \geq 3$. Observe that $e \in T_{t(m)}$ and $m \in T_{t(e)}$, and that $T_{s(m)} = T(\alpha_E)_{s(m)}$, $T_{s(e)} = T(\alpha_E)_{s(e)}$. We want to prove that

$$\left(1_{M(T_{s(e)})} \otimes \theta_{T_{t(e)},(\alpha_E)_{t(e)}}\right) \circ \theta_{T,e} = \left(1_{M(T_{s(m)})} \otimes \theta_{T_{t(m)},(\alpha_E)_{t(m)}}\right) \circ \theta_{T,m} \quad (*).$$

Recall that the definition of θ is inductive, so for example we can write

$$\theta_{T_{t(e)},(\alpha_E)_{t(e)}} = \left(1_{M((T_{t(e)})_{s(m)})} \otimes \theta_{(T_{t(e)})_{t(m)},((\alpha_E)_{t(e)})_{t(m)}}\right) \circ \theta_{T_{t(e)},m}$$

and similarly for $\theta_{T_{t(m)},(\alpha_E)_{t(m)}}$.

Observe that

$$(T_{t(e)})_{s(m)} = T_{s(m)}, \qquad (T_{t(m)})_{s(e)} = T_{s(e)}, \qquad \tilde{T} \coloneqq (T_{t(e)})_{t(m)} = (T_{t(m)})_{t(e)}$$
$$((\alpha_E)_{t(e)})_{t(m)} = ((\alpha_E)_{t(m)})_{t(e)} = \alpha_{E\setminus\{e,m\}}$$

In light of this, we observe that in $(*)$ we can write:

$$\begin{aligned}
\text{LHS} &= \left\{1_{M(T_{s(e)})} \otimes \left[\left(1_{M(T_{s(m)})} \otimes \theta_{\tilde{T},\alpha_{E\setminus\{m,e\}}}\right) \circ \theta_{T_{t(e)},m}\right]\right\} \circ \theta_{T,e} = \\
&= \left(1_{M(T_{s(e)})} \otimes 1_{M(T_{s(m)})} \otimes \theta_{\tilde{T},\alpha_{E\setminus\{m,e\}}}\right) \circ \left(1_{M(T_{s(e)})} \otimes \theta_{T_{t(e)},m}\right) \circ \theta_{T,e} \\
\text{RHS} &= \left\{1_{M(T_{s(m)})} \otimes \left[\left(1_{M(T_{s(e)})} \otimes \theta_{\tilde{T},\alpha_{E\setminus\{m,e\}}}\right) \circ \theta_{T_{t(m)},e}\right]\right\} \circ \theta_{T,m} = \\
&= \left(1_{M(T_{s(m)})} \otimes 1_{M(T_{s(e)})} \otimes \theta_{\tilde{T},\alpha_{E\setminus\{m,e\}}}\right) \circ \left(1_{M(T_{s(m)})} \otimes \theta_{T_{t(m)},e}\right) \circ \theta_{T,m}.
\end{aligned}$$

By inductive hypothesis, the thesis holds for $\theta_{\tilde{T},\alpha_{E\setminus\{m,e\}}}$, and the coassociativity property for the collection of the $\theta_{T,x}$ implies that the two blue blocks in the LHS and RHS are equal, and therefore the thesis. $\square$

This definition allows to prove that θ can be defined without having to worry about admissible edges.

Proposition 3.7. *For any tree T and any $E \subseteq E^{int}(T)$, for any edge $e \in E$, we can write*

$$\theta_{T,\alpha_E} = \left(\theta_{T_{s(e)},(\alpha_E)_{s(e)}} \otimes \theta_{T_{t(e)},(\alpha_E)_{t(e)}}\right) \circ \theta_{T,e}.$$

Proof. If $e \in E$ is admissible, the decomposition map $(\alpha_E)_{s(e)}$ is the trivial decomposition morphism of T, hence $\theta_{T_{s(e)},(\alpha_E)_{s(e)}} = \theta_{T_{s(e)},C_{T_{s(e)}} \to T_{s(e)}} = \epsilon_{M,T_{s(e)}} = 1_{M(T_{s(e)})}$, as in the original definition.

Consider a non admissible edge $e \in E$; then there exists $m \in E$ with $m > e$, and we can choose m to be maximal. In particular, m is admissible. Proving the thesis then boils down to proving that

$$\left(\theta_{T_{s(e)},(\alpha_E)_{s(e)}} \otimes \theta_{T_{t(e)},(\alpha_E)_{t(e)}}\right) \circ \theta_{T,e} = \left(\theta_{T_{s(m)},(\alpha_E)_{s(m)}} \otimes \theta_{T_{t(m)},(\alpha_E)_{t(e)}}\right) \circ \theta_{T,m} \quad (*).$$

If $\#E = 2$, equality in $(*)$ corresponds to coassociativity.

Spécifica è induction utilisée

Suppose that $\#E \geq 3$. Since $m > e$, we have $m \in T_{s(e)}$, and therefore by definition

$$\theta_{T_{s(e)},(\alpha_E)_{s(e)}} = \left(1_{M((T_{s(e)})_{s(m)})} \otimes \theta_{(T_{s(e)})_{s(m)},(\alpha_{E\,s(e)})_{t(m)}}\right) \circ \theta_{T_{s(e)},m},$$

and we can plug this inside the LHS of (*). On the other hand, we can apply the inductive hypothesis on $\theta_{T_{t(m)},(\alpha_E)_{t(m)}}$, and by observing that $(T_{t(m)})_{t(e)} = T_{t(e)}$ and $(\alpha_E)_{t(m)_{t(e)}} = \alpha_{t(e)}$ we have:

$$\mathrm{LHS} = \left[\left(1_{M(T_{s(m)})} \otimes \theta_{T_{t(e)},\alpha_{t(e)}}\right) \otimes \theta_{T_{t(e)},(\alpha_E)_{t(e)}}\right] \circ \left(\theta_{T_{s(e)},m} \otimes 1_{M(T_{s(e)})}\right) \circ \theta_{T,e}$$

$$\mathrm{RHS} = \left[1_{M(T_{s(m)})} \otimes \left(\theta_{T_{t(e)},\alpha_{t(e)}} \otimes \theta_{T_{t(e)},(\alpha_E)_{t(e)}}\right)\right] \circ \left(1_{M(T_{s(m)})} \otimes \theta_{T_{t(m)},e}\right) \circ \theta_{T,m}.$$

We conclude that the two expressions are equal thanks to coassociativity. □

Corollary 3.8. *More generally, for any two nested decompositions of T represented by the inert morphism $(\gamma, id)\colon (T,\beta) \to (T,\alpha)$, we have*

$$\theta_{T,\beta} = \left(\bigotimes_{v \in dom\,\alpha} \theta_{T(\gamma)_v,\beta_v}\right) \circ \theta_{T,\alpha}$$

Proposition 3.9. *Consider an ∞-preoperad $(M \in Ch(\mathbb{A}), \{\theta_{T,e}\}_{T,e\in E^{int}(T)})$. Then the assignment $\theta\colon M \to KM$ just constructed is a natural transformation endowing (M,θ) ~~of~~ the structure of a K-coalgebra.*
with

Proof. We check naturality. Given $\beta\colon S \to T$ in $\mathbb{A}$, we want to see that $KM(\beta) \circ \theta_T = \theta_S \circ M(\beta)$. Consider a morphism $\alpha\colon R \to S$, then

$$\begin{aligned}
&\mathrm{proj}_\alpha \circ KM(\beta) \circ \theta_T = && \\
&= \bigotimes_{v \in R} M(\beta_v) \circ \theta_{T,\beta\alpha} && \text{naturality of projection} \\
&= \theta_{S,\alpha} \circ M\Big(\bigcup_{v \in V(R)} \beta_v\Big) && \text{Corollary 3.8} \\
&= \theta_{S,\alpha} \circ M(\beta) && \beta = \bigcup_{v \in V(R)} \beta_v \\
&= \mathrm{proj}_\alpha \circ \theta_S \circ M(\beta). && \text{definition}
\end{aligned}$$

Since this is true for any $\alpha \in \mathbb{A}/S$, naturality of the θ_T's is proven.

Consider now the comonadic identities. By the very definition of θ, it is clear that $\theta \circ \epsilon_M = \mathrm{id}_M$.

We need to check compatibility of θ with the comultiplication of the comonad, namely that $\Delta_M \circ \theta = K\theta \circ \theta$, and we do this tree-wise and component-wise. Fix a tree $T \in \mathbb{A}$ and $\alpha\colon R \to T$ factoring as $\alpha = \beta \circ \gamma$, $\beta\colon S \to T$, so that (α, β) determines a component of $K^2M(T)$. By the definition of the comultiplication,

$$\mathrm{proj}_{\alpha,\beta} \circ \Delta_{M,T} \circ \theta_T = \mathrm{proj}_\beta \circ \theta_T.$$

with

On the other hand, commutativity of

$$\begin{tikzcd}
M(T) \arrow[r, "\theta_T"] & KM(T) \arrow[r, "(K\theta)_T"] \arrow[d, "\mathrm{proj}_\alpha"'] & K^2M(T) \arrow[d, "\mathrm{proj}_\alpha"] \arrow[dd, bend left, "\mathrm{proj}_{\alpha,\beta}"] \\
& \bigotimes_{u\in R} M(T(\alpha)_u) \arrow[r, "\bigotimes_{u\in R} \theta_{T(\alpha)_u}"] & \bigotimes_{u\in R} KM(T(\alpha)_u) \arrow[d, "\bigotimes_{u\in R} \mathrm{proj}_{\beta_u}"] \\
& & \bigotimes_{v\in S} M(T(\beta)_v)
\end{tikzcd}$$

Pour éviter le chevauchement

tells us that proving the thesis reduces to proving

$$\mathrm{proj}_\beta \circ \theta_T = \left[\bigotimes_{u\in R} (\mathrm{proj}_{\beta_u} \circ \theta_{T(\alpha)_u})\right] \circ \theta_{T,\alpha},$$

which is precisely the content of Corollary 3.8. ☐

Proposition 3.10. *The functors*

$$\infty preOperads \longrightarrow K\text{-}coalgebras \qquad\qquad K\text{-}coalgebras \longrightarrow \infty preOperads$$

are ~~one the~~ inverse of ~~the~~ other.
each

Proof. It is clear that the composition

$$\infty\text{preOperads} \longrightarrow K\text{-coalgebras} \longrightarrow \infty\text{preOperads}$$

is the identity. On the other hand, proving that the composition

$$K\text{-coalgebras} \longrightarrow \infty\text{preOperads} \longrightarrow K\text{-coalgebras}$$

equals the identity amounts to proving that, given (M,θ) a K-coalgebra, $\alpha\colon S \to T$ and an admissible edge $e \in \alpha(\mathrm{E}^{\mathrm{int}}(S)) \subseteq \mathrm{E}^{\mathrm{int}}(T)$, one has that $\theta_{T,\alpha} = (1_{M(T(\alpha)_{s(e)})} \otimes \theta_{T(\alpha)_{t(e)},\alpha_{t(e)}}) \circ \theta_{T,e}$. This is true thanks to the comonadic identity $K\theta \circ \theta = \Delta_M \circ \theta$.

☐

Definition 3.5 (linear ∞-operad)**.** A *linear ∞-operad* is a linear ∞-preoperad $(M, \theta\colon M \to KM)$ for which the structure maps for binary decompositions are quasi-isomorphisms. In other words, (M,θ) is a linear ∞-operad if, for any tree T and any inner edge $e \in \mathrm{E}^{\mathrm{int}}(T)$, the map $\theta_{T,e} = \mathrm{proj}_{\alpha_e} \circ \theta_T$ is a quasi-isomorphism.

Donner + haut la définition $\theta_{T,\alpha} \stackrel{=}{\Delta} \mathrm{proj}_\alpha \circ \theta_T$ (même avant la Prop. 3.3 !)

Corollary 3.11. *Suppose that the ground ring R is a field, and consider (M,θ) a K-coalgebra. Then M is an ∞-operad if and only if $\theta_{T,\alpha}$ is a quasi-isomorphism for any T and any $\alpha\colon S \to T$.*

Proof. Of course if every component of θ is a quasi-isomorphism, then in particular the structural maps $\theta_{T,e} = \theta_{T,\alpha_e}$ are quasi-isomorphisms.

For the reverse implication, recall that we can recursively write

$$\theta_{T,\alpha} = (1_{M(T(\alpha)_{s(e)})} \otimes \theta_{T(\alpha)_{t(e)},\alpha_{t(e)}}) \circ \theta_{T,e}.$$

We can then proceed by an inductive argument, observing that by hypothesis $\theta_{T,e}$ is a quasi-isomorphism and, thanks to the Künneth theorem, the functor $-\otimes A$ preserves quasi-isomorphisms for any chain complex A.

☐

Remark 3.12. If R is not a field, then Corollary 3.11 holds for any linear ∞-preoperad M having the property that $M(T)$ is flat for any tree T.

Remark 3.13. Being an ∞-operad does *not* imply that the coalgebra map $\theta\colon M \to KM$ is a tree-wise quasi-isomorphism. Indeed, by the 2-ouf-3 property this would be equivalent to stating that for any $\alpha\colon S \to T$, the projection $\mathrm{proj}_\alpha\colon KM(T) \to \bigotimes_{v\in S} M(T(\alpha)_v)$ is a quasi-isomorphism, which in turn implies, since homology commutes with direct sum, that all the homologies of the complexes $M(R)$ are trivial for any $R \in \mathbb{A}$.
out of

Proposition 3.14. *The category* ∞-PreOps *is symmetric monoidal closed. In other words, the tensor product*

$$-\otimes-\colon \infty\text{-}\mathsf{PreOps} \times \infty\text{-}\mathsf{PreOps} \longrightarrow \infty\text{-}\mathsf{PreOps}$$

is cocontinuous in each variable.

Il faut déjà définir le produit tensoriel de 2 coalgèbres.

Proof. ☐

$$KM(T) \coloneqq \left(\prod_{\alpha \in \mathbb{A}/T} \bigotimes_{v \in \operatorname{dom} \alpha} M(T(\alpha)_v) \right)^{\mathrm{inv}},$$

4. Properties of ∞-PreOps

Since $(\mathsf{Ch}(R), \otimes)$ is a monoidal category, the functor category $\mathrm{Ch}(\mathbb{A})$ becomes a symmetric monoidal category with the tree-wise tensor product of chain complexes, namely

$$\text{for every } M, N \in \mathrm{Ch}(\mathbb{A}) \text{ and every } T \in \mathbb{A} \text{ we define } (M \otimes N)(T) \coloneqq M(T) \otimes N(T).$$

We denote by τ the swap map which establishes the isomorphism $A \otimes B \xrightarrow{\simeq} B \otimes A$ for any two chain complexes A, B. Explicitly, $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$, where $|a| = k$ if and only if $a \in A_k$.

We can extend τ to a natural isomorphism $M \otimes N \xrightarrow{\simeq} N \otimes M$ for all functors $M, N \in \mathrm{Ch}(\mathbb{A})$ (and similarly when we consider the functor category $\mathsf{Fun}(\mathbb{A}, \mathsf{Ch}(R))$).

Remark 4.1. We point out that there is generally *not* an isomorphism

$$K(M \otimes N) \not\simeq KM \otimes KN.$$

Notation non introduite

Indeed, we see that

$$K(M \otimes N)(T) = \bigoplus_{(T,\alpha)} \overline{M}(T, \alpha) \otimes \overline{N}(T, \alpha),$$

while the tensor product of KM with KN gives

$$KM(T) \otimes KN(T) = \left(\bigoplus_{(T,\alpha)} \overline{M}(T, \alpha)\right) \otimes \left(\bigoplus_{(T,\beta)} \overline{N}(T, \beta)\right) \simeq \bigotimes_{(T,\alpha),(T,\beta)} \overline{M}(T, \alpha) \otimes \overline{N}(T, \beta).$$

In particular, we only have a diagonal inclusion

$$K(M \otimes N) \hookrightarrow KM \otimes KN,$$

which means that K is a colax monoidal functor. Il faudrait vérifier les conditions de "colaxité" pour l'affirmer --

Proposition 4.2. *There exists a symmetric monoidal structure on the categories of linear ∞-pre(co)operads such that the forgetful functors*

$$V \colon \infty\text{-}\mathsf{PreCoops} \longrightarrow Ch(\mathbb{A}) \quad \text{and} \quad U \colon \infty\text{-}\mathsf{PreOps} \longrightarrow Ch(\mathbb{A})$$

are monoidal.

If the ground ring is a field, then the subcategories of linear ∞-operads and linear ∞-cooperads are symmetric monoidal as well.

Proof. Consider $(M, \theta), (N, \gamma)$ linear ∞-preoperads. The functor $M \otimes N \in \mathrm{Ch}(\mathbb{A})$ has a natural structure of linear ∞-preoperad: for any tree $T = R \cup_a S$, the structure map

$$\delta_{T,a} \colon M(T) \otimes N(T) \longrightarrow M(R) \otimes N(R) \otimes M(S) \otimes N(S)$$

is given by the tensor product of the structure maps of M and N followed by a twist that puts the factors in the correct order:

$$\delta_{T,a} \coloneqq (1 \otimes \tau \otimes 1) \circ (\theta_{T,a} \otimes \gamma_{T,a}).$$

Naturality and coassociativity of the structure map are easy to check, hence we omit it. With this tensor product, the forgetful functor $U \colon \infty\text{-}\mathsf{PreOps} \to \mathrm{Ch}(\mathbb{A})$ is a monoidal functor.

If the ground ring is a field, the tensor product of quasi-isomorphisms is again a quasi-isomorphism; since moreover τ is an isomorphism, it follows that if (M, θ) and (N, γ) are ∞-operads, $(M \otimes N, \delta)$ is an ∞-operad as well.

The same definitions yield the symmetric monoidal structure on the category of linear ∞-precooperads, which restricts to a symmetric monoidal structure on ∞-cooperads when the ground ring is a field.

□

5. The model structure

In this section we endow the category of linear ∞-preoperads of a Quillen model category structure. ~~This section is devoted to the proof of the following Theorem~~.

We start by collecting some observations. We refer to Appendix A for the model categorical background (definitions, references and proofs) needed for this section.

Remark 5.1.

- There is a model category structure on $\mathsf{Ch}(R)$ where cofibrations are monomorphisms and weak equivalences are quasi-isomorphisms. If the ground ring R is a field, then fibrations are precisely the surjections. We call this model structure the *standard model structure* on chain complexes.
- The standard model structure on $\mathsf{Ch}(R)$ is combinatorial and the category $\mathbb{A}$ is small: as a consequence the functor category $\mathrm{Ch}(\mathbb{A})$ admits the injective model structure, where weak equivalences and cofibrations are defined objectwise. This means that weak equivalences (resp. cofibrations) are those maps $f\colon X \to Y$ such that for any tree T the map of chain $f_T\colon X(T) \to Y(T)$ is a quasi-isomorphism (resp. monomorphism). This model structure is again combinatorial.
- The tree category $\mathbb{A}$ is a dualizable Reedy category with trivial left class of morphisms (see Example A.5). By Proposition A.8, the injective model structure on $\mathrm{Ch}(\mathbb{A})$ coincides with the injective Reedy model structure.

  In particular, there is an explicit characterization of (trivial) fibrations as those morphisms $f\colon Y \to X$ in $\mathrm{Ch}(\mathbb{A})$ such that for any tree $T \in \mathbb{A}$, the $\mathrm{Aut}(T)$-equivariant map

  $$Y(T) \longrightarrow M_T Y \times_{M_T X} X(T)$$

  is a (trivial) fibration in $\mathsf{Ch}(R)^{\mathrm{Aut}(T)}$ with the injective model structure. We recall that the object $M_T Y$ is defined as the limit of $Y(S)$ over all face maps $S \to T$ which are not isomorphisms, i.e.

  matching

  $$M_T Y = \lim_{S \xrightarrow{+} T} Y(S)$$

The main theorem of this section is the following.

Theorem 5.2. *Suppose that the ground ring is a field. There exists a left proper, accessible monoidal model structure on the category of linear ∞-preoperads left transferred along the forgetful-cofree adjunction*

$$U\colon \infty\text{-}\mathit{PreOpds} \rightleftarrows \mathit{Ch}(\mathbb{A}) : \mathscr{F}^c,$$

where on $\mathit{Ch}(\mathbb{A})$ we consider the injective model structure with respect to the standard model structure on $\mathsf{Ch}(R)$.

To prove Theorem 5.2, we will make use of the following results. To do: merge the following two theorems into one.

Theorem 5.3 ([Hes+17, Proposition 2.1.4, Corollary 3.3.4])**.**

Suppose that $(\mathscr{M}, \mathscr{C}, \mathscr{W}, \mathscr{F})$ is an accessible model category, $\mathscr{K}$ is a bicomplete, locally presentable category and there exists an adjunction

$$U\colon \mathscr{K} \rightleftarrows \mathscr{M} : \mathscr{F}^c.$$

Définir

If the left induced factorization system exists on $\mathscr{K}$, then the left-induced model structure on $\mathscr{K}$ exists if and only if

$$(U^{-1}\mathscr{C})^{\boxtimes} \subseteq U^{-1}\mathscr{W} \quad (*).$$

introduire le symbole

If it is the case, the model structure on $\mathscr{K}$ is again accessible.

Condition $(*)$ is called the *acyclicity condition*.

Theorem 5.4 ([Hes+17, Theorem 2.2.1])**.**

Consider an adjunction between locally presentable categories

$$\mathscr{U}: \mathscr{K} \underset{\perp}{\rightleftarrows} \mathscr{M} : \mathscr{F}^c,$$

where $\mathscr{M}$ is an accessible model category. If

an object QX and

(1) for every object X in $\mathscr{K}$, there exists a morphism $\epsilon_X : QX \to X$ such that $U\epsilon_X$ is a weak equivalence and $U(QX)$ is cofibrant in $\mathscr{M}$,
(2) for each morphism $f : X \to Y$ in $\mathscr{K}$ there exists a morphism $Qf : QX \to QY$ satisfying $\epsilon_Y \circ Qf = f \circ \epsilon_X$, and
(3) for every object X in $\mathscr{K}$ there exists a factorization

$$QX \bigsqcup QX \xrightarrow{j} \mathsf{Cyl}(QX) \xrightarrow{p} QX$$

of the fold map such that Uj is a cofibration and Up is a weak equivalence,

If K is moreover bicomplete...

then the acyclicity condition holds for the *left-induced weak factorization systems on $\mathscr{K}$ and thus the left-induced model structure on $\mathscr{K}$ exists.*

We need to show that the hypotheses of both theorems are satisfied in the case of the adjunction

$$U : \infty\text{-}\mathsf{PreOps} \underset{\perp}{\rightleftarrows} \mathrm{Ch}(\mathbb{A}) : \mathscr{F}^c.$$

We start with checking the hypotheses of Theorem 5.3.

Lemma 5.5. *If the ground ring is a field, the category of ∞-PreOps is bicomplete.*

Proof. Cocompleteness holds because the forgetful functor creates colimits and $\mathrm{Ch}(\mathbb{A})$ is cocomplete. By [Adá77], if $\mathscr{C}$ is a well-powered category (namely, the subobjects of any object form a set) and H is a comonad on $\mathscr{C}$, then the category of H-coalgebras is complete if the comonad H on $\mathscr{C}$ preserves monomorphisms. Since any locally presentable category is in particular well-powered (see [AR94]), ~~we can apply the result to~~ $\mathscr{C} = \mathrm{Ch}(\mathbb{A})$ and $H = K$. If the ground ring is a field, the comonad K preserves monomorphisms, hence the thesis. ∎

Pourquoi?

Lemma 5.6. *The category ∞-PreOps is locally presentable.*

Proof. By precise ref of the theorem in [Bir84], the category of coalgebras over an accessible comonad on a locally presentable category is again locally presentable.

Since $\mathrm{Ch}(\mathbb{A})$ is locally presentable, it remains to observe that K is accessible, namely that it preserves filtered colimits. This is true, since finite products and finite tensor products of chain complexes over a field do. ∎

As a consequence, we obtain the following.

left induced
$(U^{-1}(\mathrm{Cof}), U^{-1}(\mathrm{Fib} \cap W))$

!

Corollary 5.7. *There exists the left induced weak factorization system* (S) *on the category of linear ∞-preoperads.*

Qu'est-ce que ça veut dire exactement : est-ce que les sont effectivement des WFS ?

Proof. Since both $\mathrm{Ch}(\mathbb{A})$ and ∞-PreOps are locally presentable categories, we can apply [GKR19, Therem 2.6]. ∎

o

At this point, the existence of the left induced model structure on ∞-PreOps is equivalent to the acyclicity condition, and Theorem 5.4 states that the existence of *U-cofibrant replacement objects* (points (1) and (2)) and of *U-cylinder objects* (point (3)) are sufficient for the acyclicity condition to hold. We devote the next subsection to the construction of such objects.

5.1. **U-cofibrant replacements and U-cylinder objects.** The construction of the required objects of Theorem 5.4 relies on the cobar-bar adjunction between linear ∞-(pre)cooperads and linear ∞-(pre)operads,

$$\mathbb{B}^\vee : \infty\text{-}\mathsf{PreCoops} \rightleftarrows \infty\text{-}\mathsf{PreOps} : \mathbb{B},$$

for which we refer to [HM21]. More precisely:

(1) For any linear ∞-preoperad X, we set

$$QX \coloneqq \mathbb{B}^\vee \mathbb{B} X,$$

and we define $\epsilon_X : QX \to X$ as the counit of the cobar-bar adjunction.

(2) Given a map of linear ∞-operads $f : X \to Y$, we set $Qf \coloneqq \mathbb{B}^\vee \mathbb{B} f$.

Since all objects in $\mathrm{Ch}(\mathbb{A})$ are cofibrant, in particular $U(QX)$ is cofibrant ~~as well~~, and ϵ_X is a weak equivalence (i.e. tree-wise quasi-isomorphisms) by [HM21, Theorem 8.1]. Moreover, by naturality of the counit, it holds that $\epsilon_Y \circ \mathbb{B}^\vee \mathbb{B} f = f \circ \epsilon_X$, so this shows points (1) and (2) of Theorem 5.4.

Observe that Theorem 5.4 does not require the U-cofibrant replacement to be functorial, but it is the case here, since Q is given by a functor $Q = \mathbb{B}^\vee \mathbb{B} : \infty\text{-}\mathsf{PreOps} \to \infty\text{-}\mathsf{PreOps}$.

We now proceed to prove point (3) of Theorem 5.4. What we want is:

- for any linear ∞-preoperad X, a linear ∞-preoperad $\mathsf{Cyl}(\mathbb{B}^\vee \mathbb{B} X)$, and
- morphisms of linear ∞-preoperads $i_0, i_1 : \mathbb{B}^\vee \mathbb{B} X \to \mathsf{Cyl}(\mathbb{B}^\vee \mathbb{B} X)$, $h : \mathsf{Cyl}(\mathbb{B}^\vee \mathbb{B} X) \to \mathbb{B}^\vee \mathbb{B} X$ such that:

$$\begin{aligned} &hi_0 = hi_1 = \mathrm{id}, \\ &U(i_0), U(i_1) \text{ are cofibrations, and} \\ &Uh \text{ is a weak equivalence.} \end{aligned}$$

We now proceed to the construction ~~of the construction~~ of the cylinder object ~~of point (3) in Theorem 5.~~4. What we actually construct is a *cylinder functor* (see Definition A.4)

$$I \otimes - : \infty\text{-}\mathsf{PreCoops} \to \infty\text{-}\mathsf{PreCoops}.$$

We then restrict it to the ∞-precooperads in the image of the bar construction, and by applying the cobar functor we get a cylinder object for ∞-preoperads of the form QX.

Let I be the model for the interval on chain complexes, namely it is the graded module

$$\begin{aligned} I_0 &= R\sigma^0 \oplus R\sigma^1 \\ I_1 &= R\sigma^{01} \\ I_k &= 0 \quad \text{for } k \neq 0, 1 \end{aligned}$$

with differential $d(\sigma^{01}) = \sigma^1 - \sigma^0$. I may be thought of as the oriented interval.

The chain complex I comes equipped with a diagonal map

$$\mathsf{diag} : I \to I \otimes I$$

mais d'abord, expliquer en quoi diag munit I (en tant que foncteur constant) d'une structure de pré-coopérade. (et faire voir que c'est une fonctorialité)

defined as

$$\begin{aligned} \mathsf{diag}_1(\sigma^{01}) &= (\sigma^0 \otimes \sigma^{01}) + (\sigma^{01} \otimes \sigma^1) \\ \mathsf{diag}_0(\sigma^0) &= \sigma^0 \otimes \sigma^0 \\ \mathsf{diag}_0(\sigma^1) &= \sigma^1 \otimes \sigma^1. \end{aligned}$$

The chain complex I may be considered as a constant functor $I : \mathbb{A} \to \mathsf{Ch}(R)$, and $\mathsf{diag} : I \to I \otimes I$ as a natural transformation of functors.

Proposition 5.8. (I, diag) *is a linear ∞-cooperad.*

RAPPELER la def de ∞-précoopérade

Proof. Consider the map $j\colon I \to I \otimes I$ defined by $j(\sigma^0) = \sigma^0 \otimes \sigma^0$, $j(\sigma^1) = \sigma^1 \otimes \sigma^0$. It is known reference that j is a quasi-isomorphism, and we show that diag is homotopic to j. The homotopy is given by the degree -1 morphism $q\colon I \to (I \otimes I)$ defined as:

$$\begin{aligned} q(\sigma^0) &= 0 \\ q(\sigma^1) &= \sigma^1 \otimes \sigma^{01} \\ q(\sigma^{01}) &= -\sigma^{01} \otimes \sigma^{01}. \end{aligned}$$

It is easy to check that $\mathsf{diag} - j = d_{I\otimes I} \circ q + q \circ d_I$. Hence diag is a quasi-iso ... $\square$

Consider now a linear ∞-precooperad $(M\colon \mathbb{A} \to \mathsf{Ch}(R), \theta)$. We define two morphisms of linear ∞-precooperads

$$i_0, i_1\colon (M, \theta) \longrightarrow (I, \mathsf{diag}) \otimes (M, \theta)$$

as follows: for any tree $T \in \mathbb{A}$, any integer number n, any $x \in M(T)_n$, we define

$$(i_0)_{T,n}(x) = \sigma^0 \otimes x \quad \text{and} \quad (i_1)_{T,n}(x) = \sigma^1 \otimes x.$$

If we fix $T \in \mathbb{A}$, the maps $(i_0)_T, (i_1)_T\colon M(T) \to M(T) \otimes I$ coincide with the usual maps of chains in the cylinder object for the projective? model structure on chain complexes, so in particular they are cofibrations (see precise reference in Weibel for this). It is immediate to check that i_0, i_1 are natural in tree morphisms, therefore we are left to proving that they are morphisms of ∞-precooperads.

Lemma 5.9. *Let (M, θ) be a linear ∞-precooperad and $T = S \cup_a R$, then the following diagram commutes:*

$$\begin{tikzcd} M(T) \arrow[r, "(i_0)_T"] \arrow[d, "\theta_{T,a}"'] & I \otimes M(T) \arrow[d, "(1\otimes\tau\otimes1)\circ(\mathsf{diag}\otimes\theta_{T,a})"] \\ M(S) \otimes M(R) \arrow[r, "(i_0)_S\otimes(i_0)_R"'] & I \otimes M(S) \otimes I \otimes M(R) \end{tikzcd}$$

The same holds for i_1.

Proof. We check the condition for i_0, and we omit the subscripts to avoid cluttering. We use the Sweedler notation for the decomposition map: for $x \in M(T)_n$, we write $\theta(x) = x^{(1)} \otimes x^{(2)}$. Consider hence $x \in M(T)_n$, then we have

$$(i_0 \otimes i_0)(\theta(x)) = i_0 \otimes i_0(x^{(1)} \otimes x^{(2)}) = (-1)^{|i_0||x^{(1)}|}\sigma^0 \otimes x^{(1)} \otimes \sigma^0 \otimes x^{(2)} = \sigma^0 \otimes x^{(1)} \otimes \sigma^0 \otimes x^{(2)},$$

where the last equality holds because i_0 is a morphism of chain complexes~~,~~ and hence ~~of~~ has degree 0. On the other hand, we have:

$$\begin{aligned} (1 \otimes \tau \otimes 1) \circ (\mathsf{diag} \otimes \theta)(\sigma^0 \otimes x) &= (1 \otimes \tau \otimes 1)((-1)^{|\theta||\sigma^0|}\sigma^0 \otimes \sigma^0 \otimes x^{(1)} \otimes x^{(2)}) = \\ &= (-1)^{|\sigma^0||x^{(1)}|}\sigma^0 \otimes x^{(1)} \otimes \sigma^0 \otimes x^{(2)} = \sigma^0 \otimes x^{(1)} \otimes \sigma^0 \otimes x^{(2)}, \end{aligned}$$

where signs disappear because the degree of θ is 0, and that of σ^0 as well. $\square$

We now define the map $h\colon I \otimes M \to M$. Observe that, for any integer n, we have

$$(I \otimes M)(T)_n = I_0 \otimes M(T)_n \oplus I_1 \otimes M(T)_{n-1},$$

so a general element in $(I \otimes M)(T)_n$ is given by a finite sum of elements of the form $\sigma^0 \otimes x + \sigma^1 \otimes y + \sigma^{01} \otimes z$. For this element, we define

$$h_{T,n}(\sigma^0 \otimes x + \sigma^1 \otimes y + \sigma^{01} \otimes z) = x + y.$$

Again, we observe that, if we fix a tree $T \in \mathbb{A}$, the map h_T is the one for cylinder objects of chain complexes, and in particular, it is a quasi-isomorphism (see reference??). It is also evident that the h_T's are natural in tree-morphisms, ~~so we need to~~ . We check that h is a map of linear ∞-precooperads.

Lemma 5.10. *Let (M, θ) be a linear ∞-precooperad and $T = S \cup_a R$, then the following diagram commutes:*

$$\begin{array}{ccc} I \otimes M(T) & \xrightarrow{h_T} & M(T) \\ {\scriptstyle (1\otimes\tau\otimes 1)\circ(\mathsf{diag}\otimes\theta_{T,a})}\downarrow & & \downarrow{\scriptstyle \theta_{T,a}} \\ I \otimes M(S) \otimes I \otimes M(R) & \xrightarrow[h_S\otimes h_R]{} & M(S)\otimes M(R) \end{array}$$

Proof. Again, we omit the subscripts to avoid cluttering. Moreover, call δ the structure map $(1\otimes\tau\otimes 1)\circ(\mathsf{diag}\otimes\theta)$. So on the one hand we have

$$\theta(h(\sigma^0\otimes x+\sigma^1\otimes y+\sigma^{01}\otimes z)=\theta(x+y)=\theta(x)+\theta(y)=x^{(1)}\otimes x^{(2)}+y^{(1)}\otimes y^{(2)}.$$

On the other hand, we compute:

$$h\otimes h(\delta((\sigma^0\otimes x+\sigma^1\otimes y+\sigma^{01}\otimes z))=x^{(1)}\otimes x^{(2)}+y^{(1)}\otimes y^{(2)}+z^{(1)}\otimes 0+0\otimes z^{(2)}=x^{(1)}\otimes x^{(2)}+y^{(1)}\otimes y^{(2)}.$$

□

Summing up, what we have proven so far is the following proposition.

Proposition 5.11. *For any linear ∞-precooperad (M,θ), there exist maps of linear ∞-precooperads*

$$(M,\theta)\bigsqcup(M,\theta)\xrightarrow{i_0\sqcup i_1}(I,\mathsf{diag})\otimes(M,\theta)\xrightarrow{h}(M,\theta)$$

which factor the fold map and such that i_0, i_1 are tree-wise monomorphisms and h is a tree-wise quasi-isomorphism.

tu n'as pas dit pourquoi ce sont des monos

If we apply the cobar functor $\mathbb{B}^\vee$ to the diagram in Proposition 5.11, we obtain a diagram of linear ∞-preoperads of the form

$$\mathbb{B}^\vee(M,\theta)\bigsqcup\mathbb{B}^\vee(M,\theta)\xrightarrow{\mathbb{B}^\vee i_0\sqcup\mathbb{B}^\vee i_1}\mathbb{B}^\vee((I,\mathsf{diag})\otimes(M,\theta))\xrightarrow{\mathbb{B}^\vee h}\mathbb{B}^\vee(M,\theta),$$

where the first factor is still a coproduct because the cobar functor is a left adjoint.

When (M,θ) is of the form $(M,\theta)=\mathbb{B}(X,\delta)$, for (X,δ) a linear ∞-preoperad,the above diagram can be written as

$$QX\sqcup QX\xrightarrow{j_0\sqcup j_1}\mathsf{Cyl}(X)\xrightarrow{p}QX,$$

where

$$\mathsf{Cyl}(QX):=\mathbb{B}^\vee((I,\mathsf{diag})\otimes\mathbb{B}(X,\delta)).$$

If the cobar construction preserves tree-wise monomorphisms and quasi-isomorphisms, we have constructed a cylinder object with properties as in (3) of Theorem 5.4, as wanted. This is actually the case, as we show in the next propositions.

ici

Proposition 5.12. *The cobar construction preserves quasisonorphisms.*

Proof. We make use of the following.

Proposition 5.13 ([Bro12, p. 2.6])**.** *Let C, C' be two chain complexes and let C, C' have bounded finite increasing filtrations, where a filtration $\{F^kC_m\}_{k,m}$ of C is bounded if, for any m, there exists $s<t$ such that $0=F^sC_m\subseteq\cdots\subseteq F^tC_m=C_m$. Let $f\colon C\to C'$ be a filtration-preserving chain map. If the induced map of spectral sequences $E^r(f)\colon E^r_{\bullet,\bullet}(C)\to E^r(C')_{\bullet,\bullet}$ is an isomorphism for some r, then f is a quasisonorphism.*

etc

So consider $f\colon M\to N$ a quasisonorphism between covariant functor $M, N\in\mathsf{Fun}(\mathbb{A},Ch)$, we want to prove that $\mathbb{B}^\vee(f)$ is a quasisomorphism. ça semble être 2.2.5 de la version publiée

Fix a tree $S\in\mathbb{A}$. In analogy with [LV12, Proposition 2.2.7], we define the filtration of $\mathbb{B}^\vee M(S)$ by setting

$$F^p(\mathbb{B}^\vee M(S))=\{(\alpha\colon S\to T, d|e, x\in s^{-1}M_n(T))\mid\ \dim T\le p\},$$

where recall that $\dim T=|\mathrm{E}^{\mathrm{int}}(T)|$.

? expliquer

We observe (write explicit verification?) that ~~any~~ all element in an equivalence class of $\mathbb{B}^\vee M(S)$ are in the same stage of the filtration, so it is legit to work with representatives.

This filtration satisfies the hypothesis of Proposition 5.13:

- $F^p(\mathbb{B}^\vee M(S)) \subseteq F^{p+1}(\mathbb{B}^\vee M(S))$;
- the filtration is *finite*. Indeed, for any $\alpha\colon S \to T$ morphism in $\mathbb{A}$, since α preserves the set of leaves and T has no nullary nor unary vertices, we get $\dim T \leq l(T) - 2 = l(S) - 2$. In particular, the filtration is *bounded*.
- The filtration is exhaustive, since $\mathbb{B}^\vee M(S) = F^{q_S}\mathbb{B}^\vee M(S)$, where $q_S = l(S) - 2$.
- For any p, the module $F^p\mathbb{B}^\vee M(S)$ is a subchain complex of $\mathbb{B}^\vee M(S)$, since

$$\partial_{ext}\colon F^p\mathbb{B}^\vee_m M(S) \longrightarrow F^p\mathbb{B}^\vee_{m-1}M(S)$$

$$\partial_{\text{int}}\colon F^p\mathbb{B}^\vee_m M(S) \longrightarrow F^{p-1}\mathbb{B}^\vee_{m-1}M(S) \subseteq F^p\mathbb{B}^\vee_{m-1}M(S).$$

By the convergence theorem for spectral sequences, there exists a converging spectral sequence such that

$$E^0_{p,q} = F^p\mathbb{B}^\vee_{p+q}M(S)/F^{p-1}\mathbb{B}^\vee_{p+q}M(S)$$

and

$$E^1_{p,q} = H_{p+q}(F^p\mathbb{B}^\vee_{p+q}M(S)/F^{p-1}\mathbb{B}^\vee_{p+q}M(S)) \implies H_{p+q}(\mathbb{B}^\vee M(S)).$$

Observe that

$$E^0_{p,q} = \{(\alpha, d|e, x \in s^{-1}M_{2p+q+1}(T)) \mid \dim T = p\}$$

and when we consider the differential $d_0\colon E^0_{p,q} \to E^0_{p,q-1}$ we have that

$$\operatorname{Ker}(d_0\colon E^0_{p,q} \to E^0_{p,q-1}) = \bigoplus_{(\alpha\colon S\to T, d|e)} \operatorname{Ker}(s^{-1}\partial_{M(T)_{2p+q+1}})$$

and

$$\operatorname{Imm}(d_0\colon E^0_{p,q+1} \to E^0_{p,q}) = \bigoplus_{\alpha\colon S\to T, d|e} \operatorname{Imm}(s^{-1}\partial_{M(T)_{2p+q+2}}),$$

and therefore we can compute

$$E^1_{p,q} = \bigoplus_{(\alpha\colon S\to T, d|e)} H_{2p+q}(s^{-1}M(T)).$$

Now, for any morphism $f\colon M \to N$ inside $\mathsf{Fun}(\mathbb{A}, Ch)$ and any tree S, the map of chain complexes $\mathbb{B}^\vee(f)_S\colon \mathbb{B}^\vee M(S) \to \mathbb{B}^\vee N(S)$ respects the filtration, since

$$(\mathbb{B}^\vee f)_{S,m} = \bigoplus_{m=n-\dim S-q}\ \bigoplus_{\alpha\colon S\to T, d|e, \text{codim}\alpha=q} (f_T)_n$$

$$\mathbb{B}^\vee_m M(S) = \bigoplus_{m=n-\dim S-q}\ \bigoplus_{\alpha\colon S\to T, d|e, \text{codim}\alpha=q} s^{-1}M_n(T) \longrightarrow \bigoplus_{m=n-\dim S-q}\ \bigoplus_{\alpha\colon S\to T, d|e, \text{codim}\alpha=q} s^{-1}N_n(T) = \mathbb{B}^\vee_m N(S).$$

Denote by h the map $\mathbb{B}^\vee f_S$, and consider the map induced on the first page $E^1_{p,q}(h)$, we see that

$$E^1_{p,q}(h) = \bigoplus_{\alpha\colon S\to T, d|e} H_{2p+q}(s^{-1}f_T)$$

But since f_T is a quasisomorphism for every T, we get that $E^1_{p,q}$ is an isomorphism, as wanted. $\square$

Remark 5.14. By a similar filtration argument, one can prove that, if the ground ring is a field, the bar construction preservers tree-wise quasi-isomorphisms.

Proposition 5.15. *If the groundring is a field, the cobar functor preserves tree-wise monomorphisms.*

Proof. Being a monomorphism of chain complexes does not depend on differentials, so we can just observe that over a field direct sums and finite tensor products preserves monomorphisms. $\square$

5.2. **(Semi)simplicial model category structure ?** As pointed out in [HA, Warning 1.3.5.4] and explained in more details in Appendix A.5, the category of chain complexes is enriched in simplicial sets but does not have the structure of a simplicial model category. However, it is a *weak simplicial model category,* in the sense of [Hin15]. We recall the definition below, and then we ask ourselves whether the category of ∞-preoperads is a weak simplicial model category as well.

Definition 5.3. Consider a model category $\mathscr{C}$ enriched over simplicial sets, with mapping spaces denoted by $\mathsf{Map}_{\mathscr{C}}(-,-)$. Then it is a *weak simplicial model category* if the following two conditions are satisfied:

(1) *Existence of weak path functors:* for any $n \geq 0$ and any $X, Y \in \mathscr{C}$, the functor

$$Y \to \mathsf{Hom}_{\mathbf{sSets}}(\Delta^n, \mathsf{Map}_{\mathscr{C}}(Y, X))$$

is representable;

(2) *Dual of the pushout-product axiom:* for any cofibration $i\colon A \to B$ and any fibration $p\colon X \to Y$ in $\mathscr{C}$, the map of simplicial sets

$$\mathsf{Map}_{\mathscr{C}}(B, X) \longrightarrow \mathsf{Map}_{\mathscr{C}}(A, X) \times_{\mathsf{Map}_{\mathscr{C}}(A,Y)} \mathsf{Map}_{\mathscr{C}}(B, Y)$$

is a Kan fibration, trivial whenever i or p is.

cooperad?

Consider the functor of normalized chains on simplicial sets

$$\mathscr{N}_\bullet\colon \mathbf{sSets} \longrightarrow \mathsf{Ch}(R)_{\geq 0} \hookrightarrow \mathsf{Ch}(R).$$

For any simplicial set X, the complex $\mathscr{N}_\bullet(X)$ is a coalgebra on the surjection operad (ref?). In particular it is a dg coalgebra, ~~where the coalgebra map is given by the diagonal followed by the~~ Alexander-Whitney map:

$$\mathscr{N}_\bullet(X) \longrightarrow \mathscr{N}_\bullet(X \times X) \longrightarrow \mathscr{N}_\bullet(X) \otimes \mathscr{N}_\bullet(X).$$

As a consequence, if we consider $\mathscr{N}_\bullet(X)$ as a constant contravariant functor on trees, we have the structure of an ∞-preoperad on $\mathscr{N}_\bullet(X)\colon \mathbb{A}^{\mathrm{op}} \to \mathsf{Ch}(R)$. We define the simplicial enrichment of ∞-PreOps as

$$\mathsf{Map}_{\infty\text{-}\mathsf{PreOps}}(X, Y)_n \coloneqq \mathsf{Hom}_{\infty\text{-}\mathsf{PreOps}}(X \otimes \mathscr{N}_\bullet(\Delta^n), Y).$$

Now, since the functor $- \otimes \mathscr{N}(\Delta^n)$ is cocontinuous it admits a right adjoint, which we denote by $(-)^{\mathscr{N}(\Delta^n)}$. In particular, we see that

$$\mathsf{Map}_{\infty\text{-}\mathsf{PreOps}}(X, Y)_n \simeq \mathsf{Hom}_{\infty}\text{-}\mathsf{PreOps}(X, Y^{\mathscr{N}()}),$$

so the functor $\mathsf{Map}_{\infty\text{-}\mathsf{PreOps}}(-, Y)_n\colon \infty\text{-}\mathsf{PreOps}^{\mathrm{op}} \to \mathbf{Sets}$ is represented by the ∞-preoperad $Y^{\mathscr{N}()}$. We are left with checking the dual of the pushout-product axiom.

Appendix A.

A.1. **Accessible model structures, left-induced factorization systems.** For a precise account on accessible model categories, we suggest [Ros15].

accessible model category: it is definition 3.1.6 in [Hes+17]

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Definition A.2. A model category $(\mathscr{M}, \mathscr{C}, \mathscr{W}, \mathscr{F})$ is *accessible* if it is locally presentable and its factorizations into the classes $(\mathscr{C}, \mathscr{F} \cap \mathscr{W})$ and $(\mathscr{C} \cap \mathscr{W}, \mathscr{F})$ can be realized by accessible functors, namely functors preserving λ-filtered colimits for some regular cardinal λ.

In particular, any combinatorial model category (locally presentable and cofibrantly generated) is accessible ([Hes+17, Corollary 3.1.7]).

Definition A.3 (Left induced factorization system.). remove "."

Definition A.4. A *cylinder functor* on a category $\mathscr{C}$ is a functor

$$c\colon \mathscr{C} \longrightarrow \mathscr{C}$$

Et la structure standard ?

equipped with natural transformations

$$i_0, i_1 \colon \mathrm{id}_{\mathscr{C}} \Rightarrow c \quad \text{and} \quad h\colon c \Rightarrow \mathrm{id}_{\mathscr{C}}$$

such that $hi_0 = hi_1 = \mathrm{id}_{\mathscr{C}}$.

A.5. **Model category structure(s) on chain complexes.** We recall some properties of different model category structures on $\mathsf{Ch}(R)$, where R is an associative unital ring. We rely on [Hov07].

où se servent ces ? structures

Theorem A.1 ([Hov07][2.3.11])**.** *There exists a finitely generated model category structure on* $\mathsf{Ch}(R)$*, called the* projective model structure*, where weak equivalences are quasi-isomorphisms, fibrations are surjections and trivial cofibrations and cofibrations are generated by* $I = \{0 \hookrightarrow D^n(R)\}_n$ *and* $J = \{S^n(R) \hookrightarrow D^n(R)\}_n$ *respectively. Here* $S^n(R)$ *has* R *in degree* n *and* 0 *elsewhere, while* $D^n(R)$ *has* R *in degrees* n *and* $n-1$*, with differential the identity, and zero elsewhere.*

fibrant w.r.t. which notion of fibration ?!

Theorem A.2 ([Hov07][2.3.13])**.** *There exists a cofibrantly generated model category structure on* $\mathsf{Ch}(R)$*, called the* injective model structure*, where weak equivalences are quasi-isomorphisms, cofibrations are injections and fibrations can be characterized as those surjections having fibrant kernel.*

Remark A.3. If R is a field, then the projective and injective model structure coincide. TO DO: recollect the informations needed to prove this.

The category of chain complex(es) is enriched over itself, hence via the Dold-Kan correspondence it is enriched in simplicial sets. Indeed, the enrichment is obtained by using the functor given by the composite

$$\mathsf{Ch}(R) \xrightarrow{U} \mathsf{Ch}(Ab) \xrightarrow{\tau_{\geq 0}} \mathsf{Ch}(Ab)_{\geq 0} \xrightarrow{DK} \mathsf{Fun}(\Delta^{\mathrm{op}}, Ab) \to \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set}) = \mathbf{sSets},$$

which is right-lax monoidal thanks to the Alexander-Whitney construction.

Following [HA, Warning 1.3.5.4], we observe that this simplicial enrichment does *not* make $\mathsf{Ch}(R)$ a simplicial model category, because it is not tensored over **sSets**. This is essentially due to the fact that the Alexander-Whitney map is not in general an isomorphism. Indeed, for every simplicial set K and any pair of complexes $M, M' \in \mathsf{Ch}(R)$, there is a canonical bijection

$$\mathsf{Hom}_{\mathsf{Ch}(R)}(\mathscr{N}_\bullet(K) \otimes M, M') \simeq \mathsf{Hom}_{\mathbf{sSets}}(K, \mathsf{Map}_{\mathsf{Ch}}(R)(M, M')),$$

((R) en indice !)

$\mathsf{Map}_{\mathsf{Ch}(R)}$

and this bijection extends to a map of simplicial sets

$$\mathsf{Map}_{\mathsf{Ch}}(R)(\mathscr{N}_\bullet(K) \otimes M, M') \longrightarrow \mathsf{Map}_{\mathbf{sSets}}(K, \mathsf{Map}_{\mathsf{Ch}}(R)(M, M')). \quad (*)$$

Since the AW map $\mathscr{N}_\bullet(K \times K') \to \mathscr{N}_\bullet(K) \otimes \mathscr{N}_\bullet(K')$ is not in general an isomorphism, the map $(*)$ is not in general an isomorphism.

trop elliptique ... K' n'apparaît pas dans $(*)$

A.6. **Reedy model category structure.**

Definition A.7 ((Dualizable) generalized Reedy category)**.**

Remark A.4. If R is a generalized Reedy category, then not necessarily R^{op} is as well. However, it is the case if R is dualizable.

Ce serait mieux d'avoir une fonte différente pour R = ring, R = (generalized) Reedy cat.

Example A.5.

- Δ is a strict Reedy category, where positive morphisms are face maps, negative morphisms are degeneracies and the degree function is given by $d([n]) = n$.
- By [HM22, Proposition 3.9], the dendroidal category $\mathbf{\Omega}$ is a dualizable generalized Reedy category. The degree function assigns to a tree T the number of its vertices, namely $d(T) = |V(T)|$. Positive morphisms are generated by face maps and isomorphisms, while negative morphisms are generated by degeneracies and isomorphisms. It generalizes the Reedy structure on Δ.

- The categories $\mathbf{\Omega}_r^o, \mathbb{B}, \mathbb{A}, \mathbb{C}$ inherit from $\mathbf{\Omega}$ the structure of dualizable generalized Reedy categories. In particular, the category $\mathbb{C}$ is a *strict* Reedy category, and all these four categories have a trivial left classe~~s~~ of morphisms.

Definition A.8 (Matching and latching objects)**.** Consider R a generalized Reedy category and $r \in R$. We form the categories $R(r)^+$ and $R(r)^-$ as follows:

- $R(r)^+$ is the full subcategory of the arrow category R/r spanned by those morphisms $y \to r$ which are positive and *not* isomorphisms.
- $R(r)^-$ is the full subcategory of the arrow category r/R spanned by those morphisms $r \to x$ which are negative and *not* isomorphisms.

For any model category $\mathscr{C}$ and any $X \in \mathsf{Fun}(R, \mathscr{C})$,

- the *latching object of X at r* is the colimit:
$$L_r X = \mathsf{colim}_{\phi \in R(r)^+} X(\mathsf{dom}(\phi)).$$
- The *matching object of X ar r* is the limit
$$M_r X = \mathsf{lim}_{\phi \in R(r)^-} X(\mathsf{cod}(\phi)).$$

Definition A.9. Let R be a generalized Reedy category. A model category $\mathscr{E}$ is called *R-projective* if, for any $r \in R$, the category $\mathscr{E}^{Aut(r)}$ admits the projective model structure. Dually, $\mathscr{E}$ is said to be *R-injective* if $\mathscr{E}^{Aut(r)}$ admits the injective model structure for any $r \in R$.

For instance, any cofibrantly generated model category is R-projective; if it is moreover combinatorial, then it is also R-injective.

Definition A.10. Let R be a Reedy category and $\mathscr{C}$ a R-projective (dually R-injective) model category. A map $f\colon X \to Y$ in $\mathscr{C}^R$ is called:

- *p-Reedy cofibration* if, for each r, the relative latching map
$$X_r \cup_{L_r X} L_r Y \longrightarrow Y_r$$
is a cofibration in $\mathscr{C}^{\mathrm{Aut}(r)}$ with the projective model structure.

  Dually, it is called *i-Reedy cofibration* if it is a cofibration in $\mathscr{C}^{\mathrm{Aut}(r)}$ with the injective model structure.
- *p-Reedy fibration* if, for each r, the relative matching map
$$X_r \longrightarrow M_r(X) \times_{M_r(Y)} Y_r$$
is a fibration in $\mathscr{C}^{\mathrm{Aut}(r)}$ with the projective model structure.

  Dually, we call it a *i-Reedy fibration* if it is a fibration in $\mathscr{C}^{\mathrm{Aut}(r)}$ with the injective model structure.

Theorem A.6 (Theorem 1.6, [BM10])**.** *Let R be a generalized Reedy category and let $\mathscr{E}$ be an R-projective Quillen model category. Then the classes of p-Reedy cofibrations, Reedy weak equivalences and p-Reedy fibrations endow the functor category $\mathsf{Fun}(R, \mathscr{E})$ ~~of~~ a Quillen model category structure. We call this model structure the* projective Reedy model structure.

with

Theorem A.7 (Corollary 8.6, [Rie17])**.** *Let R be a dualizable generalized Reedy category and $\mathscr{E}$ a R-injective Quillen model category. Then the functor category $\mathsf{Fun}(R, \mathscr{E})$ admits a Quillen model structure where cofibrations, weak equivalences and fibrations are, respectively, i-Reedy cofibrations, Reedy weak equivalences, i-Reedy fibrations. We call this above model structure the* injective Reedy model structure.

Proposition A.8.

(1) If the right (positive) class of maps in R is trivial, then the injective Reedy model structure on $\mathsf{Fun}(R, \mathscr{C})$ coincides with the injective model structure.

(2) Dually, if the left (negative) class of maps in R is trivial, then the projective Reedy model structure on $\mathsf{Fun}(R, \mathscr{C})$ coincides with the projective model structure.

initiales | full first name

Harmoniser

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