On positive opetopes, positive opetopic cardinals and positive opetopic sets

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Abstract

We introduce the notion of a positive opetope and positive opetopic cardinals as certain finite combinatorial structures. The positive opetopic cardinals to positive-to-one polygraphs are like simple ω-graphs to free ω-categories over ω-graphs, c.f. [MZ]. In particular, they allow us to give an explicit combinatorial description of positive-to-one polygraphs. Using this description we show, among other things, that positive-to-one polygraphs form a presheaf category with the exponent category being the category of positive opetopes. We also show that the category ωCat of ω-categories is monadic over the category **pPoly** of positive-to-one polygraphs with the 'free functor' being the inclusion **pPoly** $\to \omega Cat$.

Contents

Foreword by Pierre-Louis Curien

The present paper was submitted to the Cahiers by Marek Zawadowski in April 2023. Marek sadly passed away in early March 2024 without having the chance to revise the paper after he had received my anonymous report. After discussions between the editors-in-chief (Andrée C. Ehresmann, Marino Gran and René Guitart), the handling editor (Clemens Berger) and myself, it was decided to publish the paper in a form very close to the original manuscript submitted (available as `https://arxiv.org/abs/0708.2658`). I chose to implement in the text only the most obvious corrections and harmonizations. Some unnumbered notes are also included in the hope of offering some additional guidance through the article. They are written by me, and I am the only one to be blamed if they contain mistakes!

With this publication, we wish to pay a tribute to Marek, our dear friend and outstanding colleague. I'll miss his warm personality, his deep insights, his humour, his kindness, his elegance, his culture, and so will all his numerous colleagues and friends, with a special mention of his last PhD student Wojciech Dulínski, who helped me a lot while preparing this publised version.

1 Introduction

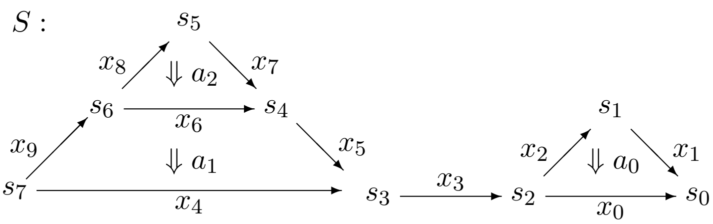
In this paper we present a combinatorial description of the category of the positive-to-one polygraphs **pPoly**. We show that this category is a presheaf category and we describe its exponent category in a combinatorial way as the category of positive opetopes **pOpe**, see Section 3. However the proof of that requires some extended studies of the larger category of all positive opetopic cardinals. Intuitively, the (isomorphism classes of) positive opetopic cardinals correspond to the shapes of arbitrary cells in positive-to-one polygraphs. The notion of a positive opetopic cardinal is the main notion introduced in this paper. We describe in a combinatorial way the embedding functor $\mathbf{e} : \mathbf{pPoly} \to \omega Cat$ of the category of positive-to-one polygraphs into the category of ω-categories as the left Kan extension along a suitable functor $\mathbf{j}$, and its right adjoint as the restriction along $\mathbf{j}$. We end by adapting an argument due to Victor Harnik [H] to show that the right adjoint to $\mathbf{e}$ is monadic. This approach does not cover the problem of the cells with empty domains which is important for both Makkai's multitopic categories and Baez-Dolan's opetopic categories.† However, it keeps something from the simplicity of Joyal's θ-categories, i.e., the category $\mathbf{pOpeCard}_\omega$ of positive opetopic cardinals with o-omega functors as morphisms is not much more complicated than the category of simple ω-categories, the dual of the category of disks, c.f. [J], [MZ], [Be]. In this sense this paper may be considered as a step towards a comparison of globular and opetopic approaches.

† The positive restriction is lifted in [Z2], where Marek proves that many-to-one polygraphs form a presheaf category with the exponent category being the category of all opetopes.

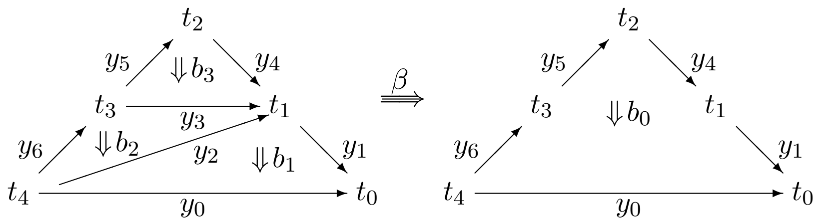
This paper is an extended and improved version of [Z1]. The terminology, notation, and proofs are changed and adjusted in many cases.

Positive opetopic cardinals

Positive opetopic cardinals represent all possible shapes of cells in positive-to-one polygraphs. A positive opetopic cardinal S of dimension 2 can be pictured as a figure



and a positive opetopic cardinal T of dimension 3 can be pictured as a figure



They have faces of various dimensions that fit together so that it makes sense to compose them in a unique way. By S_n we denote the set of faces of dimension n in S. Each face a has a face $\gamma(a)$ as its codomain and a *non-empty set* of faces $\delta(a)$ as its domain. In S above we have for a_1

$$\gamma(a_1) = x_4 \quad \text{and} \quad \delta(a_1) = \{x_5, x_6, x_9\}$$

and in T we have for β

$$\gamma(\beta) = b_0 \quad \text{and} \quad \delta(\beta) = \{b_1, b_2, b_3\}$$

These are all the data we need. Moreover, these (necessarily finite) data satisfy four conditions (see Section 3 for details). Below we explain them in an intuitive way.

Globularity. This is the main condition. It relates the sets that are obtained by double application of γ and δ. They are

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a) \qquad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a).$$

Let us look how it works for a_1 and β. In case of the face a_1 we have

$$\gamma\delta(a_1) = \{s_3, s_4, s_6\} \qquad \delta\delta(a_1) = \{s_4, s_6, s_7\}$$

$$\gamma\gamma(a_1) = s_3 \qquad \delta\gamma(a_1) = \{s_7\}.$$

So we have indeed

$$\delta\delta(a_1) - \gamma\delta(a_1) = \{s_4, s_6, s_7\} - \{s_3, s_4, s_6\} = \{s_7\} = \delta\gamma(a_1)$$

$$\gamma\delta(a_1) - \delta\delta(a_1) = \{s_3, s_4, s_6\} - \{s_4, s_6, s_7\} = \{s_3\} = \{\gamma\gamma(a_1)\}.$$

Similarly for the face β we have

$$\gamma\gamma(\beta) = y_0 \qquad \delta\gamma(\beta) = \{y_1, y_4, y_5, y_6\}$$

$$\gamma\delta(\beta) = \{y_0, y_2, y_3\} \qquad \delta\delta(\beta) = \{y_1, y_2, y_3, y_4, y_5, y_6\}.$$

and hence

$$\gamma\delta(\beta) - \delta\delta(\beta) = \{y_0, y_2, y_3\} - \{y_1, y_2, y_3, y_4, y_5, y_6\} = \{y_0\} = \{\gamma\gamma(\beta)\}$$

$$\delta\delta(\beta) - \gamma\delta(\beta) = \{y_1, y_2, y_3, y_4, y_5, y_6\} - \{y_0, y_2, y_3\} = \{y_1, y_4, y_5, y_6\} = \delta\gamma(\beta).$$

Using δ's and γ's we can define two binary relations $<^+$ and $<^-$ on faces of the same dimension which are the transitive closures of the relations $\lhd^+$ and $\lhd^-$, respectively, defined as follows: $a \lhd^+ b$ holds iff there is a face α such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$, and $a \lhd^- b$ holds iff $\gamma(a) \in \delta(b)$. We call $<^+$ the *upper order* and $<^-$ the *lower order*. For example, referring to the picture for T above, we have

$$b_3 \lhd^- b_2 \lhd^- b_1 \qquad y_5 \lhd^+ y_3 \lhd^+ y_2 \lhd^+ y_0.$$

The following three conditions refer to these relations.

Strictness. In each dimension, the relation $<^+$ is a strict order. The relation $<^+$ on 0-dimensional faces is required to be a linear order.

Disjointness. This condition says that no two faces can be comparable with respect to both orders $<^+$ and $<^-$.

Pencil linearity. This final condition says that the sets of cells with common codomain (γ-*pencil*) and the sets of cells that have the same distinguished cell in the domain (δ-*pencil*) are linearly ordered by $<^+$.

The morphisms of positive opetopic cardinals are functions that preserve dimensions and operations γ and δ. The size of a positive opetopic cardinal S is defined as an infinite sequence of natural numbers $size(S) = \{size(S)_k\}_{k\in\omega} = \{S_k - \delta(S_{k+1})\}_{k\in\omega}$ (almost all equal 0). We order the sequences lexicographically with higher dimensions being more important. The induction on the size of positive opetopic cardinals provides a convenient way of reasoning about them. The dimension of a positive opetopic cardinal S is the index of the largest non-zero number in the sequence $size(S)$. If for all $k \leq dim(S)$ (resp. for all $k < dim(S)$), $size(S)_k = 1$, then S is *principal* (resp. *normal*).[†] The normal positive opetopic cardinals play the role of the pasting diagrams of [HMP] and the principal positive opetopic cardinals play the role of the (positive) multitopes.[‡] On positive opetopic cardinals we define domain and codomain operations, as well as special pushouts which play the role of composition. With these operations (isomorphisms classes of) the positive opetopic cardinals form the terminal positive-to-one polygraph, and at the same time a monoidal globular category in the sense of Batanin.

Categories and functors

We shall work with the following categories and functors:

[†] As an illustration, in reference to the opetopic cardinals S, T drawn above, T is principal, the left 2-dimensional part of the picture of T (with b_1, b_2, b_3 as top dimensional faces) is normal and not principal, and the opetopic cardinal S is not normal.

[‡] Optimistically, Marek had added: "the precise connection between these approaches will be described elsewhere".

$$
\begin{array}{ccc}
\mathbf{pOpe} & & \mathcal{S} \\
\downarrow{\scriptstyle \mathbf{i}} & & \downarrow{\scriptstyle \mathbf{k}} \\
\mathbf{pOpeCard} & \xrightarrow{\mathbf{j}} & \mathbf{pOpeCard}_\omega \\
\downarrow{\scriptstyle (-)^*} & & \downarrow \\
\mathbf{pPoly} & \xrightarrow[\mathbf{e}]{} & \omega Cat
\end{array}
$$

where **pOpe** is the category of principal positive opetopic cardinals, **pOpeCard** is the category of positive opetopic cardinals, $\mathcal{S}$ is the category of simple categories c.f. [MZ], $(-)^*$ is the embedding functor of positive opetopic cardinals into positive-to-one polygraphs, $\mathbf{e}$ is the inclusion functor, $\mathbf{pOpeCard}_\omega$ is the full image of the composition functor $(-)^*; \mathbf{e}$, with the non-full embedding $\mathbf{j}$, and where also $\mathbf{i}$ and $\mathbf{k}$ are inclusions. Having these functors, we can form the following diagram

$$
\begin{array}{ccc}
\mathbf{pPoly} & \xrightarrow{\mathbf{e}} & \omega Cat \\
\widehat{(-)}\downarrow\ \uparrow\widetilde{(-)} & & \widehat{(-)}\downarrow\ \uparrow\widetilde{(-)} \\
sPb((\mathbf{pOpeCard})^{op}, Set) & \underset{\mathbf{j}^*}{\overset{Lan_{\mathbf{j}}}{\rightleftarrows}} & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \\
\mathbf{i}^*\downarrow\ \uparrow Ran_{\mathbf{i}} & & \mathbf{k}^*\downarrow\ \uparrow Ran_{\mathbf{k}} \\
\widehat{\mathbf{pOpe}} & & sPb(\mathcal{S}^{op}, Set)
\end{array}
$$

in which all the vertical arrows come in pairs and are adjoint equivalences of categories, and in which the categories $sPb((\mathbf{pOpeCard})^{op}, Set)$, $sPb(\mathcal{S}^{op}, Set)$ and $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ are the full categories of presheaves that preserve special pullbacks. The functors

$$\widehat{(-)} : \mathbf{pPoly} \to sPb((\mathbf{pOpeCard})^{op}, Set)$$

and

$$\widehat{(-)} : \omega Cat \to sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

are defined similarly, using the bottom vertical functors from the previous diagram: for a polygraph Q and an ω-category C, $\widehat{Q}$ and $\widehat{C}$ are presheaves so that for a positive opetopic cardinal S we have

$$\widehat{Q}(S) = \mathbf{pPoly}(S^*, Q) \qquad\qquad \widehat{C}(S) = \omega Cat(S^*, C)\ ^{\dagger}$$

The adjoint functors $\widetilde{(-)}$ that produce ω-categories are slightly more complicated. They are defined in Sections 13 and 15. The other functors are standard. The functors $\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$ are the functors $(\mathbf{i}; -)$, $(\mathbf{j}; -)$, $(\mathbf{k}; -)$, respectively. $Ran_{\mathbf{i}}$ and $Ran_{\mathbf{k}}$ are the right Kan extensions along $\mathbf{i}$, $\mathbf{k}$, respectively and $Lan_{\mathbf{j}}$ is the left Kan extension along $\mathbf{j}$.

Since we have $\mathbf{e}; \widehat{(-)} = \widehat{(-)}; Lan_{\mathbf{j}}$, and since the functors $\widehat{(-)}$ are equivalences of categories, the functor $Lan_{\mathbf{j}}$ is like $\mathbf{e}$ but moved into a more manageable context. In fact, we have a very neat description of this functor.‡

† Slowly, $\widehat{C}(\mathbf{j}(S)) = \widehat{C}(\mathbf{e}(S^*)) = \omega Cat(\mathbf{e}(S^*), C) = \omega Cat(S^*, C)$.

‡ The left vertical column in the last diagram exhibits **pPoly** as a presheaf category with **pOpe**

The content

Since the paper is quite long, I describe below the content of each section to help the reading. Sections 2 and 3 introduce the notion of a positive hypergraph and positive opetopic cardinal. Section 4 is concerned with establishing what kind of inclusions hold between iterated applications of γ's and δ's. Section 5 contains many statements concerning positive opetopic cardinals. All of them are there because they are needed afterwards. But it is not recommended to read the whole section at once. One of the main tools is the so called Path Lemma 5.7. Section 6 describes the embedding $(-)^* : \mathbf{pOpeCard} \to \omega Cat$, i.e., its main goal is to define an ω-category S^* for any positive opetopic cardinal S. Section 7 describes useful properties of normal positive opetopic cardinals.[†] In Section 8 we study a way in which we can decompose positive opetopic cardinals if they are at all decomposable. Any positive opetopic cardinal is either principal or decomposable. This provides a way of proving the properties of positive opetopic cardinals by induction on the size. Using this in Section 9 we show that the ω-category S^* and in fact the whole functor $(-)^*$ end up in **pPoly**. Section 10 describes the inner-outer factorization and its refinements, i.e., a further factorization of inner maps into inner epi and inner monos. These factorizations will play an important role in describing the strongly cartesian monad (c.f. [BMW]) T_ω on opetopic sets for ω-categories and its decomposition into two simpler monads ($T_\omega = T_\iota \circ T_c$), together with a distributive law combining them.

The next short section 11 describes the terminal positive-to-one polygraph as an ω-category in terms of positive opetopic cardinals. Section 12 gives an explicit description of all the cells in a given positive-to-one polygraph with the help of positive opetopic cardinals. Section 13 establishes the equivalence of categories between **pPoly** and the category of presheaves over **pOpe**. In Section 14, the principal pullbacks are introduced and an adaptation of Harnik's argument to the opetopic context is presented. The original argument was expressed in a different setting and was supposed to show the monadicity of the category of all ω-categories and ω-functors ωCat. However, this original proof contains a gap [H]. Section 15 describes a full nerve functor

$$\widehat{(-)} : \omega Cat \longrightarrow \widehat{\mathbf{pOpeCard}}_\omega$$

and identifies its essential image as the special pullbacks preserving functors. Section 16 describes the inclusion functor $\mathbf{e}$ as the left Kan extension

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

with formulas involving just coproducts (and no other colimits). This gives as a corollary the fact that $\mathbf{e} : \mathbf{pPoly} \to \omega Cat$ preserves connected limits. Then we show that the right adjoint to $Lan_{\mathbf{j}}$

$$\mathbf{j}^* : sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set)$$

(and hence the right adjoint to $\mathbf{e} : \mathbf{pPoly} \to \omega Cat$) is monadic. It is still an open question, c.f. [M], whether the category ωCat is monadic over *all* polygraphs. In the appendix we recall the definition of the category of positive-to-one polygraphs.

as exponent category. The right top vertical equivalence is proved by a 3-out-of-2 argument, relying on a direct equivalence between ωCat and $sPb(\mathcal{S}^{op}, Set)$ proved in [MZ], and on the right bottom equivalence (Proposition 15.1 – this is where Harnik's result 14.3 is invoked). Once the right top equivalence is known, one may concentrate on the top rectangle allowing us to look at $\mathbf{e}$ as $Lan_{\mathbf{j}}$.

[†] As a hint for the importance of normal positive opetopic cardinals, we refer to the observation in the proof of Proposition 11.1. Normal positive opetopic cardinals appear also in the definition of principal pushouts (see Proposition 14.1 and Corollary 14.2).

In Section 17, we describe in detail the strongly cartesian monad T_ω on opetopic sets and its decomposition into two other strongly cartesian monads. We finish the introduction stating an open problem.

Problem. What are the full subcategories of the category of polygraphs $\mathcal{X} \hookrightarrow$ **Poly** that are coreflective as subcategories of $\mathcal{X} \hookrightarrow$ **Poly** $\hookrightarrow \omega Cat$ with coreflector $\omega Cat \to \mathcal{X}$ being monadic?

This paper shows that positive-to-one polygraphs form one such category.

Acknowledgments

I am very grateful to Victor Harnik and Mihaly Makkai for the conversations concerning matters of this paper.

2 Positive hypergraphs

We write ω for the set of natural numbers. For a family $\{X_k\}_{k\in\omega}$ of sets, we write, for all k, $X_{\geq k} = \bigcup_{i\geq k} X_i$ and $X_{\leq k} = \bigcup_{i\leq k} X_i$.

A *positive hypergraph* S is a family $\{S_k\}_{k\in\omega}$ of finite sets of faces, a family of functions $\{\gamma_k : S_{k+1} \to S_k\}_{k\in\omega}$, and a family of total relations $\{\delta_k : S_{k+1} \to S_k\}_{0\leq k<n}$. Moreover, $\delta_0 : S_1 \to S_0$ is a function and only finitely many among sets $\{S_k\}_{k\in\omega}$ are non-empty. As it is always clear from the context, we shall never use the indices of the functions γ and δ. We shall write $\delta(a) = \{b | \, (a, b) \in \delta\}$.

A *morphism of positive hypergraphs* $f : S \longrightarrow T$ is a family of functions $f_k : S_k \longrightarrow T_k$, for $k \in \omega$, such that the diagrams

$$\begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \gamma\downarrow & & \downarrow\gamma \\ S_k & \xrightarrow[f_k]{} & T_k \end{array} \qquad\qquad \begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \delta\downarrow & & \downarrow\delta \\ S_k & \xrightarrow[f_k]{} & T_k \end{array}$$

commute, for $k \in \omega$. The commutation of the left hand square is the commutation of the diagram of sets of functions but in case of the right hand square we mean more than commutation of a diagram of relations, i.e., we demand that for any $a \in S_{\geq 1}$, $f_a : \delta(a) \longrightarrow \delta(f(a))$ be a bijection, where f_a is the restriction of f to $\delta(a)$. The category of positive hypergraphs is denoted by **pHg**.

Some notions and notation. Let S be a positive hypergraph.

1. When convenient and not leading to confusions, if $a \in S_k$, i.e., a is k-dimensional face in S, we sometime treat $\gamma(a)$ as an element of S_{k-1} and sometimes as a subset $\{\gamma(a)\}$ of S_{k-1} (following the coercion from functions to relations).

2. The dimension of S is $max\{k \in \omega : S_k \neq \emptyset\}$, and it is denoted by $dim(S)$.

3. The sets of faces of different dimensions are assumed to be disjoint (i.e., $S_k \cap S_l = \emptyset$, for $k \neq l$). S is also used to mean the set of all faces of S, i.e., $\bigcup_{k=0}^n S_k$; the notation $A \subseteq S$ means that A is a set of some faces of S; $A_k = A \cap S_k$, for $k \in \omega$.

4. For $a \in S_{\geq 1}$, the set $\partial(a) = \delta(a) \cup \gamma(a)$ is *the boundary of A*, i.e., the set of codimension 1 faces in a.

5. The set $S_{\leq k}$ is closed under δ and γ so it is a sub-hypergraph of S, called k-truncation of S.

6. The image of $A \subseteq S$ under δ and γ will be denoted by
$$\delta(A) = \bigcup_{a\in A} \delta(a), \qquad \gamma(A) = \{\gamma(a) \,:\, a \in A\},$$
respectively. In particular, $\delta\delta(a) = \bigcup_{x\in\delta(a)} \delta(x)$, $\gamma\delta(a) = \{\gamma(x) : x \in \delta(a)\}$.

7. $\iota(a) = \delta\delta(a) \cap \gamma\delta(a)$ is *the set of internal faces* of the face $a \in S_{\geq 2}$.

8. On each set S_k we introduce two binary relations $<^{S_k,-}$ and $<^{S_k,+}$, called *lower* and *upper order*, respectively. We usually omit k and even S in the superscript.

   (a) $<^{S_0,-}$ is the empty relation. For $k > 0$, $<^{S_k,-}$ is the transitive closure of the relation $\lhd^{S_k,-}$ on S_k, such that $a \lhd^{S_k,-} b$ iff $\gamma(a) \in \delta(b)$. We write $a \bowtie^- b$ iff either $a <^- b$ or $b <^- a$, and we write $a \leq^- b$ iff either $a = b$ or $a <^- b$.[†] Of course these notations apply to $<^+$, etc. as well.

   (b) $<^{S_k,+}$ is the transitive closure of the relation $\lhd^{S_k,+}$ on S_k, such that $a \lhd^{S_k,+} b$ iff there is $\alpha \in S_{k+1}$, such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. We write $a \bowtie^+ b$ iff either $a <^+ b$ or $b <^+ a$, and we write $a \leq^+ b$ iff either $a = b$ or $a <^+ b$.

   (c) $a \not\bowtie b$ if both conditions $a \not\bowtie^+ b$ and $a \not\bowtie^- b$ hold.

9. Let $a, b \in S_k$. A *lower path* $a_0, \ldots, a_m$ *from* a *to* b in S is a sequence of faces $a_0, \ldots, a_m \in S_k$ such that $a = a_0$, $b = a_m$ and for $\gamma(a_{i-1}) \in \delta(a_i)$, $i = 1, \ldots, m$.

10. Let $x, y \in S_k$. An *upper path* $x, a_0, \ldots, a_m, y$ *from* x *to* y in S is a sequence of faces $a_0, \ldots, a_m \in S_{k+1}$ such that $x \in \delta(a_0)$, $y = \gamma(a_m)$ and $\gamma(a_{i-1}) \in \delta(a_i)$, for $i = 1, \ldots, m$.

11. The iterations of γ and δ will be denoted in two different ways. By γ^k and δ^k we mean k applications of γ and δ, respectively. By $\gamma^{(k)}$ and $\delta^{(k)}$ we mean the application as many times γ and δ, respectively, to get faces of dimension k. For example, if $a \in S_5$, then $\delta^3(a) = \delta\delta\delta(a) \subseteq S_2$ and $\delta^{(3)}(a) = \delta\delta(a) \subseteq S_3$.

12. For $l \leq k$, $a, b \in S_k$, we define $a <_l b$ iff $\gamma^{(l)}(a) <^- \gamma^{(l)}(b)$.

13. A face a is *unary* iff $\delta(a)$ is a singleton.

Lemma 2.1 *If S is a hypergraph and $k \in \omega$, then $<^{S_{k+1},-}$ is a strict partial order iff $<^{S_k,+}$ is a strict partial order.*

3 Positive opetopic cardinals

To simplify the notation, we treat both δ and γ as functions acting on faces as well as on sets of faces, which means that sometimes we confuse elements with singletons. Clearly, both δ and γ, when considered as functions on sets of faces, are monotone.

A positive hypergraph S is a *positive opetopic cardinal* if it is non-empty, i.e., $S_0 \neq \emptyset$, and if the following conditions hold:

[†] In the original submission, the symbol $\perp$ was used in place of $\bowtie$. In January 2024, following referees' recommendations, Marek decided to replace this symbol by the more evocative one here – a change he had the time to implement himself in [Z3].

1. *Globularity:* for $a \in S_{\geq 2}$:

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a) \qquad\qquad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a).$$

2. *Strictness*: for $k \in \omega$, the relation $<^{S_k,+}$ is a strict order; $<^{S_0,+}$ is linear.[†]

3. *Disjointness*: for $k > 0$,

$$\bowtie^{S_k,-} \cap \bowtie^{S_k,+} = \emptyset.$$

4. *Pencil linearity*: for any $k > 0$ and $x \in S_{k-1}$, the sets

$$\{a \in S_k \mid x = \gamma(a)\} \quad \text{and} \quad \{a \in S_k \mid x \in \delta(a)\}$$

are linearly ordered by $<^{S_k,+}$.

Remarks.

1. The reason why we call the first condition 'globularity' is that it will imply the usual globularity condition in the ω-categories generated by positive opetopic cardinals.

2. Note that if we were to assume that each positive opetopic cardinal has a single cell of dimension -1, then linearity of $<^{S_0,+}$ would become a special case of pencil linearity.

3. The fact that, for $x \in S_{k-1}$, the set $\{a \in S_k \mid x = \gamma(a)\}$ is linearly ordered will sometimes be referred to as γ-*linearity* of $<^{S_k,+}$, and the fact that the set $\{a \in S_k \mid x \in \delta(a)\}$ is linearly ordered is sometimes referred to as δ-*linearity* of $<^{S_k,+}$.

4. The *size of positive opetopic cardinal* S is the sequence of natural numbers $size(S) = \{|S_n - \delta(S_{n+1})|\}_{n\in\omega}$, with almost all being equal 0. We have an order $<$ on such sequences, so that $\{x_n\}_{n\in\omega} < \{y_n\}_{n\in\omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and for all $l > k$, $x_l = y_l$. This order is well founded and many facts about positive opetopic cardinals will be proved by induction on the size.

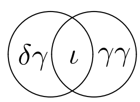
5. The *category of positive opetopic cardinals* is the full subcategory of positive hypergraphs **pHg** whose objects are the positive opetopic cardinals and is denoted by **pOpeCard**.

6. Let S be a positive opetopic cardinal. S is k-principal iff $size(S)_l = 1$, for $l \leq k$. S is a *positive opetope* iff S is $dim(S)$-principal. S is *normal* iff S is $(dim(S)-1)$-principal. By **pOpe** we denote the full subcategory of **pOpeCard** whose objects are positive opetopes.

4 Atlas for γ and δ

Lemma 4.1 *Let S be a positive opetopic cardinal, $a \in S_n$, $n > 1$. Then*

1. *the sets $\delta\gamma(a)$, $\iota(a)$, and $\gamma\gamma(a)$ are disjoint;*



[†] In particular $\lhd^{S_k,+}$ is irreflexive, i.e., for every face a of S, $\gamma(a)$ and $\delta(a)$ are disjoint.

2. $\delta\delta(a) = \delta\gamma(a) \cup \iota(a)$;

3. $\gamma\delta(a) = \gamma\gamma(a) \cup \iota(a)$.

Proof. These are immediate consequences of globularity. $\Box$

Lemma 4.2 *Let S be a positive opetopic cardinal, $a \in S_n$, $n > 2$. Then we have*

1. $\delta\gamma\gamma(a) \subseteq \delta\gamma\delta(a) \subseteq \delta\gamma\gamma(a) \cup \iota\gamma(a) = \delta\delta\gamma(a) = \delta\delta\delta(a)$;

2. $\gamma\gamma\gamma(a) \subseteq \gamma\gamma\delta(a) \subseteq \gamma\gamma\gamma(a) \cup \iota\gamma(a) = \gamma\delta\gamma(a) = \gamma\delta\delta(a)$.

Proof. From globularity we have $\gamma\gamma(\alpha) \subseteq \gamma\delta(\alpha)$. Thus by monotonicity of δ and γ we get

$$\gamma\gamma\gamma(\alpha) \subseteq \gamma\gamma\delta(\alpha) \quad \text{and} \quad \delta\gamma\gamma(\alpha) \subseteq \delta\gamma\delta(\alpha) \quad \text{and} \quad \gamma\gamma\delta(\alpha) \subseteq \gamma\delta\delta(\alpha).$$

Similarly, as we have $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$ by globularity, it follows by monotonicity of δ and γ:

$$\gamma\delta\gamma(\alpha) \subseteq \gamma\delta\delta(\alpha) \quad \text{and} \quad \delta\delta\gamma(\alpha) \subseteq \delta\delta\delta(\alpha) \quad \text{and} \quad \delta\gamma\delta(\alpha) \subseteq \delta\delta\delta(\alpha).$$

The equalities

$$\delta\gamma\gamma(a) \cup \iota\gamma(a) = \delta\delta\gamma(a) \quad \text{and} \quad \gamma\gamma\gamma(a) \cup \iota\gamma(a) = \gamma\delta\gamma(a)$$

follow from Lemma 4.1. Thus it remains to show that:

1. $\delta\delta\gamma(a) \supseteq \delta\delta\delta(a)$,

2. $\gamma\delta\gamma(a) \supseteq \gamma\delta\delta(a)$.

Both inclusions can be proved similarly. We shall show the first only. Suppose that there is $u \in \delta\delta\delta(a) - \delta\delta\gamma(a)$. Let $x \in \delta(a)$ be $<^-$-minimal element in $\delta(a)$ such that there is $s \in \delta(x)$ with $u \in \delta(s)$. If $s \in \delta\gamma(a)$, then $u \in \delta\delta\gamma(a)$, contrary to the supposition. Thus $s \notin \delta\gamma(a)$. Since $\delta\gamma(a) = \delta\delta(a) - \gamma\delta(a)$ it follows that $s \in \gamma\delta(a)$. Hence there is $x' \in \delta(a)$ with $\gamma(x') = s$. In particular, $x' <^- x$. Moreover

$$u \in \delta(s) = \delta\gamma(x') \subseteq \delta\delta(x').$$

Then there is $s' \in \delta(x')$ so that $u \in \delta(s')$. This contradicts the $<^-$-minimality of x. $\Box$

Corollary 4.3 *Let S be a positive opetopic cardinal, $a \in S_n$, $n > 2$, $k < n$. Then, with ξ^l and ξ'^l being two fixed strings of γ's and δ's of length l, we have*

1. $\gamma^k(a) \subseteq \gamma\xi^{k-1}(a)$;

2. $\delta\xi^{k-1}(a) \subseteq \delta^k(a)$;

3. $\delta^k(a) \cap \gamma^k(a) = \emptyset$;

4. $\xi^k(a) \subseteq \gamma^k(a) \cup \delta^k(a)$;

5. $\delta^2\xi^{k-2}(a) = \delta^2\xi'^{k-2}(a)$, *(e.g.* $\delta^k(a) = \delta^2\gamma^{k-2}(a)$*)*;

6. $\gamma\delta\xi^{k-2}(a) = \gamma\delta\xi'^{k-2}(a)$, *(e.g.* $\gamma\delta^{k-1}(a) = \gamma\delta\gamma^{k-2}(a)$*)*;

7. $\xi^{k-2}\delta\gamma(a) = \xi^{k-2}\delta^2(a)$, *for* $k > 2$;

8. $\delta^k(a) = \delta\gamma^{k-1}(a) \cup \iota\gamma^{k-2}(a)$, *for* $k > 1$.

5 Combinatorial properties of positive opetopic cardinals

Local properties

Proposition 5.1 *Let S be a positive opetopic cardinal, $k > 0$ and $\alpha \in S_k$, $a_1, a_2 \in \delta(\alpha)$, $a_1 \neq a_2$. Then we have*

1. $a_1 \not\bowtie^+ a_2$;
2. $\delta(a_1) \cap \delta(a_2) = \emptyset$ *and* $\gamma(a_1) \neq \gamma(a_2)$.

Proof. Ad 1. Suppose on the contrary that there are $a_1, a_2 \in \delta(\alpha)$ such that $a_1 <^+ a_2$. So we have an upper path

$$a_1, \beta_1, \ldots, \beta_r, a_2$$

and hence a lower path

$$\beta_1, \ldots, \beta_r, \alpha.$$

In particular, $\beta_1 <^- \alpha$. As $a_1 \in \delta(\beta_1) \cap \delta(\alpha)$ by δ-linearity, we have $\beta_1 \bowtie^+ \alpha$. But then $(\alpha, \beta_1) \in \bowtie^+ \cap \bowtie^- \neq \emptyset$, i.e., S does not satisfy the disjointness. This shows 1.

Ad 2. This is an immediate consequence of 1. If $a_1, a_2 \in \delta(\alpha)$ and either $\gamma(a_1) = \gamma(a_2)$ or $\delta(a_1) \cap \delta(a_2) \neq \emptyset$, then by pencil linearity we get that $a_1 \bowtie^+ a_2$, contradicting 1. $\Box$

We next introduce more notation. Let S be a positive opetopic cardinal, $n \in \omega$.

1. For a face $\alpha \in S_{n+2}$, we shall denote by $\rho(\alpha) \in \delta(\alpha)$ the only face in $\delta(\alpha)$, such that $\gamma(\rho(\alpha)) = \gamma\gamma(\alpha)$.
2. $X \subseteq S_{n+1}$, $a, b \in S_n$ and $a, \alpha_1, \ldots, \alpha_k, b$ be an upper path in S. We say that it is *a path in X* (or X-path) if $\{\alpha_1, \ldots, \alpha_k\} \subseteq X$.

Lemma 5.2 *Let S be a positive opetopic cardinal, $n \in \omega$, $\alpha \in S_{n+2}$, $a, b \in S_{n+1}$, $y \in \delta\delta(\alpha)$. Then*

1. *there is a unique upper $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$;*
2. *there is a unique $x \in \delta\gamma(\alpha)$ and an upper $\delta(\alpha)$-path from x to y such that $\gamma(x) = \gamma(y)$;*
3. *if $t \in \delta(y)$, there are a unique $x \in \delta\gamma(\alpha)$ such $t \in \delta(x)$ and an upper $\delta(\alpha)$-path from x to y;*
4. *If $a <^+ b$, then $\gamma(a) \leq^+ \gamma(b)$.*

Proof. Ad 1. The uniqueness follows from Proposition 5.1.2. To show the existence, let us suppose on the contrary that there is no $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$. We shall construct an infinite upper $\delta(\alpha)$-path from y

$$y, a_1, a_2, \ldots$$

As $y \in \delta\delta(\alpha)$, there is $a_1 \in \delta(\alpha)$ such that $y \in \delta(a_1)$. So now suppose that we have already constructed $a_1, \ldots, a_k$. By assumption $\gamma(a_k) \neq \gamma\gamma(\alpha)$ so, by globularity, $\gamma(a_k) \in \delta\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_k) \in \delta(a_{k+1})$. This ends the construction of the path. This however contradicts strictness and, in fact, there is a $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$.

Ad 2. Suppose that there is no $x \in \delta\gamma(\alpha)$ as claimed. We shall construct an infinite descending lower $\delta(\alpha)$-path

$$\ldots \lhd^- a_1 \lhd^- a_0$$

such that $\gamma(a_0) = y$, $\gamma\gamma(a_n) = \gamma(y) = t$, for $n \in \omega$.

By assumption $y \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $y \in \gamma\delta(\alpha)$. Hence there is $a_0 \in \delta(\alpha)$ such that $\gamma(a_0) = y$. Now, suppose that the lower $\delta(\alpha)$-path

$$a_k \lhd^- a_{k-1} \lhd^- \ldots \lhd^- a_0$$

has been already constructed. By globularity, we can pick $z \in \delta(a_k)$ such that $\gamma(z) = t$. By assumption $z \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $z \in \gamma\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_{k+1}) = z \in \delta(a_k)$. Clearly, $\gamma\gamma(a_{k+1}) = t$. This ends the construction of the path. But by strictness such a path has to be finite, so there is x as needed.

Ad 3. This case is similar. We prove it for completeness. Suppose that there is no $x \in \delta\gamma(\alpha)$ as above. We shall construct an infinite descending lower $\delta(\alpha)$-path

$$\ldots \lhd^- a_1 \lhd^- a_0$$

such that $\gamma(a_0) = y$, $t \in \delta\gamma(a_n)$, for $n \in \omega$.

By assumption $y \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $y \in \gamma\delta(\alpha)$. Hence there is $a_0 \in \delta(\alpha)$ such that $\gamma(a_0) = y$. Now, suppose that the lower $\delta(\alpha)$-path

$$a_k \lhd^- a_{k-1} \lhd^- \ldots \lhd^- a_0$$

has been already constructed. By globularity, we can pick $z \in \delta(a_k)$, such that $t \in \delta(z)$. By assumption $z \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $z \in \gamma\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_{k+1}) = z \in \delta(a_k)$. Clearly, $t \in \delta\gamma(a_{k+1})$. This ends the construction of the path. But by strictness such a path has to be finite, so there is x as needed. The uniqueness again follows from Proposition 5.1.2.

Ad 4. The essential case is when $a \lhd^+ b$. This follows from 1. Then use the induction on the length of the upper path from a to b. □

Lemma 5.3 *Let S be a positive opetopic cardinal, $n > 1$, $\alpha \in S_{n+1}$, and $a, b \in S_n$ such that $a <^+ b$. Then*

1. $\iota\delta(\alpha) = \iota\gamma(\alpha)$;
2. $\iota(a) \subseteq \iota(b)$;
3. $\iota(a) \cup \gamma\gamma(a) \subseteq \iota(b) \cup \gamma\gamma(b)$;
4. $\iota(a) \cup \delta\gamma(a) \subseteq \iota(b) \cup \delta\gamma(b)$;
5. $\partial\partial(a) \subseteq \partial\partial(b)$.

Proof. Ad 1. First we show $\iota\delta(\alpha) \subseteq \iota\gamma(\alpha)$. Fix $a \in \delta(\alpha)$ and $t \in \iota(a)$. Thus there are $x, y \in \delta(a)$ such that $\gamma(x) = t \in \delta(y)$. By Lemma 5.2 2,3 there are $x', y' \in \delta\gamma(\alpha)$ such that $x' \leq^+ x$, $y' \leq^+ y$ and $\gamma(x') = t \in \delta(y')$. Thus $t \in \iota\gamma(\alpha)$ and the first inclusion is proved.

Now we prove the converse inclusion $\iota\delta(\alpha) \supseteq \iota\gamma(\alpha)$. Fix $t \in \iota\gamma(\alpha)$. In particular, there are $x, y \in \delta\gamma(\alpha)$, so that $\gamma(x) = t \in \delta(y)$. Suppose that $t \not\in \iota\delta(\alpha)$. We shall build an infinite $\delta(\alpha)$-path

$$a_1 \lhd^- a_2 \ldots$$

such that $\gamma\gamma(a_i) = t$ for $i \in \omega$.

Since $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$, there is $a_1 \in \delta(\alpha)$ such that $x \in \delta(a_1)$. Since $t \not\in \iota\delta(\alpha)$, it follows that $\gamma\gamma(a_1) = t$. Suppose now that we have already constructed the path

$$a_1 \lhd^- a_2 \lhd^- \ldots \lhd^- a_k$$

with the stated properties. We have $\gamma\gamma(a_k) = t \lhd^+ \gamma(y) \leq^+ \gamma\gamma\gamma(\alpha)$. So, by strictness, $\gamma(a_k) \neq \gamma\gamma(\alpha)$ and $\gamma(a_k) \in \delta\delta(\alpha)$. Then there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_k) \in \delta(a_{k+1})$. Again, as $t \not\in \iota\delta(\alpha)$, it follows that $\gamma\gamma(a_{k+1}) = t$. This ends the construction of the path. Since, by strictness, such a path cannot exist, we get the other inclusion.

Ad 2. Since the inclusion is transitive, it is enough to consider the case $a \lhd^+ b$, i.e., there is an $\alpha \in S_{n+1}$ such that $a \in \delta(\alpha)$ and $b = \gamma(\alpha)$. Then by 1. we have

$$\iota(a) \subseteq \iota\delta(\alpha) = \iota\gamma(\alpha) = \iota(b).$$

Ad 3. Again it is enough to consider the case $a \lhd^+ b$, i.e., that there is $\alpha \in S_{n+1}$ such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. By 2. we need to show that $\gamma\gamma(a) \in \iota(b) \cup \gamma\gamma(b)$. Using Lemma 4.2.2 we have

$$\gamma\gamma(a) \in \gamma\gamma\delta(\alpha) \subseteq \iota\gamma(\alpha) \cup \gamma\gamma\gamma(\alpha) = \iota(b) \cup \gamma\gamma(b).$$

Ad 4. This is similar to 3 and uses Lemma 4.2.1.

Ad 5. This follows from 3. and 4 and Lemma 4.1. □

Global properties

Lemma 5.4 *Let S be a positive opetopic cardinal, $n \in \omega$, $a, b \in S_n$, $a <^+ b$. Then, there is an upper $(S_{n+1} - \gamma(S_{n+2}))$-path from a to b.*

Proof. Let $a, \alpha_1, \ldots, \alpha_k, b$ be an upper path is S. By Lemma 5.2 we can replace each face α_i in this path which is not in $S - \gamma(S)$ by a sequence of faces which are $<^+$-smaller. Just take $\Gamma \in S_{n+2}$, such that $\gamma(\Gamma) = \alpha_i$ and take instead of α_i a path in $\delta(\Gamma)$ from $\gamma(\alpha_{i-1})$ (if $i = 0$ then from a) to $\gamma(\alpha_i)$. Repeated application of this procedure will eventually yield the required path. □

Lemma 5.5 *Let S be a positive opetopic cardinal, $n > 0$, $a \in S_n$, $\alpha \in S_{n+1}$, such that either $\gamma(a) \in \iota(\alpha)$ or $\delta(a) \cap \iota(\alpha) \neq \emptyset$. Then $a <^+ \gamma(\alpha)$. Moreover, if $\alpha \in S - \gamma(S)$, then there is a unique $a' \in \delta(\alpha)$ such that $a \leq^+ a'$.*

Proof. If $a \in \delta(\alpha)$, there is nothing to prove. So we assume $a \not\in \delta(\alpha)$. We begin with the second part of the statement, i.e., we assume $\alpha \in S_{n+1} - \gamma(S_{n+2})$. Let $\gamma(a) \in \iota(\alpha)$. Thus there are $b, c \in \delta(\alpha)$ such that $\gamma(a) = \gamma(b) \in \delta(c)$. In particular, $a <^- c$. By γ-linearity either $b <^+ a$ or $a <^+ b$. Suppose that $b <^+ a$. Then we have an $(S - \gamma(S))$-upper path $b, \beta_0, \ldots, \beta_r, a$. As $b \in \alpha \cap \beta_0$ and $\alpha, \beta_0 \in S - \gamma(S)$, we have $\alpha = \beta_0$. But then $c \in \delta(\alpha) = \delta(\beta_0)$ and hence $c <^+ \gamma(\beta_0) \leq^+ a$. So we get $a <^- c$ and $c <^+ a$, contradicting the disjointness of $\bowtie^+$ and $\bowtie^-$. Thus we can put $a' = b$ and we have $a <^+ a'$. The uniqueness of a' follows from the fact that $\gamma(a) = \gamma(a')$.

The case $\delta(a) \cap \iota(\alpha) \neq \emptyset$ is similar and we put it for completeness. There are $b, c \in \delta(\alpha)$ such that $\gamma(b) \in \delta(a) \cap \delta(c)$. In particular, $b <^- a$. By δ-linearity either $c <^+ a$ or $a <^+ c$. Suppose that $c <^+ a$. Then we have an $(S - \gamma(S))$-upper path $c, \beta_0, \ldots, \beta_r, a$. As $c \in \alpha \cap \beta_0$ and $\alpha, \beta_0 \in S - \gamma(S)$, we have $\alpha = \beta_0$. But then $b \in \delta(\alpha) = \delta(\beta_0)$ and hence $b <^+ \gamma(\beta_0) \leq^+ a$. So we get $b <^- a$ and $b <^+ a$, contradicting the disjointness of $\bowtie^+$ and $\bowtie^-$. Thus we can put $a' = c$ and we have $a <^+ a'$. The uniqueness of a' follows from the fact that $\gamma(b) \in \delta(a')$ and $a' \in \delta(\alpha)$ and Proposition 5.1.

The first part of the statement follows from the above, Lemma 5.2.4 and the following claim.

Claim. If $\alpha \in S_{n+1}$ and $x \in \iota(\alpha)$, then there is an $\alpha' \in S_{n+1}$ such that $\alpha' \leq^+ \alpha$, $x \in \iota(\alpha')$ and $\alpha' \notin \gamma(S_{n+2})$.

Proof of the claim. Suppose the contrary. To get a contradiction, we shall build an infinite descending $\gamma(S_{n+2})$-path

$$\ldots \lhd^+ \alpha_1 \lhd^+ \alpha_0 = \alpha$$

such that $x \in \iota(\alpha_i)$, for $i \in \omega$.

We put $\alpha_0 = \alpha$. Suppose that we have already constructed $\alpha_0, \ldots, \alpha_k \in \gamma(S_{n+2})$. Hence there is $\beta \in S_{n+2}$ such that $\gamma(\beta) = \alpha_k$. Since, by Lemma 5.3.1, $\iota\delta(\beta) = \iota\gamma(\beta) = \iota(\alpha_k)$, there is $\alpha_{k+1} \in \delta(\beta)$ such that $x \in \iota(\alpha_{k+1})$. This ends the construction of the infinite path and the proof of the claim and the lemma. $\Box$

Corollary 5.6 *Let S be a positive opetopic cardinal. If $a \in S - \delta(S)$, then $\gamma(a) \in S - \iota(S)$ and $\delta(a) \subseteq S - \iota(S)$.*

Proof. Let $a \in S_n$ and $\alpha \in S_{n+2}$. If either $\gamma(a) \in \iota(\alpha)$ or $\delta(a) \cap \iota(\alpha) \neq \emptyset$, then by Lemma 5.5 $a <^+ \gamma(\alpha)$. Thus $a \in \delta(S)$. $\Box$

A lower path $b_0, \ldots, b_m$ is a *maximal path* if $\delta(b_0) \subseteq \delta(S) - \gamma(S)$ and $\gamma(b_m) \in \gamma(S) - \delta(S)$, i.e., if it can't be extended either way.

Lemma 5.7 (Path Lemma) *Let $k \geq 0$, $B = (a_0, \ldots, a_k)$ be a maximal lower S_n-path in a positive opetopic cardinal S, $b \in S_n$, $0 \leq s \leq k$, $a_s <^+ b$. Then there are $0 \leq l \leq s \leq p \leq k$ such that*

1. *$a_i <^+ b$ for $i = l, \ldots, p$;*
2. *$\gamma(a_p) = \gamma(b)$;*
3. *either $l = 0$ and $\delta(a_0) \subseteq \delta(b)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(b)$;*
4. *$\gamma(a_i) \in \iota(S)$, for $l \leq i < p$.*

Proof. Let $0 \leq l \leq p \leq k$ be such that $a_i <^+ b$, for $l \leq i \leq p$ and either $l = 0$ or $a_{l-1} \not<^+ b$ and either $p = k$ or $a_{p+1} \not<^+ b$. We shall show that l and p have the properties stated in the lemma. From the very definition the property 1 holds.

We shall next show 2. Take an upper $(S - \gamma(S))$-path from a_p to b: $a_p, \beta_0, \ldots, \beta_r, b$. If $\gamma(a_p) = \gamma\gamma(\beta_i)$, for $i = 0, \ldots, r$, then $\gamma(a_p) = \gamma\gamma(\beta_r) = \gamma(b)$ and we are done. So suppose the contrary and let

$$i_0 = min\{i : \gamma(a_p) \neq \gamma\gamma(\beta_i)\}.$$

Then there are $a, c \in \delta(\beta_{i_0})$ such that $\gamma(a_p) = \gamma(a) \in \delta(c)$ (NB. $a = a_p$ if $i_0 = 0$ and $a = \gamma(\beta_{i_0-1})$, otherwise). In particular, $\gamma(a_p) \in \iota(\beta_{i_0})$. As $\gamma(a_p) \in \delta(S)$, we have $p < k$. Thus $\gamma(a_p) \in \delta(a_{p+1}) \cap \iota(\beta_{i_0})$, and by Lemma 5.5 $a_{p+1} <^+ c <^+ b$. But this contradicts the choice of p. So the property 2. holds.

Now we shall show 3. Take an upper $(S-\gamma(S))$-path from a_l to b: $a_l, \beta_0, \ldots, \beta_r, b$. We have two cases: $l = 0$ and $l > 0$.

If $l=0$, then there is no face $a\in S$ such that $\gamma(a)\in\delta(a_l)$. As $\delta(a_l)\subseteq\delta\delta(\beta_0)$, we must have $\delta(a_l)\subseteq\delta\gamma(\beta_i)$, for $i=0,\ldots,r$. Hence $\delta(a_l)\subseteq\delta\gamma(\beta_r)=\delta(b)$ and 3. holds in this case.

Now suppose that $l>0$. If $\gamma(a_{l-1})\in\delta\gamma(\beta_i)$, for $i=0,\ldots,r$, then $\gamma(a_{l-1})\in\delta\gamma(\beta_r)=\delta(b)$ and 3. holds again. So suppose the contrary and let

$$i_1 = min\{i : \gamma(a_{l-1})\not\in\delta\gamma(\beta_i)\}.$$

Then there are $a,c\in\delta(\beta_{i_1})$ such that $\gamma(a_{l-1})=\gamma(a)\in\delta(c)$ (NB: $c=a_l$ if $i_1=0$ and $c=\gamma(\beta_{i_1-1})$ otherwise). In particular, $\gamma(a_{l-1})\in\iota(\beta_{i_1})$, and by Lemma 5.5 we have $a_{l-1}<^+ a<^+ b$ contrary to the choice of l. Thus 3. holds in this case as well.

Finally, we shall show 4. Let $l\leq j<p$ and $a_j,\beta_0,\ldots,\beta_r,b$ be an upper $(S-\delta(S))$-path from a_j to b. As $a_j<^- a_p$ and $a_p<^+ b$, we have $\gamma(a_j)\neq\gamma(b)$. So we can put

$$i_2 = min\{i : \gamma(a_j)\neq\gamma\gamma(\beta_i)\}.$$

But then $\gamma(a_j)\in\gamma\delta(\beta_{i_2})-\gamma\gamma(\beta_{i_2})=\iota(\beta_{i_2})$. Therefore $\gamma(a_j)\in\iota(S)$ and 4. holds. □

Lemma 5.8 *Let S be a positive opetopic cardinal, $n\in\omega$, $x,y\in S_n$, $x<^+ y$. If $x,y\not\in\iota(S_{n+2})$, then there is an upper $S_{n+1}-\delta(S_{n+2})$-path from x to y.*

Proof. Assume $x,y\in(S-\iota(S))$ and $x<^+ y$. Let

$$x,b_0,\ldots,b_k,y$$

be an upper path from x to y with the longest possible initial segment $b_0,\ldots,b_l$ in $S-\delta(S)$. As $x\not\in\iota(S_{n+2})$, such a non-empty path exists. We need to show that $k=l$. Let a be the $<^+$-largest element of the set $\{b\in S:\gamma(b_l)\in\delta(b)\}$. Then $b_{l+1}\leq^+ a$ and $a\not\in\delta(S)$. Since $y\not\in\iota(S)$, by Lemma 5.7.4 there is p such that $l+1\leq p\leq k$ such that $\gamma(b_p)=\gamma(a)$. Thus we have an upper path from x to y, $x,b_0,\ldots,b_l,a,b_{p+1},\ldots,b_k,y$ with a longer initial segment in $S-\delta(S)$. But this is a contradiction with the choice of the path $x,b_0,\ldots,b_k,y$, and it means that in fact $l=k$, as required. □

Order

Lemma 5.9 *Let S be a positive opetopic cardinal, $n\in\omega$, $a,b\in S_n$. Then we have*

1. *if $a<^+ b$, then for any $x\in\delta(a)$ there is $y\in\delta(b)$ such that $y\leq^+ x$;*
2. *if $a<^+ b$ and $\gamma(a)=\gamma(b)$, then for any $y\in\delta(b)$ there is $x\in\delta(a)$ such that $y\leq^+ x$;*
3. *if $\gamma(a)=\gamma(b)$, then either $a=b$ or $a\bowtie^+ b$;*
4. *if $\gamma(a)<^+\gamma(b)$ then either $a<^+ b$ or $a<^- b$;*
5. *if $a<^+ b$ then $\gamma(a)\leq^+\gamma(b)$;*
6. *if $a<^- b$ then $\gamma(a)<^+\gamma(b)$;*
7. *if $\gamma(a)\bowtie^-\gamma(b)$ then $a\not\bowtie^- b$ and $a\not\bowtie^+ b$.*

Proof. Ad 1. Let $a <^+ b$ and $x \in \delta(a)$. We have two cases: either $x \in \gamma(S)$ or $x \notin \gamma(S)$.

In the first case there is $a' \in S - \gamma(S)$ such that $\gamma(a') = x$. Let $a_0, \ldots, a_k$ be a maximal path containing a', a, say $a_{s-1} = a'$ and $a_s = a$, where $0 < s \leq k$. As $a_s <^+ b$, by Lemma 5.7 there is $l \leq s$ and $y \in \delta(a_l) \cap \delta(b)$. Clearly, $y \leq^+ x$.

In the second case consider an upper path from a to b: $a, \beta_0, \ldots, \beta_r, b$. We have $x \in \delta(a) \subseteq \delta\delta(\beta_0)$. As $x \notin \gamma(S)$ so $x \notin \gamma\delta(\beta_0)$, and hence $x \in \delta\delta(\beta_0) - \gamma\delta(\beta_0) = \delta\gamma(\beta_0)$. Thus we can define

$$r' = max\{i : x \in \delta\gamma(\beta_i)\}.$$

If we had $r' < r$, then again we would have $x \in \delta\delta(\beta_{r'+1}) - \gamma\delta(\beta_{r'+1}) = \delta\gamma(\beta_{r'+1})$, contrary to the choice of r'. So $r' = r$ and $x \in \delta\gamma(\beta_r) = \delta(b)$. Thus we can put $y = x$.

Ad 2. Fix $a <^+ b$ such that $\gamma(a) = \gamma(b)$ and $y \in \delta(b)$. We need to find $x \in \delta(a)$ with $y \leq^+ x$. Take a maximal $(S - \gamma(S))$-path $a_0, \ldots, a_k$ passing through y, i.e., there is $0 \leq j \leq k$ such that $y \in \delta(a_j)$ and if $y \in \gamma(S)$, then moreover $j > 0$ and $\gamma(a_{j-1}) = y$. Since $a_j \notin \gamma(S)$ by δ-linearity we have $a_j <^+ b$. Thus by Lemma 5.7 there is $j \leq p \leq k$ such that $\gamma(a_p) = \gamma(b) = \gamma(a)$. Since $a_p \notin \gamma(S)$ by γ-linearity we have $a_p \leq^+ a$. If $a_p = a$, then we can take as the face x either y if $p = 0$ or $\gamma(a_{p-1})$ if $p > 0$. So assume now $a_p <^+ a$. Again by Lemma 5.7 there is $0 \leq l \leq p$ such that either $l = 0$ and $\delta(a_0) \subseteq \delta(a)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(a)$. As a_l is the first face in the path $a_0, \ldots, a_k$ such that $a_l <^+ a$ and a_j is the first face in the path $a_0, \ldots, a_k$ such that $a_j <^+ b$ and moreover $a <^+ b$, it follows that $j \leq l$. Thus in this case we can take as the face x either y if $l = 0$ or $\gamma(a_{l-1})$ if $l > 0$.

Ad 3. This is an immediate consequence of γ-linearity.

Ad 4. Suppose $\gamma(a) <^+ \gamma(b)$. So there is an upper path

$$\gamma(a), c_1, \ldots, c_k, \gamma(b)$$

with $k > 0$. We put $c_0 = a$. We have $\gamma(c_k) = \gamma(b)$ so by γ-linearity $c_k \bowtie^+ b$ or $c_k = b$. In the latter case $a <^- b$. In the former case, we have two possibilities: either $b <^+ c_k$ or $c_k <^+ b$.

If $b <^+ c_k$, then by Lemma 5.7 for any maximal path that contains b and the face c_k we get that $c_{k-1} <^- b$. Thus we have $a <^- b$.

If $c_k <^+ b$, then by Lemma 5.7 for any maximal path that extends $c_0, c_1, \ldots, c_k$ and face b we get that either there is $0 \leq i < k$ such that $\gamma(c_i) \in \delta(b)$ and then $a <^- b$ or else $a = c_0 <^+ b$.

Ad 5. This is repeated from Lemma 5.2.

Ad 6. Suppose $a <^- b$. Then there is a lower path

$$a = a_0, a_1, \ldots, a_k = b$$

with $k > 0$. Then we have an upper path

$$\gamma(a) = \gamma(a_0), a_1, \ldots, a_k, \gamma(a_k) = \gamma(b).$$

Hence $\gamma(a) <^+ \gamma(b)$.

Ad 7. Easily follows from 5 and 6. □

Proposition 5.10 *Let S be a positive opetopic cardinal, $a, b \in S_n$, $a \neq b$. Let $\{a_i\}_{0\leq i\leq n}$, $\{b_i\}_{0\leq i\leq n}$ be the two sequences of codomains of a and b, respectively, so that*

$$a_i = \gamma^{(i)}(a) \qquad\qquad b_i = \gamma^{(i)}(b)$$

(i.e., $dim(a_i) = i$), for $i = 0, \ldots, n$. Then there are two numbers $0 \leq l \leq k \leq n$ such that

1. $a_i = b_i$, *for* $i < l$,

2. $a_i <^+ b_i$, *for* $l \leq i \leq k$,

3. $a_i <^- b_i$, *for* $k+1 = i \leq n$,

4. $a_i \not\bowtie b_i$, *for* $k+2 \leq i \leq n$,

or the roles of a *and* b *are interchanged.*

Proof. We can present the above conditions more visually as:

$$a_0 = b_0, \ldots, a_{l-1} = b_{l-1} \quad a_l <^+ b_l, \ldots a_k <^+ b_k$$

$$a_{k+1} <^- b_{k+1} \quad a_{k+2} \not\bowtie b_{k+2}, \ldots, a_n \not\bowtie b_n.$$

We will verify these conditions from the bottom up. Note that by strictness $<^{S_0,+}$ is a linear order. So either $a_0 = b_0$ or $a_0 \bowtie^+ b_0$. In the latter case $l = 0$. As $a \neq b$, then there is $i \leq n$ such that $a_i \neq b_i$. Let l be minimal such, i.e., $l = min\{i : a_i \neq b_i\}$. By Lemma 5.9 3., $a_l \bowtie^+ b_l$. So assume $a_l <^+ b_l$. We put $k = max\{i \leq n : a_i <^+ b_i\}$. If $k = n$, we are done. If $k < n$, then by Lemma 5.9 4., we have $a_{k+1} <^- b_{k+1}$. Then if $k+1 < n$, by Lemma 5.9 5. 6. 7., $a_i \not\bowtie b_i$ for $k+2 \leq i \leq n$. This ends the proof. □

Having Proposition 5.10 we can define a relation $<_l^-$ (or simply $<_l$) on k-faces of any positive opetopic cardinal S, $l < k$, as follows: for $a, b \in S_k$, $a <_l b$ iff $\gamma^{(l)}(a) <^- \gamma^{(l)}(b)$.

Corollary 5.11 *Let* S *be a positive opetopic cardinal,* $a, b \in S_n$*,* $a \neq b$*. Then either* $a \bowtie^+ b$ *or there is a unique* $0 \leq l \leq k$ *such that* $a \bowtie_l^- b$*, but not both.*

The above corollary allows us to define an order $<^S$ (also denoted $<$) on all cells of S as follows: for $a, b \in S_n$,

$$a <^S b \quad \text{iff} \quad a <^+ b \ \text{ or } \ \exists_l \ a <_l^- b.$$

Corollary 5.12 *For any positive opetopic cardinal* S*, and* $n \in \omega$*, the relation* $<^S$ *restricted to* S_n *is a linear order.*

Proof. We need to verify that $<^S$ is transitive.

Let $a, b, c \in S_n$. There are some cases to consider.

If $a <^+ b <^+ c$, then clearly $a <^+ c$.

If $a <^+ b <_l^- c$, then, by Lemma 5.2.4., we have $\gamma^{(l)}(a) \leq^+ \gamma^{(l)}(b) <^- \gamma^{(l)}(c)$, and by transitivity of $<^-$ we have $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$. Hence $a <_l^- c$.

Now assume $a <_l^- b <^+ c$ and consider a lower path in S_l containing $\gamma^{(l)}(a)$ and $\gamma^{(l)}(b)$. By Lemma 5.9.5 $\gamma^{(l)}(b) <^+ \gamma^{(l)}(c)$, and hence by Lemma 5.7, either $\gamma^{(l)}(a) <^+ \gamma^{(l)}(c)$ or $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$. In the latter case, by transitivity of $<_l$ we have $a <_l c$, and we are done. In the former case, by Proposition 5.10, either $a <^+ c$ and we are done, or there is $k > l$ such that $\gamma^{(k)}(a) <^- \gamma^{(k)}(c)$, i.e. $a <_k c$, as required.

The last case $a <_k^- b <_l^- c$ has three subcases.

If $k = l$, then clearly $a <_l c$.

If $k > l$, then $\gamma^{(l)}(a) \leq^+ \gamma^{(l)}(b) <^- \gamma^{(l)}(c)$ and, by the previous argument, $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$, i.e., $a <_l^- c$.

Finally, assume $k < l$. Then $\gamma^{(k)}(a) <^- \gamma^{(k)}(b) <^+ \gamma^{(k)}(c)$. By Path Lemma, either $\gamma^{(k)}(a) <^- \gamma^{(k)}(c)$ or $\gamma^{(k)}(a) <^+ \gamma^{(k)}(c)$. In the former case we are done. In the latter case, by Proposition 5.10, either $a <^+ c$ or there is k', such that $k < k' \leq n$ and $\gamma^{(k')}(a) <^+ \gamma^{(k')}(c)$, as required. □

Lemma 5.13 *Let S be a positive opetopic cardinal, $a \in S_n$. Then the set*

$$\{b \in S_n \,:\, a \leq^+ b\}$$

is linearly ordered by $\leq^+$.

Proof. Suppose $a \leq^+ b, b'$. If we were to have $b <^-_l b'$ for some $l \leq n$ then, by Corollary 5.12 we would have $a <^-_l b'$ which is a contradiction. □

Corollary 5.14 *Any morphism of positive opetopic cardinals is one-to-one. Moreover, any automorphism of positive opetopic cardinals is an identity.*

Proof. By Corollary 5.12, the (strict, linear in each dimension) order $<^S$ is defined internally using relations $<^-$ and $<^+$ that are preserved by any morphism. Hence $<^S$ must be preserved by any morphism, as well. From this observation the corollary follows. □

Lemma 5.15 *Let S be a positive opetopic cardinal, $a, b \in S_n$. Then*

1. *if $\iota(a) \cap \iota(b) \neq \emptyset$, then $a \bowtie^+ b$;*
2. *if $\emptyset \neq \iota(a) \subset \iota(b) \neq \iota(a)$, then $a <^+ b$;*
3. *if $a \bowtie^- b$, then $\iota(a) \cap \iota(b) = \emptyset$.*

Proof. 2. is an easy consequence of 1. and Lemma 5.3. 3. is an easy consequence of 1. and Disjointness. We shall show 1.

Assume $u \in \iota(a) \cap \iota(b)$. Thus there are $x, y \in \delta(a)$ and $x', y' \in \delta(b)$ such that $\gamma(x) = \gamma(x') = u \in \delta(y) \cap \delta(y')$. If $x = x'$, then by pencil linearity $a \bowtie^+ b$, as required. So assume that $x \neq x'$. Again by pencil linearity $x \bowtie^+ x'$, say $x' <^+ x$. Thus there is an upper $(T - \gamma(T))$-path $x', a_1, \ldots, a_k, x$. As, for $i = 1, \ldots, k$, $\gamma\gamma(a_i) = u$ and $\gamma\gamma(b) \notin \iota(b) \ni u$, we have that $\gamma(a_i) \neq \gamma(b)$ and $a_i \neq b$. Once again by pencil linearity $a_1 \bowtie^+ b$ and by Path Lemma $a_i < b$, for $i = 1, \ldots, k$ with $\gamma(a_k) \neq \gamma(b)$. As $\gamma(a_k) = x \in \delta(a)$, again by Path Lemma $a <^+ b$, as well. □

Proposition 5.16 *Let S be a positive opetopic cardinal, $a, b \in S_k$, $\alpha \in S_{k+1}$, so that α is a $<^+$-minimal element in S_{k+1}, and $a \in \delta(\alpha)$, $b = \gamma(\alpha)$. Then b is* **a** *$<^+$-successor of a.*

Proof. Assume that α is a $<^+$-minimal element in S_{k+1}. Suppose that there is $c \in S_k$ such that $a <^+ c <^+ b$. Thus we have an upper path

$$a, \beta_1, \ldots, \beta_i, c, \beta_{i+1}, \ldots, \beta_l, b.$$

Hence $\beta_1 <^- \beta_l$. Moreover, $a \in \delta(\beta_1) \cap \delta(\alpha)$ and $\gamma(\beta_l) = b = \gamma(\alpha)$. Thus both β_1 and β_l are $<^+$-comparable with α. Since α is $<^+$-minimal, we have $\alpha <^+ \beta_1, \beta_l$. By Lemma 5.13, $\beta_1 \bowtie^+ \beta_l$. But then we have $(\beta_1, \beta_l) \in \bowtie^+ \cap \bowtie^- \neq \emptyset$, contradicting disjointness. □

Proposition 5.17 *Let T be a positive opetopic cardinal and $X \subseteq T$ a subhypergraph of T. Then X is a positive opetopic cardinal iff the relation $<^{X,+}$ is the restriction of $<^{T,+}$ to X.*

Proof. Assume that X is a subhypergraph of a positive opetopic cardinal T. Then X satisfies axioms of globularity, disjointness, and strictness of the relations $<^{X_k,+}$ for $k > 0$.

Clearly, if $<^{X_k,+} = <^{T_k,+} \cap (X_k)^2$, then the relation $<^{X_0,+}$ is linear, the relations $<^{X_k,+}$, for $k > 0$, satisfy pencil linearity, i.e., X is a positive opetopic cardinal.

Now we assume that the subhypergraph X of positive opetopic cardinal T is a positive opetopic cardinal. We shall show that for $k \in \omega$, $a, b \in X_k$, we have $a <^{X_k,+} b$ iff $a <^{T_k,+} b$. Since X is a subhypergraph, $a <^{X_k,+} b$ implies $a <^{T_k,+} b$. Thus it is enough to show that if $a <^{T_k,+} b$ then $a \bowtie^{X_k,+} b$. We shall prove this by induction on k. For $k = 0$, it is obvious, since $<^{X_0,+}$ is linear. So assume that for faces $x, y \in X_l$, with $l < k$, we already know that $x <^{X_l,+} y$ iff $x <^{T_l,+} y$. Fix $a, b \in T_k$ such that $a <^{T_k,+} b$. Then by Lemma 5.9.2 $\gamma(a) \leq^{T_{k-1},+} \gamma(b)$ and hence by inductive hypothesis $\gamma(a) \leq^{X_{k-1},+} \gamma(b)$. Thus we have an upper $(X - \gamma(X))$-path $a = a_r, \gamma(a), a_{r-1} \ldots, a_1, \gamma(b)$, with $r \geq 1$. Since X is a positive opetopic cardinal and $\gamma(a_1) = \gamma(b)$, by pencil linearity we have $a_1 \leq^+ b$. By Path Lemma 5.7, either $a <^{X,-} b$ or $a <^{X,+} b$. Since the first option is impossible, we have $a <^{X,+} b$, as required. □

Lemma 5.18 *Let T be a positive opetopic cardinal, $a, b, \alpha \in T$. If $a \in \delta(\alpha)$ and $a <^+ b <^+ \gamma(\alpha)$, then $b \in \iota(T)$.*

Proof. Assume that $a, b, \alpha \in T$ are as in the assumption of the lemma. Thus we have an upper path $a, \alpha_0, \ldots, \alpha_r, b$. As $a \in \delta(\alpha) \cap \delta(\alpha_0)$, by pencil linearity we have $\alpha \bowtie^+ \alpha_0$. If $\alpha <^+ \alpha_0 <^- \alpha_r$, then $\gamma(\alpha) \leq^+ \gamma(\alpha_r) = b$, contradicting our assumption. Thus $\alpha_0 <^+ \alpha$. Then by Path Lemma 5.7, since $b = \gamma(\alpha_r) <^+ \gamma(\alpha)$, we have $\alpha_r <^+ \alpha$ and $b \in \iota(T)$, as required. □

Some equations

Proposition 5.19 *Let S be a positive opetopic cardinal $0 < k \in \omega$. Then*

1. $\iota(S_{k+1}) = \iota(S_{k+1} - \delta(S_{k+2}))$ *and* $\iota(S_{k+1}) = \iota(S_{k+1} - \gamma(S_{k+2}))$;
2. $\delta(S_k) = \delta(S_k - \gamma(S_{k+1}))$;
3. $\gamma(S_k) = \gamma(S_k - \gamma(S_{k+1}))$;
4. $\delta(S_k) = \delta(S_k - \iota(S_{k+2}))$;
5. $\delta(S_k) = \delta(S_k - \delta(S_{k+1})) \cup \iota(S_{k+1})$.

Proof. In all the above equations the inclusion $\supseteq$ is obvious. So in each case we need to check the inclusion $\subseteq$ only.

Ad 1. Both equalities follow from Lemma 5.3.

To prove the first equality, let $s \in \iota(S_{k+1})$, i.e., there is $a \in S_{k+1}$ such that $s \in \iota(a)$. By strictness, there is $b \in S_{k+1}$ such that $a \leq^+ b$ and $b \notin \delta(S_{k+1})$. By Lemma 5.3, we have

$$s \in \iota(a) \subseteq \iota(b) \subseteq \iota(S_{k+1} - \delta(S_{k+2}))$$

as required.

To prove the second equality, suppose on the contrary that there is $x \in \iota(S_{n+1})$ such that $x \notin \iota(S_{n+1} - \gamma(S_n))$. Let $a \in S_{n+1}$ be a $<^+$-minimal face such that $x \in \iota(a)$. Since $x \notin \iota(S_{n+1} - \gamma(S_n))$, there is $\alpha \in S_n$ such that $a = \gamma(\alpha)$. By Lemma 5.3 we have

$$\iota(a) = \iota\gamma(\alpha) = \iota\delta(\alpha).$$

Therefore, there is $a' \in \delta(\alpha)$ such that $x \in \iota(a')$. Clearly $a' \lhd^+ a$, and hence a is not $<^+$-minimal, contrary to the supposition. This ends the proof of the first equality above.

Ad 2. Let $x \in \delta(S_k)$. Let $a \in S_k$ be the $<^+$-minimal element in S_k such that $x \in \delta(a)$. We shall show that $a \in S_k - \gamma(S_{k+1})$. Suppose on the contrary that there is an $\alpha \in S_{k+1}$ such that $a = \gamma(\alpha)$. Then by globularity

$$x \in \delta(a) = \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

So there is $b \in \delta(\alpha)$ such that $x \in \delta(b)$. As $b <^+ a$, this contradicts the minimality of a.

Ad 3. This is similar to the previous one but simpler.

Ad 4. Since $\iota(S_{k+2}) \subseteq \gamma(S_{k+1})$ 4. follows from 2.

Ad 5. Let $x \in \delta(S_k)$. Let $a \in S_k$ be the $<^+$-largest element in S_k such that $x \in \delta(a)$. If $a \notin \delta(S_{k+1})$, then $x \in \delta(S_k - \delta(S_{k+1}))$, as required. So assume that $a \in \delta(S_{k+1})$, i.e., there is $\alpha \in S_{k+1}$ such that $a \in \delta(\alpha)$. Thus $x \in \delta\delta(\alpha)$. As $a <^+ \gamma(\alpha)$, by choice of a we have $x \notin \delta\gamma(\alpha)$ $(= \delta\delta(\alpha) - \gamma\delta(\alpha))$. So $x \in \gamma\delta(\alpha)$ and hence $x \in \iota(\alpha) \subseteq \iota(S_{k+1})$, as required. $\Box$

6 The ω-categories generated by the positive opetopic cardinals

The main objective of this section is to construct an embedding

$$(-)^* : \mathbf{pOpeCard} \longrightarrow \omega Cat$$

of the category of opetopic cardinals into the category of ω-categories. This embedding is not full. In Section 9, we shall show that the image of $(-)^*$ factorizes through $\mathbf{pPoly} \to \omega Cat$ as a full functor.

Let T be a positive opetopic cardinal. By T^*_n we denote the set of all positive opetopic cardinals contained in T of dimension at most n. If one wants to make these sets disjoint, one can think that an element of T^*_n is a pair $\langle n, S\rangle$, where S is a positive opetopic cardinal contained in T. We define below an ω-category, denoted T^*, whose set of n-cells is T^*_n. We introduce operations

$$\mathbf{d}^{(k)}, \mathbf{c}^{(k)} : T^*_n \longrightarrow T^*_k$$

of the k-th *domain* and the k-th *codomain* (of an m-dimensional cell), where $0 \leq k \leq n$. For S in $(T^*)_m$, the faces of the k-th *domain* $\mathbf{d}^{(k)}(S)$ are:

1. $(\mathbf{d}^{(k)}(S))_l = \emptyset$, for $l > k$,
2. $(\mathbf{d}^{(k)}(S))_k = S_k - \gamma(S_{k+1})$,
3. $(\mathbf{d}^{(k)}(S))_l = S_l$, for $0 \leq l < k$.

and faces of the k-th *codomain* $\mathbf{c}^{(k)}(S)$ are:

1. $(\mathbf{c}^{(k)}(S))_l = \emptyset$, for $l > k$,
2. $(\mathbf{c}^{(k)}(S))_k = S_k - \delta(S_{k+1})$,
3. $(\mathbf{c}^{(k)}(S))_{k-1} = S_{k-1} - \iota(S_{k+1})$, if $k > 0$,
4. $(\mathbf{c}^{(k)}(S))_l = S_l$, for $0 \leq l < k-1$.

Note that the definitions of $\mathbf{d}^{(k)}(S)$ and $\mathbf{c}^{(k)}(S)$, for $S \in T^*_n$ do not depend on the ambient opetopic cardinal T, nor even on $dim(S)$. Therefore we can write $\mathbf{d}^{(k)}(S)$ and $\mathbf{c}^{(k)}(S)$ without specifying T. If $n \in \omega$ and $S \in T^*_{n+1}$, we write $\mathbf{d}(S)$ for $\mathbf{d}^{(n)}(S)$, and $\mathbf{c}(S)$ for $\mathbf{c}^{(n)}(S)$.

Lemma 6.1 *Let S and T be positive opetopic cardinals and $k \in \omega$. Then*

1. *if $k < dim(S)$, then both $\mathbf{d}^{(k)}(S)$, $\mathbf{c}^{(k)}(S)$ are positive opetopic cardinals of dimension k; if $k \geq dim(S)$, then $\mathbf{d}^{(k)}(S) = S = \mathbf{c}^{(k)}(S)$;*
2. *$\mathbf{dd}^{(k+1)}(S) = \mathbf{d}^{(k)}(S)$, $\mathbf{cc}^{(k+1)}(S) = \mathbf{c}^{(k)}(S)$;*
3. *if $S \in T^*_k$ and $k \geq 2$, then $\mathbf{dd}(S) = \mathbf{dc}(S)$, $\mathbf{cd}(S) = \mathbf{cc}(S)$;*
4. *for any $\alpha \in S_k$, the least sub-hypergraph of S containing the face α is again a positive opetopic cardinal of dimension k; it is denoted by $[\alpha]$. Moreover, if $k > 0$, then*
$$\mathbf{c}[\alpha] = [\gamma(\alpha)], \qquad \mathbf{d}[\alpha] = [\delta(\alpha)]$$
where $[\delta(\alpha)]$ is the least sub-hypergraph of S containing the set of faces $\delta(\alpha)$.†

Proof. Ad 1. It is obvious that $\mathbf{d}^{(k)}(S)$ is a sub-hypergraph S. By Corollary 5.6 $\mathbf{c}^{(k)}(S)$ is a sub-hypergraph S as well. Any sub-hypergraph T of a positive opetopic cardinal S satisfies the conditions of globularity, strictness (possibly without $<^{T_0,+}$ being linear), and disjointness.

By Lemma 5.4, for $a, b \in \mathbf{d}^{(k)}S_l$ we have $a <^{S_l,+} b$ iff $a <^{\mathbf{d}^{(k)}S_l,+} b$. Moreover, by Lemma 5.8, for $a, b \in \mathbf{c}^{(k)}(S)_l$ we have $a <^{S_l,+} b$ iff $a <^{\mathbf{c}^{(k)}(S)_l,+} b$. Hence by Proposition 5.17 both $\mathbf{d}^{(k)}(S)$ and $\mathbf{c}^{(k)}(S)$ are positive opetopic cardinals.

Ad 2. Fix a positive opetopic cardinal S and $k \in \omega$ such that $dim(S) > k$. Then the faces of $\mathbf{c}^{(k+1)}(S)$, $\mathbf{cc}^{(k+1)}(S)$, and $\mathbf{c}^{(k)}(S)$ are as in the table

dim	$\mathbf{c}^{(k+1)}(S)$	$\mathbf{cc}^{(k+1)}(S)$	$\mathbf{c}^{(k)}(S)$
$k+1$	$S_{k+1} - \delta(S_{k+2})$	$\emptyset$	$\emptyset$
k	$S_k - \iota(S_{k+2})$	$(S_k - \iota(S_{k+2})) - \delta(S_{k+1} - \delta(S_{k+2}))$	$S_k - \delta(S_{k+1})$
$k-1$	S_{k-1}	$S_{k-1} - \iota(S_{k+1} - \delta(S_{k+2}))$	$S_{k-1} - \iota(S_{k+1})$
l	S_l	S_l	S_l

where $l < k-1$. Moreover, the faces of $\mathbf{d}^{(k+1)}(S)$, $\mathbf{dd}^{(k+1)}(S)$, and $\mathbf{d}^{(k)}(S)$ are as in the table

dim	$\mathbf{d}^{(k+1)}(S)$	$\mathbf{dd}^{(k+1)}(S)$	$\mathbf{d}^{(k)}(S)$
$k+1$	$S_{k+1} - \gamma(S_{k+2})$	$\emptyset$	$\emptyset$
k	S_k	$S_k - \gamma(S_{k+1} - \gamma(S_{k+2}))$	$S_k - \gamma(S_{k+1})$
l	S_l	S_l	S_l

where $l < k$. Thus the equalities in question all follow from Lemma 5.19.

Ad 3. Let $dim(S) = n > 1$. Note that both $(\mathbf{dd}(S))_{n-2}$ and $(\mathbf{dc}(S))_{n-2}$ are the sets of all $<^+$-minimal elements in S_{n-2}, i.e., they are equal and the equation $\mathbf{dd}(S) = \mathbf{dc}(S)$ holds.

To see that $\mathbf{cd}(S) = \mathbf{cc}(S)$ holds, note first that both $(\mathbf{cd}(S))_{n-2}$ and $(\mathbf{cc}(S))_{n-2}$ are the sets of all $<^+$-maximal elements in S_{n-2}. Moreover

$$(\mathbf{cd}(S))_{n-3} = S_{n-3} - \iota(S_{n-1} - \gamma(S_n))$$

† As usual, parentheses are sometimes omitted: $\mathbf{dd}(S)$ (or even $\mathbf{dd}S$) stands for $\mathbf{d}(\mathbf{d}(S))$, $\mathbf{c}[\alpha]$ stands for $\mathbf{c}([\alpha])$, etc.

$$(\mathbf{cc}(S))_{n-3} = S_{n-3} - \iota(S_{n-1} - \delta(S_n)).$$

Now the equality $\mathbf{cd}(S) = \mathbf{cc}(S)$ follows from the following equalities

$$\iota(S_{n-1} - \gamma(S_n)) = \iota(S_{n-1}) = \iota(S_{n-1} - \delta(S_n)),$$

that themselves follow from Lemma 5.19.1.

Ad 4. Fix $\alpha \in S_k$. We need to show that $[\alpha]$ is a positive opetopic cardinal. The globularity, strictness (except for linearity of $<^{[\alpha]_0,+}$), and disjointness are clear.

The linearity of $<^{[\alpha]_0,+}$. If $k \leq 2$, it is obvious. Put $a = \gamma^{(k+2)}(\alpha)$. Using Corollary 4.3, we have

$$[\alpha]_k = \delta^{(k)}(\alpha) \cup \gamma^{(k)}(\alpha) =$$
$$= \delta\delta(\gamma^{(k+2)}(\alpha)) \cup \gamma\gamma(\gamma^{(k+2)}(\alpha)) = \delta\delta(a) \cup \gamma\gamma(a).$$

Thus it is enough to check the linearity of $<^{[\alpha]_0,+}$ for α of dimension $k = 2$. But in this case, as we mentioned, the linearity of $<^{[\alpha]_0,+}$ is obvious.

The γ-linearity of $[\alpha]$. The proof proceeds by induction on $k = dim(\alpha)$. For $k \leq 2$, the γ-linearity is obvious. So assume that $k > 2$ and that for $l < k$ and $a \in S_l$ γ-linearity holds in $[a]$.

First we shall show that $\mathbf{c}[\alpha] = [\gamma(\alpha)]$. We have

$$\mathbf{c}[\alpha]_{k-1} = (\gamma(\alpha) \cup \delta(\alpha)) - \delta(\alpha) = \gamma(\alpha) = [\gamma(\alpha)]_{k-1}$$

$$\mathbf{c}[\alpha]_{k-2} = (\gamma\gamma(\alpha) \cup \delta\delta(\alpha)) - \iota(\alpha) = \gamma\gamma(\alpha) \cup \delta\gamma(\alpha) = [\gamma(\alpha)]_{k-2},$$

and for $l < k-2$

$$\mathbf{c}[\alpha]_l = \gamma^{(l)}(\alpha) \cup \delta^{(l)}(\alpha) = \gamma^{(l)}(\alpha) \cup \delta^{(l)}\gamma(\alpha) = [\gamma(\alpha)]_l.$$

Note that the definition of $\mathbf{c}(H)$ makes sense for any positive hypergraph H and in the above argument we haven't used the fact (which we don't know yet) that $[\alpha]$ is a positive opetopic cardinal.

Thus, for $l < k-2$, $[\alpha]_l = [\gamma(\alpha)]_l$. By induction hypothesis, $[\gamma(\alpha)]$ is a positive opetopic cardinal, and hence $[\alpha]_l$ is γ-linear for $l < k-2$. Clearly $[\alpha]_l$ is γ-linear for $l = k-1, k$. Thus it remains to show the γ-linearity of $(k-2)$-cells in $[\alpha]$.

Fix $t \in [\alpha]_{k-3}$, and let

$$\Gamma_t = \{x \in [\alpha]_{k-2} : \gamma(x) = t\}.$$

We need to show that Γ_t is linearly ordered by $<^+$. We can assume $t \in \gamma([\alpha]_{n-2}) = \gamma\delta\delta(\alpha) = \gamma\delta\gamma(\alpha)$ (otherwise $\Gamma_t = \emptyset$ is clearly linearly ordered by $<^+$). By Proposition 5.1 there is a unique $x_t \in \delta\gamma(\alpha)$ such that $\gamma(x_t) = t$. From Lemma 5.2.2 we get easily the following claim.

Claim 1. For every $x \in \Gamma_t$ there is a unique upper $\delta(\alpha)$-path from x_t to x.

Now fix $x, x' \in \Gamma_t$. By Claim 1, we have the unique upper $\delta(\alpha)$-path

$$x_t, a_0, \ldots, a_l, x, \quad x_t, a'_0, \ldots, a'_{l'}, x'.$$

Suppose $l \leq l'$. By Proposition 5.1, for $i \leq l$, we have $a_i = a'_i$. Hence either $l = l'$ and $x = x'$ or $l < l'$ and

$$x, a_{l+1}, \ldots, a_{l'}, x'$$

is a $\delta(\alpha)$-upper path. Hence either $x = x'$ or $x \bowtie^+ x'$ and $[\alpha]_{k-2}$ satisfy the γ-linearity, as required.

The proof of the δ-linearity of $[\alpha]$ is very similar to the one above. For the same reasons the only non-trivial thing to check is the condition for $(k-2)$-faces. We pick $t \in \delta\delta(\alpha)$ and consider the set

$$\Delta_t = \{x \in [\alpha]_{k-2} : t \in \delta(x)\}.$$

Then we have a unique $y_t \in \delta\gamma(\alpha)$ such that $t \in \delta(y_t)$. From Lemma 5.2.3 we get the following claim.

Claim 2. For every $y \in \Delta_t$ there is a unique upper $\delta(\alpha)$-path from y_t to y.

The δ-linearity of the $(k-2)$-faces in $[\alpha]$ can be proved from Claim 2 in the same way as the γ-linearity was proved from Claim 1.

It remains to verify the equalities

$$\mathbf{c}[\alpha] = [\gamma(\alpha)] \qquad \mathbf{d}[\alpha] = [\delta(\alpha)].$$

We already checked the first one on the way. To see that the second equality also holds we calculate

$$\mathbf{d}[\alpha]_{k-1} = (\gamma(\alpha) \cup \delta(\alpha)) - \gamma(\alpha) = \delta(\alpha) = [\delta(\alpha)]_{k-1}$$

$$\mathbf{d}[\alpha]_{k-2} = (\gamma\gamma(\alpha) \cup \delta\delta(\alpha)) = \gamma\delta(\alpha) \cup \delta\delta(\alpha) = [\delta(\alpha)]_{k-2},$$

and for $l < k-2$

$$\mathbf{d}[\alpha]_l = \gamma^{(l)}(\alpha) \cup \delta^{(l)}(\alpha)) = \gamma^{(l)}\delta(\alpha) \cup \delta^{(l)}(\alpha) = [\delta(\alpha)]_l.$$

So the second equality holds as well. $\Box$

Remarks.

1. Inspired by the above Lemma 6.1.4 we call S a *weak positive opetopic cardinal* if S satisfies globularity, strictness, and disjointness as a positive hypergraph and if moreover for any face α in S the sub-hypergraph $[\alpha]$ is an opetope. (i.e., pencil linearity is required to hold only 'locally'). The *category of weak positive opetopic cardinals* is the full subcategory of the category of positive hypergraphs **pHg** whose objects are the weak positive opetopic cardinals and is denoted by **wpOpeCard**. For each $k \in \omega$, the k-truncation of a weak positive opetopic cardinal S is again a weak positive opetopic cardinal $S_{\leq k}$. In particular, any k-truncation of a positive opetopic cardinal S is a weak positive opetopic cardinal $S_{\leq k}$, but it does not necessarily satisfy the linearity condition.

2. From Lemma 6.1.1 we know that for any positive opetopic cardinal S the hypergraphs $\mathbf{c}^{(k)}(S)$ and $\mathbf{d}^{(k)}(S)$ are positive opetopic cardinals contained in S. We shall denote these embeddings by

$$\mathbf{c}^{(k)}(S) \xrightarrow{\mathbf{c}^{(k)}_S} S \xleftarrow{\mathbf{d}^{(k)}_S} \mathbf{d}^{(k)}(S).$$

Lemma 6.2 *Let X and Y be positive opetopic cardinals.*

1. *If $\mathbf{c}^{(k)}(X) = X \cap Y \subseteq \mathbf{d}^{(k)}(Y)$. If we have moreover $\mathbf{c}^{(k)}(X) = X \cap Y$, then the diagram*

$$\begin{array}{ccc} Y \cup X & \longleftarrow & X \\ \uparrow & & \uparrow {\scriptstyle \mathbf{c}^{(k)}_X} \\ Y & \underset{\mathbf{d}^{(k)}_Y}{\longleftarrow} & \mathbf{c}^{(k)}(X) \end{array}$$

of inclusions in **pOpeCard** *is a pushout. (Here, $\cap$ and $\cup$ are levelwise set intersection and union, respectively.)*

2. *If $\mathbf{c}^{(k)}(X)$ and $\mathbf{d}^{(k)}(Y)$ are isomorphic, then the pushout $X \oplus_k Y$ in* **pOpeCard** *of X and Y over $\mathbf{c}^{(k)}(X)$ exists.*

3. *If there exists a positive opetopic cardinal T such that $X, Y \in S^*$ and if $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$, then $\mathbf{c}^{(k)}(X) = X \cap Y$.* †

Proof. Ad 1. Assume $\mathbf{c}^{(k)}(X) = X \cap Y \subseteq \mathbf{d}^{(k)}Y$. The fact that $Y \cup X$ is a pushout in **pHg** is obvious. Thus the only thing we need to verify is that $Y \cup X$ is a positive opetopic cardinal.

First we write in detail the condition $\mathbf{c}^{(k)}(X) = X \cap Y \subseteq \mathbf{d}^{(k)}Y$:

1. $X_l \cap Y_l = \emptyset$, for $l > k$,
2. $X_k - \delta(X_{k+1}) \subseteq Y_k - \gamma(Y_{k+1})$,
3. $X_{k-1} - \iota(X_{k+1}) \subseteq Y_{k-1}$,
4. $X_l \subseteq Y_l$, for $l < k-1$.

Now we describe the orders $<^+$ in $Y \cup X$:

$$<^{(Y\cup X)_l,+} = \begin{cases} <^{X_l,+} + <^{Y_l,+} & \text{for } l > k \\ <^{X_l,+} +_{(X_k-\delta(X_{k+1}))} <^{Y_l,+} & \text{for } l = k \\ <^{X_l,+} +_{(X_{k-1}-\iota(X_{k+1}))} <^{Y_l,+} & \text{for } l = k-1 \\ <^{Y_l,+} & \text{for } l <= k-1. \end{cases}$$

We shall comment on these formulas. For $l > k$, the formulas say that the order $<^+$ in $(Y \cup X)_l$ is the disjoint sum of the orders in X_l and Y_l. This is obvious.

For $l < k-1$, the order $<^+$ in $(X \cap Y)_l$ is just the order $<^{Y_l,+}$. The only case that requires an explanation is $l = k-2$. So suppose that $a, b \in Y_{k-2}$ and $a <^{(Y\cup X)_{k-2},+} b$. So we have an upper path

$$a, \alpha_1, \ldots, \alpha_m, b$$

such that $\alpha_i \in (Y \cup X)_{k-1} = \iota(X_{k+1}) \cup Y_{k-1}$. By Lemma 5.4, we can assume that if $\alpha_i \in X_{k-1}$, then $\alpha_i \notin \gamma(X_k)$. But then $\alpha_i \notin \iota(X_{k+1})$. So in fact $\alpha_i \in Y_{k-1}$, as required.

The most involved are the formulas for $<^{(X\cap Y)_l,+}$, for $l = k$ and $l = k-1$. In both cases the comparison in $Y \cup X$ involves orders both from X and Y. In the former case, for $a, b \in (Y \cup X)_k$, we have

$$a <^{(Y\cup X)_k,+} b \text{ iff}$$

$$\begin{cases} \text{either } a, b \in Y_k & \text{and } a <^{Y_k,+} b \\ \text{or } a, b \in X_k & \text{and } a <^{X_k,+} b \\ \text{or } a \in \delta(X_{k+1}),\ b \in Y_k & \text{and } \exists_{a' \in X_k - \delta(X_{k+1})} a <^{X_k,+} a' \text{ and } a' \leq^{Y_k,+} b. \end{cases}$$

† Point 3. was not present in the original submission. It explains why the objects of the ω-category S^* defined at the end of this section can be taken to be opetopic cardinals, while the objects of the terminal polygraph described in Section 11 have to be opetopic cardinals up-to-iso. As a hint, point 3. is an easy consequence of the following properties, that hold for all opetopic cardinals S: (i) for every $x \in S_k \setminus \mathbf{d}^{(k)}S$, there exists $y \in (\mathbf{d}^{(k)}S)_k$ such that $y <^+ x$; (ii) for every of $x \in S_k \setminus \mathbf{c}^{(k)}S$, there exists $y \in (\mathbf{c}^{(k)}S)_k$ such that $x <^+ y$; (iii) for every distinct cells x_1, x_2 of $(\mathbf{d}^{(k)}S)_k$, we have that x_1 and x_2 are incomparable for $<^{S,+}$; (iv) $S_{<k} \subseteq \mathbf{d}^{(k)}S$; (v) for every $x \in S_{>k}$ and every $y \in (\mathbf{c}^{(k)}x)_k$, there exists a cell $z \in (\mathbf{d}^{(k)}x)_k$ such that $z <^+ x$; (vi) if $S \in T^*$ for some T, and if $x, y \in S$ are such that $x <^{T,+} y$, then $x <^{S,+} y$.

The orders $<^{X_k,+}$ and $\leq^{Y_k,+}$ are glued together along $X_k - \delta(X_{k+1})$ which is the set of $<^{X_k,+}$-maximal elements in X_k and at the same time it is contained in the set of $<^{Y_k,+}$-minimal elements $Y_k - \gamma(Y_{k+1})$. This is obvious when we realize that $\delta(X_{k+1}) \cap \gamma(Y_{k+1}) = \emptyset$.

In the latter case, for $x, y \in (Y \cup X)_{k-1}$, we have

$$x <^{(Y\cup X)_{k-1},+} y \text{ iff}$$

$$\begin{cases} \text{either } x, y \in X_{k-1} & \text{and } x <^{X_{k-1},+} y \\ \text{or } x, y \in Y_{k-1} & \text{and } x <^{Y_{k-1},+} y \\ \text{or } x \in \iota(X_{k+1}),\ y \in Y_k & \text{and } \exists_{x' \in X_{k-1}-\iota(X_{k+1})} x <^{X_k,+} x' \text{ and } x' \leq^{Y_k,+} y \\ \text{or } x \in Y_k,\ y \in \iota(X_{k+1}) & \text{and } \exists_{x' \in X_{k-1}-\iota(X_{k+1})} x <^{Y_k,+} x' \text{ and } x' \leq^{X_k,+} y. \end{cases}$$

The order $<^{X_{k-1},+}$ is 'plugged into' the order $\leq^{Y_{k-1},+}$, along the set $X_k - \iota(X_{k+1})$.

To show that these formulas hold true, we argue by cases. Assume that $x, y \in (Y \cup X)_{k-1}$ and that $x <^{(Y\cup X)_{k-1},+} y$, i.e., there is an upper path

$$x, a_1, \ldots, a_m, y$$

with $a_i \in (Y \cup X)_k$, for $i = 1, \ldots, m$.

First suppose $x, y \in X_{k-1}$ and $\{a_i\}_i \not\subseteq X_k$. Let $a_{i_0}, a_{i_0+1}, \ldots, a_{i_1}$ be a maximal subsequence of consecutive elements of the path $a_1, \ldots, a_m$ such that $\{a_i\}_{i_0 \leq i \leq i_1} \subseteq Y_k$. Thus it is an upper path in Y_k from $\bar{x}$ to $\bar{y} = \gamma(a_{i_1})$, where

$$\bar{x} = \begin{cases} x & \text{if } i_0 = 1 \\ \gamma(a_{i_0 - 1}) & \text{otherwise.} \end{cases}$$

Note that it follows from the maximality of the path $a_{i_0}, \ldots, a_{i_1}$ that $\bar{x}, \bar{y} \in X_{k-1} - \iota(X_{k+1})$. As we have $\bar{x} <^{Y_{k-1},+} \bar{y}$ from Corollary 5.11, we have $\bar{x} \not\bowtie^{Y_l,-} \bar{y}$, for all $l < k-1$. Clearly $\bowtie^{X_l,-} \subseteq \bowtie^{Y_l,-}$. Thus $\bar{x} \not\bowtie^{X_l,-} \bar{y}$, for all $l < k-1$, as well. But then again by Corollary 5.11 we have that $\bar{x} \bowtie^{X_{k-1},+} \bar{y}$. If we were to have $\bar{y} <^{X_{k-1},+} \bar{x}$, then, as $\bar{x}, \bar{y} \in X_{k-1} - \iota(X_{k+1})$, we would have $\bar{y} <^{Y_{k-1},+} \bar{x}$. But this would contradict the strictness of $<^{Y_{k-1},+}$. So we must have $\bar{x} <^{X_{k-1},+} \bar{y}$. In this way we can replace the upper path $a_1, \ldots, a_m$ in $(Y \cup X)_k$ from x to y by an upper path from x to y in X_k.

Next, suppose $x, y \in Y_{k-1}$ and $\{a_i\}_i \not\subseteq Y_k$. Let $a_{i_0}, a_{i_0+1}, \ldots, a_{i_1}$ be a maximal subsequence of consecutive elements of the path $a_1, \ldots, a_m$ such that $\{a_i\}_{i_0 \leq i \leq i_1} \subseteq X_k$. Thus it is an upper path in X_k from $\bar{x}$ to $\bar{y} = \gamma(a_{i_1})$, where

$$\bar{x} = \begin{cases} x & \text{if } i_0 = 1 \\ \gamma(a_{i_0 - 1}) & \text{otherwise.} \end{cases}$$

Note that $\bar{x}, \bar{y} \in X_{k-1} - \iota(X_{k+1}) \subseteq Y_{k-1}$ follows from the maximality of the sequence $a_{i_0}, \ldots, a_{i_1}$. Thus by Lemma 5.8 there is an upper path from $\bar{x}$ to $\bar{y}$ in $X_{k-1} - \delta(X_k) \subseteq Y_{k-1}$. In this way we can replace the upper path $a_1, \ldots, a_m$ in $(Y \cup X)_k$ from x to y by an upper path from x to y in Y_k.

Thus we have justified the first two cases of the above formula. The following two cases are easy consequences of these two. This ends the description of the orders in $Y \cup X$.

From these descriptions follows immediately that $<^{(Y\cup X),+}$ is strict, for all l. It remains to show the pencil linearity. Both γ- and δ-linearity of l-cells, for $l < k-1$ or $l > k$, are obvious.

To see the γ-linearity of k-cells, suppose that we have $a \in X_k$ and $b \in Y_k$ such that $\gamma(a) = \gamma(b)$. Let $\bar{a} \in X_k$ be the $<^{X_k,+}$-maximal k-cells such that $\gamma(a) = \gamma(\bar{a})$.

Then $\bar{a} \in \mathbf{c}^{(k)}(X)_k \subseteq \mathbf{d}^{(k)}(Y)_k$. So $\bar{a} \in Y_k$ is a $<^{Y_k,+}$-minimal k-cell such that $\gamma(\bar{a}) = \gamma(b)$. Thus
$$a \leq^{X_k,+} \bar{a} \leq^{Y_k,+} b.$$
Thus the γ-linearity of k-cells holds. The proof of δ-linearity of k-cells is similar.

Finally, we need to establish the γ- and δ-linearity of $(k-1)$-cells in $Y \cup X$.

In order to prove the γ-linearity, let $x \in \iota(X_{k+1})$ and $y \in Y_{k-1}$ such that $\gamma(x) = \gamma(y)$. We need to show that $x \bowtie^{(Y\cup X)_{k-1},+} y$.

Let $\alpha_0 \in X_{k+1}$ such that $x \in \iota(\alpha_0)$, $a \in \delta(\alpha_0)$ such that $x = \gamma(a)$ and let $\alpha_0, \ldots, \alpha_l$ be a lower path in X_{k+1} such that $\gamma(\alpha_l) \in Y_k$. Since $x \in \iota(\alpha_0)$, then $x \in \gamma\delta(\alpha_0)$ and, by Lemma 4.2
$$\gamma(x) \in \gamma\gamma\delta(\alpha_0) \subseteq \iota\gamma(\alpha_0) \cup \gamma\gamma\gamma(\alpha_0).$$
As $\gamma(\alpha_0) \leq^{+} \gamma(\alpha_l)$, by Lemma 5.3.3, we have $\gamma(x) \in \iota\gamma(\alpha_l) \cup \gamma\gamma\gamma(\alpha_l)$. Thus we have two cases:

1. $\gamma(x) \in \iota\gamma(\alpha_l)$,
2. $\gamma(x) = \gamma\gamma\gamma(\alpha_l)$.

Case 1: $\gamma(x) \in \iota\gamma(\alpha_l)$. By Lemma 5.2.2, there is a unique $z \in \delta\gamma(\alpha_l)$ such that $\gamma(z) = \gamma(x)$ and $z <^{+} x$. As $\gamma(\alpha_l) \in Y_k$, so $z \in Y_{k-1}$. If $y <^{Y_{k-1},+} z$, then indeed $y <^{(Y\cup X)_{k-1},+} z$, as required. By γ-linearity in Y_{k-1}, it is enough to show that it is impossible to have $z <^{Y_{k-1},+} y$.

Suppose on the contrary that there is an upper path $z, b_0, \ldots, b_r, y$ in Y. Since $\gamma(\alpha_l)$ is $<^{+}$-minimal in Y (as $\alpha_l \in X$) and $z \in \delta\gamma(\alpha_l) \cap \delta(b_0)$, by δ-linearity in Y_k we have $\gamma(\alpha_l) <^{+} b_0$. By Lemma 5.3.2, we have
$$\gamma(x) \in \iota\gamma(\alpha_l) \subseteq \iota(b_0) \subseteq \iota(b_r).$$
But $\gamma(b_r) = y$ so $\gamma\gamma(b_r) = \gamma(y) = \gamma(x)$. In particular, $\gamma(x) \notin \iota(b_r)$ and we get a contradiction.

Case 2: $\gamma(x) = \gamma\gamma\gamma(\alpha_l)$. By Lemma 5.2.2 there is $z \in \delta\gamma(\alpha_l)$ such that $\gamma(x) = \gamma(z)(= \gamma\gamma\gamma(\alpha_l))$, so that we have
$$z <^{X_{k-1},+} x <^{X_{k-1},+} \gamma\gamma(\alpha_l).$$
As $\gamma(\alpha_l) \in Y_k$ and is $<^{+}$-minimal in Y_k, by Proposition 5.16, there is no face $y' \in Y_{k-1}$ so that
$$z <^{Y_{k-1},+} y' <^{Y_{k-1},+} \gamma\gamma(\alpha_l).$$
So if $y \in Y_{k-1}$ and $\gamma(y) = \gamma(x)$, then either
$$y \leq^{Y_{k-1},+} z <^{X_{k-1},+} x \quad \text{or} \quad x <^{X_{k-1},+} \gamma\gamma(\alpha_l) \leq^{X_{k-1},+} y.$$
In either case $x \bowtie^{(Y\cup X)_{k-1},+} y$, as required. This ends the proof of γ-linearity of $(k-1)$-faces in $(Y \cup X)$.

Finally, we prove the δ-linearity of $(k-1)$-faces in $Y \cup X$. Let $x \in \iota(X_{k+1})$ and $y \in Y_{k-1}$, $t \in Y_{k-2}$ such that $t \in \delta(x) \cap \delta(y)$. We need to show that $x \bowtie^{(Y\cup X)_{k-1},+} y$.

Let $\alpha_0 \in X_{k+1}$ such that $x \in \iota(\alpha_0)$, $a \in \delta(\alpha_0)$ such that $x = \gamma(a)$, and let $\alpha_0, \ldots, \alpha_l$ be a lower path in X_{k+1} such that $\gamma(\alpha_l) \in Y_k$. As $x \in \iota(\alpha_0)$, using Lemma 4.2 we have
$$t \in \delta(x) \subseteq \delta\gamma\delta(\alpha_0) \subseteq \delta\gamma\gamma(\alpha_0) \cup \iota\gamma(\alpha_0).$$
As $\gamma(\alpha_0) <^{+} \gamma(\alpha_l)$, by Lemma 5.3.4, we have two cases:

1. $t \in \iota\gamma(\alpha_l)$,
2. $t \in \delta\gamma\gamma(\alpha_l)$.

Case 1: $t \in \iota\gamma(\alpha_l)$. By Lemma 5.2.3, there is a unique $z \in \delta\gamma(\alpha_l)$ such that $t \in \delta(z)$ and $z <^+ x$. As $\gamma(\alpha_l) \in Y_k$, so $z \in Y_{k-1}$. If $y <^{Y_{k-1},+} z$, then indeed $y <^{(Y\cup X)_{k-1},+} z$, as required. By δ-linearity in Y_k, it is enough to show that it is impossible to have $z <^{Y_{k-1},+} y$.

Suppose on the contrary that there is an upper path in Y

$$z, b_0, \ldots, b_r, y.$$

Since $\gamma(\alpha_l)$ is $<^+$-minimal in Y_k and $z \in \delta\gamma(\alpha_l) \cap \delta(b_0)$, by δ-linearity of k-faces in Y we have $\gamma(\alpha_l) <^+ b_0$. By Lemma 5.3.2, we have

$$t \in \iota\gamma(\alpha_l) \subseteq \iota(b_0) \subseteq \ldots \subseteq \iota(b_r).$$

But $\gamma(b_r) = y$, so $t \in \delta(y) \subseteq \delta\gamma(b_r)$. In particular, $t \notin \iota(b_r)$ and we get a contradiction.

Case 2: $t \in \delta\gamma\gamma(\alpha_l)$. By Lemma 5.2.3 there is $z \in \delta\gamma\gamma(\alpha_l)$ such that $t \in \delta(z)$ and we have

$$z <^{X_{k-1},+} x <^{X_{k-1},+} \gamma\gamma(\alpha_l).$$

As $\gamma(\alpha_l) \in Y_k$, and it is $<^+$-minimal face in Y_k, by Lemma 5.16, there is no face $y' \in Y_{k-1}$ such that

$$z <^{Y_{k-1},+} y' <^{Y_{k-1},+} \gamma\gamma(\alpha_l).$$

So if $y \in Y_{k-1}$ and $t \in \delta(y)$, then either

$$y \leq^{Y_{k-1},+} z <^{X_{k-1},+} x \quad \text{or} \quad x <^{X_{k-1},+} \gamma\gamma(\alpha_l) \leq^{X_{k-1},+} y.$$

In either case $x \bowtie^{(Y\cup X)_{k-1},+} y$, as required. This ends the proof of δ-linearity of $(k-1)$-faces in $(Y \cup X)$ and the whole proof that $Y \cup X$ is a positive opetopic cardinal.

Ad 2. We note that we can rename the cells of $Y \setminus (\mathbf{d}^{(k)}Y)$ to ensure disjointness with X and rename the cells of $(\mathbf{d}^{(k)}Y)$ to turn the codomain-domain isomorphism into an equality, and then apply point 1. $\Box$

Let X and Y be positive opetopic cardinals such that $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$. Then the pushout just described

$$\begin{array}{ccc} Y \oplus_k X & \longleftarrow & X \\ \uparrow & & \uparrow \mathbf{c}^{(k)}_X \\ Y & \underset{\mathbf{d}^{(k)}_Y}{\longleftarrow} & \mathbf{c}^{(k)}(X) \end{array}$$

is called *special pushout* in **pOpeCard** (or *special pullback* in $\mathbf{pOpeCard}^{op}$).

Now we shall describe an ω-category T^* generated by the positive opetopic cardinal T. The set of m-cells of T^* is T^*_m, i.e., the set of all the positive opetopic cardinals contained in T of dimension at most m, for $m \in \omega$. The *k-th domain* and *k-th codomain* operations in T^* are the operations

$$\mathbf{d}^{(k)}, \mathbf{c}^{(k)} : T^*_m \longrightarrow T^*_k$$

defined above, with $m \geq k$. The *identity* operations

$$\mathbf{i}^{(m)} : T^*_k \longrightarrow T^*_m$$

are inclusions, and the composition map

$$\mathbf{m}_{m,k,m} : T^*_m \times_{T^*_k} T^*_m \longrightarrow T^*_m,$$

where $k < m$, is the sum, i.e., if X, Y are positive opetopic cardinals contained in T of dimension at most m such that $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$, then

$$\mathbf{m}_{m,k,m}(X, Y) = X \oplus_k Y = X \cup Y.$$

Corollary 6.3 *Let T be a weak positive opetopic cardinal. Then T^* is an ω-category. In fact, we have a functor*

$$(-)^* : \mathbf{wpOpeCard} \longrightarrow \omega Cat.$$

Proof. The fact that the operations on T^* are well defined follows from Lemmas 6.1 and 6.2. The satisfaction of the laws of ω-categories is a simple matter of rearrangements of unions.

If $f : S \to T$ is a morphism of positive opetopic cardinals, $X \in S^*$, then the image $f(X) \in T^*$ is isomorphic to X. Then again using Lemmas 6.1 and 6.2, the association $X \mapsto f(X)$ is easily seen to be an ω-functor. $\Box$

7 Normal positive opetopic cardinals

Let S be a normal positive opetopic cardinal of dimension k, i.e., S is $(k-1)$-principal. By $\mathbf{p}^S_l$ we denote the unique element of the set $S_l - \delta(S_{l+1})$, for $l < k$. Moreover, as we shall show below, $\mathbf{p}^S_{k-1} \in \gamma(S_k)$ and hence the set $\{x \in S_k : \gamma(x) = \mathbf{p}^S_{k-1}\}$ is not empty. We denote by $\mathbf{p}^S_k$ the $<^+$-largest element of this set. We shall omit the superscript S if it does not lead to a confusion.

Lemma 7.1 *Let S be a $(k-1)$-principal opetope of dimension at least k, $k > 0$. Then*

1. $S_l = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) = \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$*, for* $l < k$.
2. $\delta(S_{l+1}) = \delta^{(l)}(S_k)$*, for* $l < k$.
3. $\mathbf{p}_k$ *is the* $<^-$*-largest element in* $S_k - \delta(S_{k+1})$.
4. $\gamma(\mathbf{p}_l) = \mathbf{p}_{l-1}$*, for* $0 < l \leq k$.
5. $\delta(\mathbf{p}_l) = \delta(S_l) - \gamma(S_l)$*, for* $0 < l < k$.
6. $S_l = \delta^{(l)}(\mathbf{p}_{k-1}) \cup \gamma^{(l)}(\mathbf{p}_{k-1})$*, for* $l < k-2$.

Proof. Ad 1. If H is a hypergraph of dimension greater than l and $\gamma(H_{l+1}) \subseteq \delta(H_{l+1})$, then there is an infinite lower path in H_{l+1}, i.e., $<^{H_l,+}$ is not strict. Thus, if S is a positive opetopic cardinal of dimension greater than l, we have $\delta(S_{l+1}) \subsetneq S_l$. A positive opetopic cardinal is normal iff this difference

$$S_l - \delta(S_{l+1})$$

is a singleton, for $l < k$. Thus, by the above, we must have

$$S_l = \delta(S_{l+1}) \cup \gamma(S_{l+1}). \tag{1}$$

We shall show the first equation of the statement 1. by downward induction on l. Suppose that we have $S_{l+1} = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k)$ (for $l = k-2$, it is true by the above). Then

$$S_l = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) =$$
$$= \delta(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) \cup \gamma(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) =$$
$$= \delta\delta^{(l+1)}(S_k) \cup \delta\gamma^{(l+1)}(S_k) \cup \gamma\delta^{(l+1)}(S_k) \cup \gamma\gamma^{(l+1)}(S_k) =$$
$$= \delta^{(l)}(S_k) \cup \delta\gamma^{(l)}(S_k) \cup \gamma\delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) =$$
$$= \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k),$$

where the last equation follows from Corollary 4.3.

The second equation of 1. is obvious, for $l = k-1$. So assume $l < k-1$. We have

$$\{\mathbf{p}_l\} = S_l - \delta(S_{l+1}) =$$
$$= S_l - \delta(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) =$$
$$= S_l - (\delta^{(l)}(S_k) \cup \delta\gamma^{(l+1)}(S_k)) =$$
$$S_l - \delta^{(l)}(S_k).$$

Thus

$$S_l = \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$$

as required.

Ad 2. Let $l < k$. Then using 1. we have

$$\delta^{(l)}(S_k) \subseteq \delta(S_{l+1}) \subsetneq \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}.$$

Hence

$$\delta^{(l)}(S_k) = \delta(S_{l+1}).$$

Ad 3. First we shall show that $\mathbf{p}_k \in S_k - \delta(S_{k+1})$. Suppose on the contrary that there is $\alpha \in S_{k+1}$ such that $\mathbf{p}_k \in \delta(\alpha)$. Then $\gamma(\mathbf{p}_k) \in \gamma\delta(\alpha) = \gamma\gamma(\alpha) \cup \iota(\alpha)$. If $\gamma(\mathbf{p}_k) = \gamma\gamma(\alpha)$, then $\mathbf{p}_k <^+ \gamma(\alpha)$, contradicting the choice of $\mathbf{p}_k$. If $\gamma(\mathbf{p}_k) = \iota(\alpha)$, then there is $a \in \delta(\alpha)$ such that $\gamma(\mathbf{p}_k) \in \delta(a)$. But this means that $\mathbf{p}_{k-1} = \gamma(\mathbf{p}_k) \in \delta(S_k)$, contradicting the choice of $\mathbf{p}_{k-1} \in S_{k-1} - \delta(S_k)$. This shows that $\mathbf{p}_k \in S_k - \delta(S_{k+1})$.

We need to prove that any maximal lower $(S_k - \delta(S_{k+1}))$-path ends at $\mathbf{p}_k$. By strictness, it is enough to show that if $x \in S_k - \delta(S_{k+1})$ and $x \neq \mathbf{p}_k$, then there is $x' \in S_k - \delta(S_{k+1})$ such that $\gamma(x) \in \delta(x')$. So fix $x \in S_k - \delta(S_{k+1})$. If we were to have $\gamma(x) \in \iota(\beta)$, for some $\beta \in S_{k+1}$, then by Lemma 5.5 we would have $x <^+ \gamma(\beta)$, and in particular $x \in \delta(S_{k+1})$, contrary to the assumption. Therefore $\gamma(x) \in S_{k-1} - \iota(S_{k+1})$. As $x, \mathbf{p}_k \in S_k - \delta(S_{k+1})$, by γ-linearity we have $\gamma(x) \neq \gamma(\mathbf{p}_k) = \mathbf{p}_{k-1}$. Hence by 1. the set

$$\Delta_{\gamma(x)} = \{y \in S_k : \gamma(x) \in \delta(y)\}$$

is not empty. Let x' be the $<^+$-largest element of this set. It remains to show that $x' \notin \delta(S_{k+1})$. Suppose on the contrary that there is $\alpha \in S_{k+1}$ such that $x' \in \delta(\alpha)$. As $\gamma(x) \notin \iota(S_{k+1})$ and $\gamma(x) \in \delta(x')$, so $\gamma(x) \notin \iota(\alpha)$ and $\gamma(x) \neq \gamma\gamma(\alpha)$. Thus $\gamma(x) \in \delta\gamma(\alpha)$. But this means that $x' <^+ \gamma(\alpha)$ and $\gamma(\alpha) \in \Delta_{\gamma(x)}$. This contradicts the choice of x'. This ends the proof of 3.

Ad 4. $\gamma(\mathbf{p}_k) = \mathbf{p}_{k-1}$ by definition. Fix $0 < l < k$. As $S_l = \delta(S_{l+1}) \cup \{\mathbf{p}_l\}$, $\mathbf{p}_l$ is $<^+$-greatest element in S_l. Assume $\gamma(\mathbf{p}_l) \neq \mathbf{p}_{l-1}$. Thus $\gamma(\mathbf{p}_l) <^+ \mathbf{p}_{l-1}$. Let $x \in S_l$. Then $x \leq \mathbf{p}_l$ and, by Lemma 5.9, $\gamma(x) \leq^+ \gamma(\mathbf{p}_l) <^+ \mathbf{p}_{l-1}$. Thus $\mathbf{p}_{l-1} \notin \gamma(S_l)$. So

$\gamma(S_l) \subseteq \delta(S_l)$. But this is impossible in a positive opetopic cardinal, as we noticed in the proof of 1. This ends the proof of 4.

Ad 5. Fix $l < k$. First we shall show that

$$\delta(\mathbf{p}_l) \cap \gamma(S_l) = \emptyset. \tag{2}$$

Let $z \in \gamma(S_l)$, i.e., there is $a \in S_l$ such that $\gamma(a) = z$. By 1. $a \leq^+ \mathbf{p}_l$. By Lemma 5.7, there are $x \in \delta(\mathbf{p}_l)$ and $y \in \delta(a)$ such that $x \leq^+ y$. Hence $x <^+ \gamma(a) = z$. By Proposition 5.1, since $x \in \delta(\mathbf{p}_l)$, it follows that $z \notin \delta(\mathbf{p}_l)$. This shows (2).

By Lemma 5.19, we have

$$\delta(S_l) = \delta(S_l - \delta(S_{l+1})) \cup \iota(S_{l+1}). \tag{3}$$

Since $\delta(\mathbf{p}_l) = \delta(S_l - \delta(S_{l+1}))$ and $\iota(S_{l+1}) \subseteq \gamma(S_l)$, we have by (2)

$$\delta(S_l - \delta(S_{l+1})) \cap \iota(S_{l+1}) = \emptyset. \tag{4}$$

Next we shall show

$$\iota(S_{l+1}) = \gamma(S_l) \cap \delta(S_l). \tag{5}$$

The inclusion $\subseteq$ is obvious. Let $x \in \gamma(S_l) \cap \delta(S_l)$. Hence there are $a, b \in S_l$ such that $\gamma(a) = x \in \delta(b)$. We can assume that a is $<^+$-maximal with this property. As $a <^- b$, neither a nor b is equal to the $<^+$-greatest element $\mathbf{p}_l \in S_l$. Therefore there is $\alpha \in S_{l+1}$ such that $a \in \delta(\alpha)$. If we were to have $x = \gamma(a) = \gamma\gamma(\alpha)$, then $\gamma(\alpha)$ would be a $<^+$-greater element than a with $\gamma(\gamma(\alpha)) = x$. So $\gamma(a) \neq \gamma\gamma(\alpha)$. Clearly, $x \in \gamma\delta(\alpha)$. By globularity, $x \in \delta\delta(\alpha)$ as well. Thus $x \in \iota(\alpha)$, and (5) is shown.

Using (2), (3), (4), and (5) we have

$$\begin{aligned} \delta(\mathbf{p}_l) &= \delta(S_l - \delta(S_{l+1})) = \\ &= \delta(S_l) - \iota(S_{l+1}) = \\ &= \delta(S_l) - (\gamma(S_l) \cap \delta(S_l)) = \\ &= \delta(S_l) - \gamma(S_l) \end{aligned}$$

as required.

Ad 6. By 1. and 2. it is enough to show

$$\delta^{(l)}(S_{k-1}) = \delta^{(l)}(\mathbf{p}_{k-1}),$$

for $l < k-2$. The inclusion $\supseteq$ is obvious. Pick $x \in S_{k-1}$. We have an upper path $x, a_1, \ldots, a_r, \mathbf{p}_{k-1}$. By Corollary 4.3, as $\gamma(a_i) \in \delta(a_{i+1})$, we have

$$\delta^{(l)}(a_i) = \delta^{(l)}\gamma(a_i) \subseteq \delta^{(l)}(\delta(a_{i+1})) = \delta^{(l)}(a_{i+1})$$

for $i = 0, \ldots, r-1$. Then, by transitivity of $\subseteq$ and again Corollary 4.3, we get

$$\delta^{(l)}(x) \subseteq \delta^{(l)}(a_1) \subseteq \delta^{(l)}(a_r) \subseteq \delta^{(l)}(\gamma(a_r)) = \delta^{(l)}(\mathbf{p}_{k-1}).$$

This ends the proof of the inclusion $\subseteq$ and the proof of 6. □

Lemma 7.2 *Let S be a positive opetopic cardinal of dimension at least k. Then*

1. *S is $(k-1)$-principal iff $\mathbf{d}^{(k)}(S)$ is normal iff $\mathbf{c}^{(k-1)}(S)$ is principal;*
2. *if S is normal, so is $\mathbf{d}(S)$;*
3. *if S is normal, $\mathbf{c}(S)$ is principal.*

Proof. The whole lemma is an easy consequence of Lemma 5.19. We shall show 1., leaving 2. and 3. for the reader. First note that all three conditions in 1. imply that $|S_l - \delta(S_{l+1}| = 1$, for $l < k-2$. In addition, these conditions say:

1. S is $(k-1)$-principal iff $|S_l - \delta(S_{l+1})| = 1$, for $l = k-2, k-1$.
2. $\mathbf{d}^{(k)}S$ is normal iff
   (a) $|S_{k-1} - \delta(S_k - \gamma(S_{k+1}))| = 1$, and
   (b) $|S_{k-2} - \delta(S_{k-1})| = 1$.
3. $\mathbf{c}^{(k-1)}(S)$ is principal iff
   (a) $|(S_{k-1} - \iota(S_{k+1})) - \delta(S_k - \delta(S_{k+1}))| = 1$, and
   (b) $|S_{k-2} - \delta(S_{k-1} - \iota(S_{k+1}))| = 1$.

So the equivalence of these conditions follows directly from Lemma 5.19. □

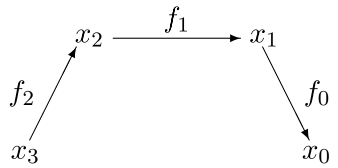
Let N be a normal positive opetopic cardinal of dimension n. We define a $(n+1)$-hypergraph $N^\bullet$ that contains two additional faces: $\mathbf{p}^{N^\bullet}_{n+1}$ of dimension $n+1$, and $\mathbf{p}^{N^\bullet}_{n}$ of dimension n. We shall drop superscripts if it does not lead to confusions. We also put

$$\delta(\mathbf{p}_{n+1}) = N_n \quad \gamma(\mathbf{p}_{n+1}) = \mathbf{p}_n$$

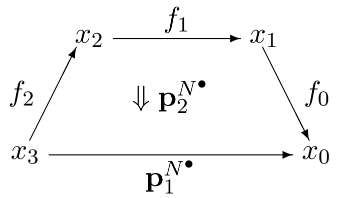
$$\delta(\mathbf{p}_n) = \delta(N_n) - \gamma(N_n) \quad \gamma(\mathbf{p}_n) = \mathbf{p}_{n-1} (= \gamma(N_n) - \delta(N_n)).$$

As N is normal, $\gamma(N_n) - \delta(N_n)$ has one element, so $\gamma(\mathbf{p}_n)$ is well defined. This determines $N^\bullet$ uniquely.† $N^\bullet$ is called *a simple extension of* N.

Example. For a normal positive opetopic cardinal N like this



the hypergraph $N^\bullet$ looks like this



Proposition 7.3 *Let N be a normal positive opetopic cardinal of dimension n. Then*

1. *$N^\bullet$ is a positive opetope of dimension $n+1$.*
2. *We have $\mathbf{d}(N^\bullet) \cong N$, $\mathbf{c}(N^\bullet) \cong (\mathbf{d}N)^\bullet$.*
3. *If N is a principal, then $N \cong (\mathbf{d}N)^\bullet$.*
4. *If T is a positive opetopic cardinal contained in $N^\bullet$, then either $T = N^\bullet$ or $T = \mathbf{c}(N^\bullet)$ or $T \subseteq N$.*

† The uniqueness is meant here with respect to the properties of being principal and having N as domain. Principality is characterized by the addition of the unique top cell p_{n+1}. Having N as domain is taken care of by the equality $\delta(p_{n+1}) = N_n$. Then all the rest is imposed. The cell p_{n+1} must have a (fresh) codomain p_n, which is itself the top cell of $(dN)^\bullet$, as specified by the rest of the definition of $N^\bullet$.

Proof. Ad 1. We shall check globularity of the new added cells. The other conditions are simple.

For $\mathbf{p}_{n+1}$, we have:

$$\gamma\gamma(\mathbf{p}_{n+1}) = \gamma(\mathbf{p}_n) =$$
$$= \gamma(N_n) - \delta(N_n) = \gamma\delta(\mathbf{p}_{n+1}) - \delta\delta(\mathbf{p}_{n+1})$$

and

$$\delta\gamma(\mathbf{p}_{n+1}) = \delta(\mathbf{p}_n) =$$
$$= \delta(N_n) - \gamma(N_n) = \delta\delta(\mathbf{p}_{n+1}) - \gamma\delta(\mathbf{p}_{n+1}).$$

So globularity holds for $\mathbf{p}_{n+1}$.

For $\mathbf{p}_n$, using Lemmas 7.1, 5.19 and normality of N, we have:

$$\gamma\gamma(\mathbf{p}_n) = \gamma(\mathbf{p}_{n-1}) = \mathbf{p}_{n-2} =$$
$$= \gamma(N_{n-1}) - \delta(N_{n-1}) =$$
$$= \gamma(N_{n-1} - \gamma(N_n)) - \delta(N_{n-1} - \gamma(N_n)) =$$
$$= \gamma(\delta(N_n) - \gamma(N_n)) - \delta(\delta(N_n) - \gamma(N_n)) =$$
$$= \gamma\delta(\mathbf{p}_n) - \delta\delta(\mathbf{p}_n),$$

and similarly

$$\delta\gamma(\mathbf{p}_n) = \delta(\mathbf{p}_{n-1}) =$$
$$= \delta(N_n) - \gamma(N_n) =$$
$$= \delta(\delta(N_n) - \gamma(N_n)) - \gamma(\delta(N_n) - \gamma(N_n)) =$$
$$= \delta\delta(\mathbf{p}_n) - \gamma\delta(\mathbf{p}_n).$$

So globularity holds for $\mathbf{p}_n$, as well.

Ad 2. The first isomorphism is obvious.

The faces of $(N^\bullet)$, $\mathbf{c}(N^\bullet)$, $\mathbf{d}N$, and $(\mathbf{d}N)^\bullet$ are as in the tables

dim	$(N^\bullet)$	$\mathbf{c}(N^\bullet)$
$n+1$	$\{\mathbf{p}^{N^\bullet}_{n+1}\}$	$\emptyset$
n	$N_n \cup \{\mathbf{p}^{N^\bullet}_n\}$	$\{\mathbf{p}^{N^\bullet}_n\}$
$n-1$	N_{n-1}	$N_{n-1} - (\gamma(N_n) \cap \delta(N_n))$
$n-2$	N_{n-2}	N_{n-2}

and

dim	$\mathbf{d}N$	$(\mathbf{d}N)^\bullet$
$n+1$	$\emptyset$	$\emptyset$
n	$\emptyset$	$\{\mathbf{p}^{(\mathbf{d}N)^\bullet}_n\}$
$n-1$	$N_{n-1} - \gamma(N_n)$	$(N_{n-1} - \gamma(N_n)) \cup \{\mathbf{p}^{(\mathbf{d}N)^\bullet}_{n-1}\}$
$n-2$	N_{n-2}	N_{n-2}

We define the isomorphism $f : \mathbf{c}(N^\bullet) \longrightarrow (\mathbf{d}N)^\bullet$ as follows

$$f_n(\mathbf{p}^{N^\bullet}_{n+1}) = \mathbf{p}^{(\mathbf{d}N)^\bullet}_{n+1}$$

$$f_{n-1}(x) = \begin{cases} \mathbf{p}^{(\mathbf{d}N)^\bullet}_{n-1} & \text{if } x = \gamma(\mathbf{p}^{N^\bullet}_n), \\ x & \text{otherwise.} \end{cases}$$

and $f_l = 1_{N_l}$ for $l < n-1$. Clearly, all f_i's are bijective. The preservation of the domains and codomains is left for the reader.

3. is left as an exercise.

Ad 4. If $\mathbf{p}_{n+1} \in T_{n+1}$, then $T = N^\bullet$. If $\mathbf{p}_n \not\in T_n$, then $T \subseteq N$.

Suppose that $\mathbf{p}_{n+1} \not\in T_{n+1}$ but $\mathbf{p}_n \in T_n$. Since $N^\bullet = [\mathbf{p}_{n+1}]$, by Lemma 6.1 it is enough to show that $T = [\mathbf{p}_n]$. Clearly $[\mathbf{p}_n] \subseteq T$. As $[\mathbf{p}_n]_l = N_l$, for $l < n-1$, we have $[\mathbf{p}_n]_l = T_l$, for $l < n-1$, as well.

Fix $x \in N_n$. As $x \in \delta(\mathbf{p}_{n+1})$ and $\gamma(\mathbf{p}_{n+1}) = \mathbf{p}_n$, we have $x <^{N^\bullet,+} \mathbf{p}_n$. So by Corollary 5.11 $x \not\bowtie_l^{N^\bullet,-} \mathbf{p}_n$, for any $l \leq n$. Thus we cannot have $x \bowtie_l^{T,-} \mathbf{p}_n$, for any $l \leq n$ either. As T is a positive opetopic cardinal, again by Corollary 5.11, $x \not\in T$. Since x was an arbitrary element of N_n, we have $T_n = \{\mathbf{p}_n\} = [\mathbf{p}_n]_n$.

It remains to show that $T_{n-1} = [\mathbf{p}_n]_{n-1}$. Suppose that $x \in N_{n-1} - (\delta(\mathbf{p}_n) \cup \gamma(\mathbf{p}_n))$. Then $x <^{N,+} \gamma(\mathbf{p}_n)$ and hence $x \not\bowtie_l^{N^\bullet,-} \gamma(\mathbf{p}_n)$, for $l \leq n$. So x and $\gamma(\mathbf{p}_n)$ cannot be $<_l^{T,-}$-comparable, for $l \leq n$. Since, as we have shown, $N_n \cap T_n = \emptyset$, it follows that x and $\gamma(\mathbf{p}_n)$ cannot be $<^{T,+}$-comparable. So by Lemma 5.11, $x \not\in T_{n-1}$, i.e., $T_{n-1} = \delta(\mathbf{p}_n) \cup \gamma(\mathbf{p}_n) = [\mathbf{p}_n]_{n-1}$. $\Box$

8 Decomposition of positive opetopic cardinals

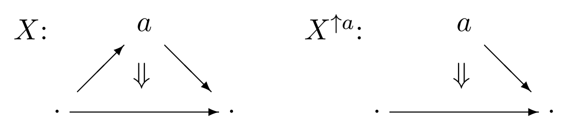
Let T be a positive opetopic cardinal, $X \subseteq T$ a subhypergraph of T, $k \in \omega$, $a \in (T_k - \iota(T_{k+2}))$. We define two subhypergraphs of T, $X^{\downarrow a}$ and $X^{\uparrow a}$, as follows:

$$X_l^{\downarrow a} = \begin{cases} \{\alpha \in X_l : \gamma^{(k)}(\alpha) \leq^+ a\} & \text{for } l > k \\ \{b \in X_k : b \leq^+ a \text{ or } \ b \not\in \gamma(X_{k+1})\} & \text{for } l = k \\ X_l & \text{for } l < k. \end{cases}$$

$$X_l^{\uparrow a} = \begin{cases} \{\alpha \in X_l : \gamma^{(k)}(\alpha) \not\leq^+ a\} & \text{for } l > k \\ \{b \in X_k : b \not<^+ a \text{ or } b \not\in \delta(X_{k+1})\} & \text{for } l = k \\ X_{k-1} - \iota(X_{k+1}^{\downarrow a}) & \text{for } l = k-1 \\ X_l & \text{for } l < k-1. \end{cases}$$

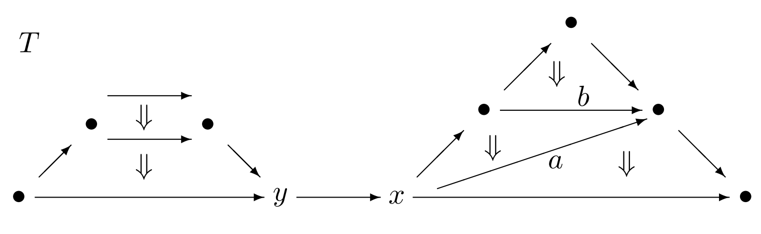
Intuitively, if X is a positive opetopic cardinal contained in T, $X^{\downarrow a}$ is the least positive opetopic cardinal contained in X that contains faces 'smaller or equal' a and can be k-pre-composed with the 'rest' to get X. $X^{\uparrow a}$ is this 'rest' or in other words it is the largest positive opetopic cardinal contained in X that can be k-post-composed with $X^{\downarrow a}$ to get X (or the largest positive opetopic cardinal contained in X that does not contain faces 'smaller' than a). Note that a does not need to be a face in X, in general.

Examples. If X is a hypergraph $a \in T$, then $X^{\downarrow a}$ is a hypergraph, as well. However, this is not the case with $X^{\uparrow a}$, if $a \in \iota(T)$, as we can see below:

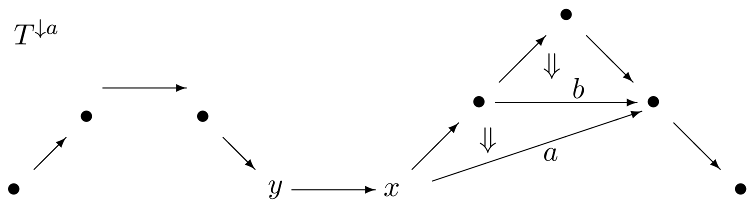


Here $X = T$. The faces in the domain of the 2-dimensional face are not in $X^{\uparrow a}$, i.e., $X^{\uparrow a}$ is not closed under δ.

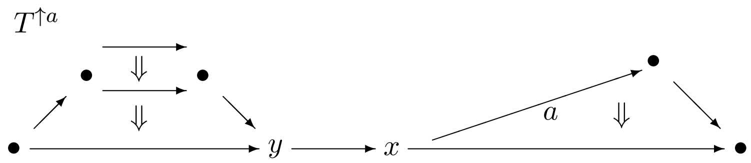
To see some real decompositions, let fix a positive opetopic cardinal T as follows:



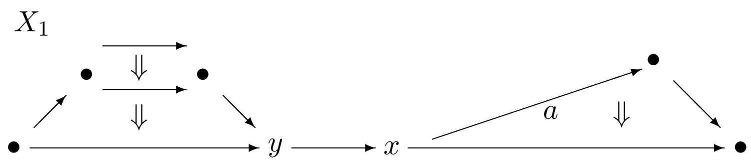
Clearly $x, y, a, b \in T - \iota(T)$. Then



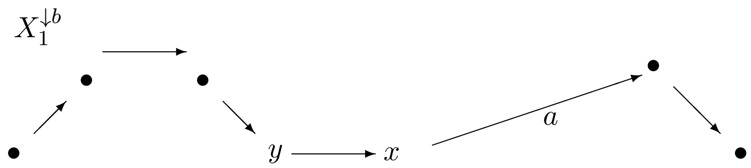
and



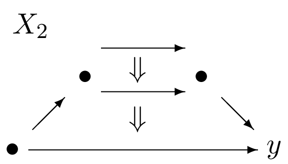
Moreover, with



we have $X_1^{\uparrow b} = X_1$ and



i.e., $X_1^{\downarrow b} = \mathbf{d}^{(1)}(X_1)$. For



we have $X_2^{\downarrow x} = X_2$ and $X_2^{\uparrow x} = \{y\}$.

Lemma 8.1 *Let T be a positive opetopic cardinal, $X \subseteq T$ a subhypergraph of T, $a \in (T - \iota(T))$, $a \in X_k$. Then*

1. *$X^{\downarrow a}$ and $X^{\uparrow a}$ are positive opetopic cardinals;*
2. *$\mathbf{c}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow a}) = X^{\downarrow a} \cap X^{\uparrow a}$;*
3. *$\mathbf{d}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X)$, $\mathbf{c}^{(k)}(X^{\uparrow a}) = \mathbf{c}^{(k)}(X)$;*
4. *$X = X^{\uparrow a} \oplus_k X^{\downarrow a} = X^{\uparrow a} \cup X^{\downarrow a}$.*

Proof. Ad 1. The verification that $X^{\downarrow a}$ and $X^{\uparrow a}$ are closed under γ and δ is routine. For any k, if $x, y \in X_k^{\downarrow a}$, then $x <^{+,X} y$ iff $x <^{+,X^{\downarrow a}} y$. Similarly, for any k, if $x, y \in X_k^{\uparrow a}$, then $x <^{+,X} y$ iff $x <^{+,X^{\uparrow a}} y$. Thus by Lemma 5.17 $X^{\downarrow a}$ and $X^{\uparrow a}$ are positive opetopic cardinals.

Ad 2. Let us spell out $\mathbf{c}^{(k)}(X^{\downarrow a})$ and $\mathbf{d}^{(k)}(X^{\uparrow a})$:
$\mathbf{c}^{(k)}(X^{\downarrow a})$:

1. $\mathbf{c}^{(k)}(X^{\downarrow a})_l = \emptyset$, for $l > k$;
2. $\mathbf{c}^{(k)}(X^{\downarrow a})_k =$
   $(\{b \in X_k : b \leq^+ a\} \cup (X_k - \gamma(X_{k+1}))) - \delta(\{\alpha \in X_{k+1} : \gamma(\alpha) \leq^+ a\})$;
3. $\mathbf{c}^{(k)}(X^{\downarrow a})_{k-1} = X_{k-1} - \iota(X^{\downarrow a}_{k+1})$;
4. $\mathbf{c}^{(k)}(X^{\downarrow a})_l = X_l$, for $l < k-1$.

$\mathbf{d}^{(k)}(X^{\uparrow a})$:

1. $\mathbf{d}^{(k)}(X^{\uparrow a})_l = \emptyset$, for $l > k$;
2. $\mathbf{d}^{(k)}(X^{\uparrow a})_k =$
   $\{b \in X_k : b \not<^+ a \text{ or } b \not\in \delta(X_{k+1})\} - \gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \leq^+ a\})$;
3. $\mathbf{d}^{(k)}(X^{\uparrow a})_{k-1} = X_{k-1} - \iota(X^{\downarrow a}_{k+1})$;
4. $\mathbf{d}^{(k)}(X^{\uparrow a})_l = X_l$, for $l < k-1$.

Thus to show that $\mathbf{c}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow a})$, we need to verify that $\mathbf{c}^{(k)}(X^{\downarrow a})_k = \mathbf{d}^{(k)}(X^{\uparrow a})_k$. As both sets are contained in X_k, we can compare their complements. We have

$$X_k - \mathbf{c}^{(k)}(X^{\downarrow a})_k = \{b \in \delta(X_{k+1}) : b <^+ a\} \cup \gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \not\leq^+ a\})$$

and

$$X_k - \mathbf{d}^{(k)}(X^{\uparrow a})_k = \{b \in \gamma(X_{k+1}) : b \not\leq^+ a\} \cup \delta(\{\alpha \in X_{k+1}\gamma(\alpha) \leq^+ a\}).$$

But it easy to see that

$$\{b \in \delta(X_{k+1}) : b <^+ a\} = \delta(\{\alpha \in X_{k+1}\gamma(\alpha) \leq^+ a\})$$

and

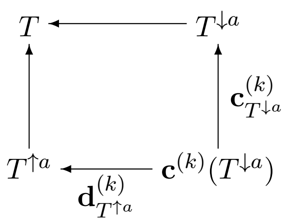
$$\gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \not\leq^+ a\}) = \{b \in \gamma(X_{k+1}) : b \not\leq^+ a\}.$$

The second equality uses the fact that $a \not\in \iota(T)$. Thus $\mathbf{c}^{(k)}(X^{\downarrow a})_k = \mathbf{d}^{(k)}(X^{\uparrow a})_k$, as required.

Ad 3. To see that $\mathbf{c}^{(k)}(X^{\uparrow a}) = \mathbf{c}^{(k)}(X)$, it is enough to note that $\iota(X_{k+1}) = \iota(X^{\downarrow a}_{k+1}) \cup \iota(X^{\uparrow a}_{k+1})$. The equation $\mathbf{d}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X)$ is even simpler.

Ad 4. Obvious. $\Box$

Corollary 8.2 *Let T be a positive opetopic cardinal, $k \in \omega$, $a \in (T_k - \iota(T_{k+2}))$. Then the square*



is a special pushout in **pOpeCard**.

Proof. Follows immediately from Lemmas 6.2 and 8.1. $\Box$

We need more notions and notations. Let X, T be positive opetopic cardinals $X \subseteq T$, $a \in (T - \iota(T))$. The decomposition $X = X^{\downarrow a} \cup X^{\uparrow a}$ is said to be *proper* iff $size(X^{\downarrow a}), size(X^{\uparrow a}) < size(X)$. If the decomposition $X = X^{\downarrow a} \cup X^{\uparrow a}$ is proper then a is said to be a *saddle face* of X. $Sd(X)$ is the set of saddle faces of X, $Sd(X)_k = Sd(X) \cap X_k$.

Lemma 8.3 *Let X, S, T be positive opetopic cardinals, $X \subseteq T$, $l \in \omega$. Then*

1. *if $a \in (T_l - \iota(T))$, then $a \in Sd(X)$ iff there are $\alpha, \beta \in X_{l+1}$ such that $\gamma(\alpha) \leq^+ a$ and $\gamma(\beta) \not\leq^+ a$;*
2. *if $\mathbf{c}^{(k)}(S) = \mathbf{d}^{(k)}(T)$, then*

$$size(S \oplus_k T)_l = \begin{cases} size(S)_l + size(T)_l & \text{if } l > k \\ size(T)_l & \text{if } l \leq k; \end{cases}$$

3. *$size(S)_k \geq 1$ iff $k \leq dim(S)$;*
4. *if $a \in Sd(S)_k$, then $size(S)_{k+1} \geq 2$;*
5. *S is principal iff $Sd(S)$ is empty.*

Proof. We shall show 5. The rest is easy.

If there is $a \in Sd(S)_k$, then by 2., 3. and Lemma 8.1 we have that $size(S)_{k+1} = size(S^{\downarrow a})_{k+1} + size(S^{\uparrow a})_{k+1} \geq 1 + 1 > 1$. So in that case S is not principal.

For the converse, assume that S is not principal. Fix $k \in \omega$ such that $size(S)_{k+1} > 1$. Thus there are $a, b \in S_{k+1}$, such that $a \neq b$. Suppose $\gamma(a) \in \iota(\alpha)$, for some $\alpha \in S_{k+2}$. Then by Lemma 5.5 we get $a <^+ \gamma(\alpha)$, contrary to the assumption on a. Hence $a \in S - \iota(S)$ and for similar reasons $b \in S - \iota(S)$. We have $a \not\lessgtr^+ b$ and, by pencil linearity, $\gamma(a) \neq \gamma(b)$. Then either $\gamma(a) \not<^+ \gamma(b)$ and then $\gamma(b) \in Sd(S)_k$, or $\gamma(b) \not<^+ \gamma(a)$ and then $\gamma(a) \in Sd(S)_k$. In either case $Sd(S)$ is not empty, as required. $\Box$

Lemma 8.4 *Let T, X be positive opetopic cardinals, $X \subseteq T$, and $a, x \in X - \iota(X)$, $k = dim(x) < dim(a) = m$.*

1. *We have the following equations of positive opetopic cardinals:*

$$X^{\downarrow x \downarrow a} = X^{\downarrow a \downarrow x} \qquad X^{\downarrow x \uparrow a} = X^{\uparrow a \downarrow x} \qquad X^{\uparrow x \downarrow a} = X^{\downarrow a \uparrow x} \qquad X^{\uparrow x \uparrow a} = X^{\uparrow a \uparrow x},$$

   i.e., 'the decompositions of different dimensions commute'.

2. *If $x \in Sd(X)$, then $x \in Sd(X^{\downarrow a}) \cap Sd(X^{\uparrow a})$.*
3. *Moreover, we have the following equations concerning domains and codomains*

$$\mathbf{c}^{(k)}(X^{\downarrow x \downarrow a}) = \mathbf{c}^{(k)}(X^{\downarrow x \uparrow a}) = \mathbf{d}^{(k)}(X^{\uparrow x \downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow x \uparrow a})$$

$$\mathbf{c}^{(m)}(X^{\downarrow x \downarrow a}) = \mathbf{d}^{(m)}(X^{\downarrow x \uparrow a}) \quad \mathbf{c}^{(m)}(X^{\uparrow x \downarrow a}) = \mathbf{d}^{(m)}(X^{\uparrow x \uparrow a}).$$

4. *Finally, we have the following equations concerning compositions*

$$X^{\downarrow x \uparrow a} \oplus_m X^{\downarrow x \downarrow a} = X^{\downarrow x} \quad X^{\uparrow x \uparrow a} \oplus_m X^{\uparrow x \downarrow a} = X^{\uparrow x}$$

$$X^{\uparrow x \downarrow a} \oplus_k X^{\downarrow x \downarrow a} = X^{\downarrow a} \quad X^{\uparrow x \uparrow a} \oplus_k X^{\downarrow x \uparrow a} = X^{\uparrow a}.$$

Proof. Simple check. $\Box$

Lemma 8.5 *Let T, X be positive opetopic cardinals, $X \subseteq T$, and $a, b \in X - \iota(X)$, $dim(a) = dim(b) = m$.*

1. *We have the following equations of positive opetopic cardinals:*

$$X^{\downarrow a \downarrow b} = X^{\downarrow b \downarrow a} \qquad X^{\uparrow a \uparrow b} = X^{\uparrow b \uparrow a},$$

i.e., 'the decompositions in the same dimension and the same directions commute'.

2. *Assume $a <^+ b$. Then we have the following further equations of positive opetopic cardinals:*

$$X^{\downarrow a} = X^{\downarrow a \downarrow b} \qquad X^{\uparrow b} = X^{\uparrow a \uparrow b} \qquad X^{\downarrow b \uparrow a} = X^{\uparrow a \downarrow b}.$$

Moreover, if $a, b \in Sd(X)$, then $a \in Sd(X^{\downarrow b})$ and $b \in Sd(X^{\uparrow a})$.

3. *Assume $a <^-_l b$, for some $l < m$. Then $X^{\uparrow b \downarrow a}$, $X^{\uparrow a \downarrow b}$, are positive opetopic cardinals, and*

$$X^{\uparrow a \downarrow b} \oplus_m X^{\downarrow a} = X^{\uparrow b \downarrow a} \oplus_m X^{\downarrow b}.$$

Moreover, if $a, b \in Sd(X)$, then either there is k such that $l-1 \leq k < m$ and $\gamma^{(k)}(a) \in Sd(X)$ or $a \in Sd(X^{\uparrow b})$ and $b \in Sd(X^{\uparrow a})$.

Proof. Simple check. □

Lemma 8.6 *Let T, X be positive opetopic cardinals, $X \subseteq T$, $dim(X) = n$, $l < n-1$, $a \in Sd(X)_l$. Then*

1. *$a \in Sd(\mathbf{c}(X)) \cap Sd(\mathbf{d}(X))$;*
2. *$\mathbf{d}(X^{\downarrow a}) = (\mathbf{d}X)^{\downarrow a}$;*
3. *$\mathbf{d}(X^{\uparrow a}) = (\mathbf{d}X)^{\uparrow a}$;*
4. *$\mathbf{c}(X^{\downarrow a}) = (\mathbf{c}(X))^{\downarrow a}$;*
5. *$\mathbf{c}(X^{\uparrow a}) = (\mathbf{c}(X))^{\uparrow a}$.*

Proof. The proof is again by a long and simple check. We shall check part of 5. We should consider separately cases: $l = n-2$, $l = n-3$, and $l < n-3$, but we shall check the case $l = n-3$ only. The other cases can be also shown by similar, but easier, checks.

$(\mathbf{c}(X))^{\uparrow a}$ is:

1. $(\mathbf{c}(X))^{\uparrow a}_l = \emptyset$, for $l \geq n$;
2. $(\mathbf{c}(X))^{\uparrow a}_{n-1} = \{x \in X_{n-1} : \gamma^{(n-3)}(x) \not\leq^+ a, \ \ x \not\in \delta(X_n)\}$;
3. $(\mathbf{c}(X))^{\uparrow a}_{n-2} = \{x \in X_{n-2} : \gamma(x) \not\leq^+ a, \ \ x \not\in \iota(X_n)\}$;
4. $(\mathbf{c}(X))^{\uparrow a}_{n-3} = \{x \in X_{n-3} : x \not<^+ a \text{ or } x \not\in \delta(X_{n-2} - \iota(X_n))\}$;
5. $(\mathbf{c}(X))^{\uparrow a}_{n-4} = X_{n-4} - \iota(\{x \in X_{n-2} : x \not\in \iota(X_n), \ \gamma(x) \leq^+ a\})$;
6. $X^{\downarrow a}_l = X_l$, for $l < n-4$.

and $\mathbf{c}(X^{\uparrow a})$ is:

1. $\mathbf{c}(X^{\uparrow a})_l = \emptyset$, for $l \geq n$;
2. $\mathbf{c}(X^{\uparrow a})_{n-1} = \{x \in X_{n-1} : \gamma^{(n-3)}(x) \not\leq^+ a\} - \delta(\{z \in X_n : \gamma^{(n-3)}(z) \not\leq^+ a\})$;

3. $\mathbf{c}(X^{\uparrow a})_{n-2} = \{x \in X_{n-2} : \gamma(x) \not\leq^+ a\} - \iota(\{z \in X_n : \gamma^{(n-3)}(z) \not\leq^+ a\})$;

4. $\mathbf{c}(X^{\uparrow a})_{n-3} = \{x \in X_{n-3} : x \not<^+ a \text{ or } x \notin \delta(X_{n-2})\}$;

5. $\mathbf{c}(X^{\uparrow a})_{n-4} = X_{n-4} - \iota(X^{\downarrow a}_{n-2})$;

6. $\mathbf{c}(X^{\uparrow a})_l = X_l$, for $l < n-4$.

We need to verify the equality $(\mathbf{c}(X))^{\uparrow a}_l = \mathbf{c}(X^{\uparrow a})_l$, for $l = n-1, \ldots, n-4$.

In dimension $n-1$, it is enough to show that if $x \in X_{n-1}$ and $z \in X_n$ so that $\gamma^{(n-3)}(x) \not\leq^+ a$ and $x \in \delta(z)$, then $\gamma^{(n-3)}(z) \not\leq^+ a$.

So assume that we have $x \in X_{n-1}$, $\gamma^{(n-3)}(x) \not\leq^+ a$, $z \in X_n$ such that $x \in \delta(z)$. Hence $x \lhd^+ \gamma(z)$. By Lemma 5.9.5, $\gamma^{(n-3)}(x) \leq^+ \gamma^{(n-3)}(z)$. Therefore $\gamma^{(n-3)}(z) \not\leq^+ a$ (otherwise we would have $\gamma^{(n-3)}(x) \not\leq^+ a$), as required.

In dimension $n-2$, it is enough to show that if $x \in X_{n-2}$ and $z \in X_n$ so that $x \not\leq^+ a$ and $x \in \iota(z)$, then $\gamma^{(n-3)}(z) \not\leq^+ a$.

So assume that $x \in X_{n-2}$, $z \in X_n$ so that $x \not\leq^+ a$ and $x \in \iota(z)$. Hence $x \leq^+ \gamma\gamma(z)$. By Lemma 5.9.5, $\gamma(x) \leq^+ \gamma^{(n-3)}(z)$. Therefore $\gamma^{(n-3)}(z) \not\leq^+ a$, as required.

The equality in dimension $n-3$ follows immediately from Lemma 5.19.4.

To show that in dimension $n-4$, the above equation also holds, we shall show that

$$\iota(X^{\downarrow a}_{n-2}) \subseteq \iota(\{x \in X_{n-2} : x \notin \iota(X_n),\ \gamma(x) \leq^+ a\}).$$

Note that, by Lemma 5.3.1, if $t \in X_{n-4}$ and $x \in X_{n-2}$, $y \in X_{n-1}$, $t \in \iota(x)$ and $\gamma(x) \leq^+ a$ and $x = \gamma(y)$, then there is $x' \in \delta(y)$ (i.e., $x' \lhd^+ x$ and hence $\gamma(x') \leq^+ a$) such that $t \in \iota(x')$.

Thus, as $<^+$ is well founded, it follows from the above observation that, for any $t \in X_{n-4}$ and $x \in X_{n-2}$ such that $t \in \iota(x)$ and $\gamma(x) \leq^+ a$, there is $x'' \leq^+ x$ such that $t \in \iota(x'')$ and $x'' \notin \gamma(X)$. Then we clearly have that $x'' \notin \iota(X)$ and $\gamma(x'') \leq^+ a$, as required. $\Box$

The following lemma describes how one can express decompositions of a special pushout in terms of decompositions of its components.

Lemma 8.7 *Let T, T_1, T_2 be positive opetopic cardinals, $dim(T_1), dim(T_2) > k$ such that $\mathbf{c}^{(k)}(T_1) = \mathbf{d}^{(k)}(T_2)$ and $T = T_2 \oplus_k T_1$. Then $\mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1) \neq \emptyset$. For any $a \in \mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1)$, we have $a \in Sd(T)_k$ and*

- *either $T_1 = T^{\downarrow a}$ and $T_2 = T^{\uparrow a}$*
- *or $a \in Sd(T_1)_k$, $T^{\downarrow a} = T_1^{\downarrow a}$ and $T^{\uparrow a} = T_2 \oplus_k T_1^{\uparrow a}$.*

Proof. By assumption, $(T_1)_{k+1} \neq \emptyset$ and $(T_2)_{k+1} \neq \emptyset$. So $\mathbf{c}^{(k)}(T_1) \cap \gamma(T_1) \neq \emptyset$. Fix $a \in \mathbf{c}^{(k)}(T_1) \cap \gamma(T_1) \neq \emptyset$. Then $T^{\downarrow a}_{k+1} \neq \emptyset$. As $T^{\downarrow a}_{k+1} \cap (T_2)_{k+1} = \emptyset$, we must have $a \in Sd(T)_k$.

Assume $T_1 \neq T^{\downarrow a}$. Then $T^{\downarrow a} \subsetneq T_1$. Hence $(T_1) - (T^{\downarrow a}) \neq \emptyset$. But this means that $a \in Sd(T_1)_k$. The verification that the equalities $T^{\downarrow a} = T_1^{\downarrow a}$ and $T^{\uparrow a} = T_2 \oplus_k T_1^{\uparrow a}$ hold in this case is left as an exercise. $\Box$

9 Positive opetopic cardinals as positive-to-one polygraphs

For the definition of positive-to-one polygraphs and related notation see Appendix. In this section we show that the image of the embedding defined in Section 6

$$(-)^* : \mathbf{pOpeCard} \longrightarrow \omega Cat$$

is in fact contained in the category of polygraphs.

Proposition 9.1 *Let S be a weak positive opetopic cardinal. Then S^* is a positive-to-one polygraph whose k-indeterminates correspond to faces in S_k.*

Proof. The proof is by induction on the dimension n of the weak positive opetopic cardinal S. For $n = 0, 1$, the proposition is obvious.

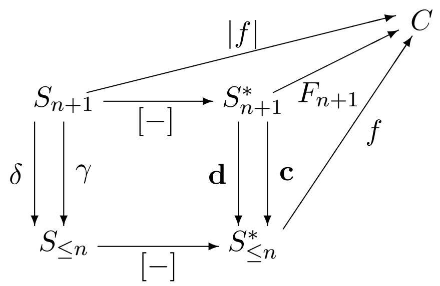
So assume that for any weak positive opetopic cardinal T of dimension n, T^* is a positive-to-one polygraph of dimension n, generated by faces in T. Suppose that S is a weak positive opetopic cardinal of dimension $n+1$. We shall show that S^* is a polygraph generated by faces in S. Since $S_{\leq n}$ is a weak positive opetopic cardinal, by inductive assumption, $S^*_{\leq n}$ is the polygraph generated by faces in $S_{\leq n}$. So we need to verify that, for any ω-functor $f : S^*_{\leq n} \longrightarrow C$ to any ω-category C and any function $|f| : S_{n+1} \longrightarrow C_{n+1}$ such that for $a \in S_{n+1}$

$$d_C(|f|(a)) = f(\mathbf{d}([a])), \qquad c_C(|f|(a)) = f(\mathbf{c}([a])),$$

there is a unique ω-functor $F : S^* \longrightarrow C$ such that

$$F_{n+1}([a]) = |f|(a), \qquad F_{\leq n} = f$$

as in the diagram



We define F_{n+1} as follows. For $X \in S^*_{n+1}$:

$$F_{n+1}(X) = \begin{cases} id_{f(X)} & \text{if } dim(X) \leq n \\ |f|(a) & \text{if } dim(X) = n+1, \ X \text{ is principal and } X = [a] \\ F_{n+1}(X^{\uparrow a}) \circ_l F_{n+1}(X^{\downarrow a}) & \text{if } dim(X) = n+1, \ a \in Sd(X)_l, \end{cases}$$

where $\circ_l$ refers to the composition in the ω-category C. Clearly $F_k = f_k$, for $k \leq n$. The above morphism, if well defined, clearly preserves identities. Uniqueness is also clear by construction. We need to verify three conditions, for $X \in S^*_{n+1}$ and $dim(X) = n+1$:

I F is well defined, i.e., $F_{n+1}(X) = F_{n+1}(X^{\uparrow a}) \circ_l F_{n+1}(X^{\downarrow a})$ does not depend on the choice of the saddle face $a \in Sd(X)$;

II F preserves the domains and codomains, i.e., $F_n(d(X)) = d(F_{n+1}(X))$ and $F_n(c(X)) = c(F_{n+1}(X))$;

III F preserves compositions, i.e., $F_{n+1}(X) = F_{n+1}(X_2) \circ_k F_{n+1}(X_1)$ whenever $X = X_2 \oplus_k X_1$ and $dim(X_1), dim(X_2) > k$.

We have an embedding $[-] : S_{\leq n} \longrightarrow S^*_{\leq n}$. So let us assume that for positive opetopic cardinals of S of size less than $size(X)$ the above assumption holds. If $size(X)_{n+1} = 0$ or X is principal, all three conditions are obvious. So assume that X is not principal and $dim(X) = n+1$. To save on notation we write F for F_{n+1}.

Ad **I**. First we will consider two saddle faces $a, x \in Sd(X)$ of different dimension $k = dim(x) < dim(a) = m$. Using Lemma 8.4 we have

$$\begin{aligned}
F(X^{\uparrow a}) \circ_m F(X^{\downarrow a}) &= \quad \textit{ind. hyp. I} \\
= (F(X^{\uparrow a \uparrow x}) \circ_k F(X^{\uparrow a \downarrow x})) \circ_m (F(X^{\downarrow a \uparrow x}) \circ_k F(X^{\downarrow a \downarrow x})) &= \\
= (F(X^{\uparrow a \uparrow x}) \circ_m F(X^{\downarrow a \uparrow x})) \circ_k (F(X^{\uparrow a \downarrow x}) \circ_m F(X^{\downarrow a \downarrow x})) &= \\
= (F(X^{\uparrow x \uparrow a}) \circ_m F(X^{\uparrow x \downarrow a})) \circ_k (F(X^{\downarrow x \uparrow a}) \circ_m F(X^{\downarrow x \downarrow a})) &= \quad \textit{ind. hyp. III} \\
= F(X^{\uparrow x}) \circ_m F(X^{\downarrow x}). &
\end{aligned}$$

Now we will consider two saddle faces $a, b \in Sd(X)$ of the same dimension $dim(a) = dim(b) = m$. We shall use Lemma 8.5. Assume $a <_l^- b$, for some $l < m$. If $\gamma^{(k)}(a) \in Sd(X)$, for some $k < m$, then this case reduces to the previous one for two pairs $a, \gamma^{(k)}(a) \in Sd(X)$ and $b, \gamma^{(k)}(a) \in Sd(X)$. Otherwise $a \in Sd(X^{\uparrow b})$ and $a \in Sd(X^{\uparrow b})$ and we have

$$\begin{aligned}
F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) &= \quad \textit{ind. hyp. I} \\
= (F(X^{\uparrow a \uparrow b}) \circ_k F(X^{\uparrow a \downarrow b})) \circ_k F(X^{\downarrow a}) &= \\
= F(X^{\uparrow a \uparrow b}) \circ_k (F(X^{\uparrow a \downarrow b}) \circ_k F(X^{\downarrow a})) &= \quad \textit{ind. hyp. III} \\
= F(X^{\uparrow b \uparrow a}) \circ_k F(X^{\uparrow a \downarrow b} \oplus_k X^{\downarrow a}) &= \\
= F(X^{\uparrow b \uparrow a}) \circ_k F(X^{\uparrow b \downarrow a} \oplus_k X^{\downarrow b}) &= \quad \textit{ind. hyp. III} \\
= F(X^{\uparrow b \uparrow a}) \circ_k (F(X^{\uparrow b \downarrow a}) \circ_k F(X^{\downarrow b})) &= \\
= (F(X^{\uparrow b \uparrow a}) \circ_k F(X^{\uparrow b \downarrow a})) \circ_k F(X^{\downarrow b}) &= \\
= F(X^{\uparrow b}) \circ_k F(X^{\downarrow b}). &
\end{aligned}$$

Finally, we consider the case $a <^+ b$. We have

$$\begin{aligned}
F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) &= \quad \textit{ind. hyp. I} \\
= (F(X^{\uparrow a \uparrow b}) \circ_k F(X^{\uparrow a \downarrow b})) \circ_k F(X^{\downarrow a}) &= \\
= (F(X^{\uparrow b}) \circ_k F(X^{\downarrow b \uparrow a})) \circ_k F(X^{\downarrow b \downarrow a}) &= \\
= F(X^{\uparrow b}) \circ_k (F(X^{\downarrow b \uparrow a}) \circ_k F(X^{\downarrow b \downarrow a})) &= \quad \textit{ind. hyp. I} \\
= F(X^{\uparrow b}) \circ_k F(X^{\downarrow b}). &
\end{aligned}$$

This shows that $F(X)$ is well defined.

Ad **II**. We shall show that the domains are preserved. The proof that the codomains are preserved is similar.

The fact that if $Sd(X) = \emptyset$, then F preserves domains and codomains follows immediately from the assumption on f and $|f|$. So assume $Sd(X) \neq \emptyset$ and let $a \in Sd(X)$, $dim(a) = k$. We use Lemma 8.6. We have to consider two cases $k < n$, and $k = n$.

If $k < n$, then

$$\begin{aligned}
F_n(d(X)) = F_n(d(X^{\uparrow a} \oplus_k X^{\downarrow a})) &= \\
= F_n(d((X^{\uparrow a}) \oplus_k d(X^{\downarrow a})) &= \\
= F_n(d(X)^{\uparrow a} \oplus_k d(X)^{\downarrow a}) &= \quad \textit{ind. hyp. III} \\
= F_n(d(X)^{\uparrow a}) \circ_k F_n(d(X)^{\downarrow a}) &= \\
= F_n(d(X^{\uparrow a})) \circ_k F_n(d(X^{\downarrow a})) &= \quad \textit{ind. hyp. II} \\
= d(F_{n+1}(X^{\uparrow a})) \circ_k d(F_{n+1}(X^{\downarrow a})) &= \\
= d(F_{n+1}(X^{\uparrow a}) \circ_k F_{n+1}(X^{\downarrow a})) &= \quad \textit{ind. hyp. I} \\
= d(F_{n+1}(X)). &
\end{aligned}$$

If $k = n$, then

$$\begin{aligned}
F_n(d(X)) = F_n(d(X^{\uparrow a} \oplus_n X^{\downarrow a})) &= \\
= F_n(d(X^{\downarrow a})) &= \quad \textit{ind. hyp. II} \\
= d(F_{n+1}(X^{\downarrow a})) &= \\
= d(F_{n+1}(X^{\uparrow a})) \circ_n F_{n+1}(X^{\downarrow a})) &= \quad \textit{ind. hyp. I} \\
= d(F_{n+1}(X)).&
\end{aligned}$$

Ad **III**. Suppose that $X = X_2 \oplus_k X_1$ and $dim(X) \leq n+1$. We shall show that F preserves this composition. If $dim(X_1) = k$, then $X = X_2$, $X_1 = \mathbf{d}^{(k)}(X_2)$. We have

$$\begin{aligned}
F_{n+1}(X) = F_{n+1}(X_2) &= \\
= F_{n+1}(X_2) \circ_k 1^{(n+1)}_{F_k(\mathbf{d}^{(k)}(X_2))} &= \\
= F_{n+1}(X_2) \circ_k 1^{(n+1)}_{F_k(X_1)} &= \\
= F_{n+1}(X_2) \circ_k F_{n+1}(X_1).&
\end{aligned}$$

The case $dim(X_2) = k$ is similar. So now assume $dim(X_1), dim(X_2) > k$. We shall use Lemma 8.7. Fix $a \in \mathbf{c}^{(k)}(X_1)_k \cap \gamma(X_1)$. So $a \in Sd(X)_k$. If $X_1 = X^{\downarrow a}$ and $X_2 = X^{\uparrow a}$, then we have

$$F(X) = F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) = F(X_2) \circ_k F(X_1).$$

If $a \in Sd(X_1)_k$, then

$$\begin{aligned}
F(X) = F(X^{\uparrow a}) \circ_k F(X^{\downarrow a}) &= \quad \textit{ind. hyp. II} \\
= (F(X_2) \circ_k F(X_1^{\uparrow a})) \circ_k F(X^{\downarrow a}) &= \\
= F(X_2) \circ_k (F(X_1^{\uparrow a}) \circ_k F(X_1^{\downarrow a})) &= \quad \textit{ind. hyp. II} \\
F(X_2) \circ_k F(X_1).&
\end{aligned}$$

So in any case the composition is preserved. This ends the proof of the lemma. □

Let $\mathbf{wpOpeCard}_n$ be the full subcategory of **wpOpeCard** whose objects have dimension at most $n \geq 0$. For $n \in \omega$, we have a functor

$$(-)^{\sharp,n} : \mathbf{wpOpeCard}_n \longrightarrow Set \downarrow D_{n-1}$$

such that, for S in $\mathbf{wpOpeCard}_n$

$$S^{\sharp,n} = (S_n, S^*_{<n}, [\delta], [\gamma])$$

and, for $f : S \to T$ in $\mathbf{wpOpeCard}_n$, we have

$$f^{\sharp,n} = (f_n, (f_{<n})^*).$$

By construction, we have

$$\overline{(-)}^n \circ (-)^{\sharp,n} = (-)^{*,n},$$

where $(-)^{*,n} : \mathbf{wpOpeCard}_n \longrightarrow \mathbf{pPoly}_n$ is the n-dimensional version of $(-)^*$.

Corollary 9.2 *For every $n \in \omega$, the functor $(-)^{\sharp,n}$ is full and faithful, and it preserves special pushouts (i.e., it maps special pushouts to pushouts).*

Proof. Fullness and faithfulness of $(-)^{\sharp,n}$ is left for the reader. We shall show that for every $n \in \omega$, $(-)^{\sharp,n}$ preserves special pushouts. For $n = 0$, there is nothing to prove. For $n = 1$, this is obvious. So assume that $n \geq 1$ and that $(-)^{\sharp,n}$ preserves special pushouts. Let

$$\begin{array}{ccc} S & \longrightarrow & S +_R T \\ \uparrow & & \uparrow \\ R & \longrightarrow & T \end{array}$$

be a special pushout in $\mathbf{wpOpeCard}_{n+1}$. Clearly its n-truncation is a pushout in $\mathbf{wpOpeCard}_n$. Hence by inductive hypothesis it is preserved by $(-)^{*,n}$. In dimension $n+1$, the functor $(-)^{\sharp,n+1}$ is an inclusion. Hence, in dimension $n+1$, this square is a pushout (of monos) in *Set*. So the whole square

$$\begin{array}{ccc} S^{\sharp,n+1} & \longrightarrow & (S +_R T)^{\sharp,n+1} \\ \uparrow & & \uparrow \\ R^{\sharp,n+1} & \longrightarrow & T^{\sharp,n+1} \end{array}$$

is a pushout in $Set \downarrow D_{n+1}$, i.e., $(-)^{\sharp,n+1}$ preserves special pushouts. □

Corollary 9.3 *The functor*

$$(-)^* : \mathbf{wpOpeCard} \longrightarrow \mathbf{pPoly}$$

is full and faithful and preserves special pushouts. In particular, it is conservative.

Proof. This follows from the previous corollary and the fact that the functor $\overline{(-)}^n$: $Set \downarrow D_{n-1} \longrightarrow \mathbf{pPoly}_n$ is an equivalence of categories. [†] □

Corollary 9.4 *The functor*

$$(-)^* : \mathbf{wpOpeCard} \longrightarrow \omega Cat$$

is faithful, conservative and preserves special pushouts.

Proof. The faithfulness and preservation of special pushout follows from the previous corollary and the fact that the functor $F_n : \mathbf{pPoly}_n \longrightarrow n\mathbf{Cat}$ (see Appendix) is faithful and a left adjoint. Conservativity follows from the previous corollary and the fact that any isomorphic ω-functor between polygraphs preserves and reflects indeterminates, i.e., is a map of polygraphs. [‡] □

Let P be a positive-to-one polygraph, a a k-cell in P. A *description of the cell* a is a pair [&]

$$< T_a, \tau_a : T_a^* \longrightarrow P >$$

[†] In reference to the description given in the appendix, $\overline{(-)}^n$ is full and essentially surjective by construction. Here is a hint on how faithfulness can be established. The morphisms of the free category are equivalence classes of formal composites in various dimensions of generators of dimension $\leq n$. One can associate to such a formal expression s the multiset $top(s)$ of the generating n-morphisms occurring in it, and one can show that this is an invariant: if two expressions s and t can be proved equal by the laws of ω-categories, then $top(s) = top(t)$. For a generator a we have $top(a) = \{a\}$, hence if two generators a, b where equated, we would have $\{a\} = top(a) = top(b) = \{b\}$, proving faithfulness.

[‡] For a proof of this, see [Poly, Proposition 16.6.3].

[&] The existence and uniqueness of descriptions is proved below (Proposition 12.2).

where T_a is a positive opetopic cardinal and τ_a is a polygraph map such that

$$\tau_a(T_a) = a.$$

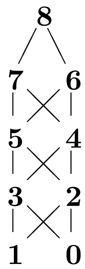
In the remainder of this section we shall define some specific positive opetopic cardinals that will be used later. First we define the globes α^n, for $n \in \omega$. We put

$$\alpha^n_l \;=\; \begin{cases} \emptyset & \text{if } l > n \\ \{2n\} & \text{if } l = n \\ \{2l+1,\, 2l\} & \text{if } 0 \leq l < n \end{cases}$$

$$d,\, c : \alpha^n_l \longrightarrow \alpha^n_{l-1}$$

$$d(x) = \{2l-1\} \qquad c(x) = 2l-2$$

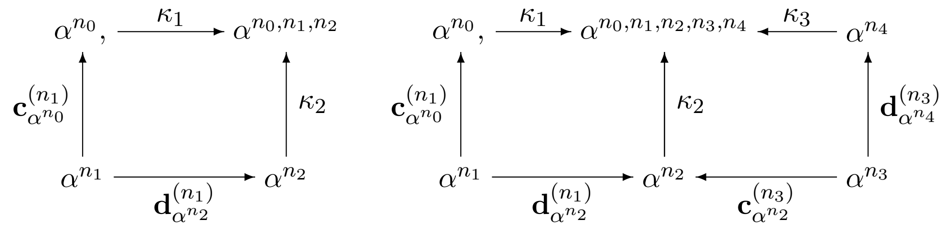
for $x \in \alpha^n_l$, and $1 \leq l \leq n$. For example, α^4 can be pictured as follows:



i.e., 8 is the unique cell of dimension 4 in α^4 that has 7 as its domain and 6 as its codomain, 7 and 6 have 5 as its domain and 4 as its codomain, and so on. Note that, for any $k \leq n$, we have

$$\mathbf{d}^{(k)}\alpha^n = \alpha^k = \mathbf{c}^{(k)}\alpha^n.$$

Let $n_1 < n_0, n_2$ and $n_3 < n_2, n_4$. We define the positive opetopic cardinals α^{n_0,n_1,n_2} and $\alpha^{n_0,n_1,n_2,n_3,n_4}$ as the following colimits in **pOpeCard**:



Proposition 9.5 *The above colimits are preserved by the functor*

$$(-)^* : \mathbf{pOpeCard} \longrightarrow \mathbf{pPoly}.$$

Moreover, for any ω-category C, we have bijective correspondences

$$\omega Cat((\alpha^n)^*, C) = C_n$$

$$\omega Cat((\alpha^{n_0,n_1,n_2})^*, C) = \{(x,y) \in C_{n_0} \times C_{n_2} : c^{(n_1)}(x) = d^{(n_1)}(y)\}$$

$$\omega Cat((\alpha^{n_0,n_1,n_2,n_3,n_4})^*, C)$$
$$= \{(x,y,z) \in C_{n_0} \times C_{n_2} \times C_{n_4} : c^{(n_1)}(x) = d^{(n_1)}(y) \text{ and } c^{(n_3)}(y) = d^{(n_3)}(z)\}$$

which are natural in C.

Proof. As both positive opetopic cardinals α^{n_0,n_1,n_2} and $\alpha^{n_0,n_1,n_2,n_3,n_4}$ are obtained via special pushout (in the second case applied twice), these colimits are preserved by $(-)^*$. $\Box$

Let T be a positive opetopic cardinal. We have a functor

$$\Sigma^T : \mathbf{pOpe} \downarrow T \longrightarrow \mathbf{pOpeCard}$$

such that

$$\Sigma^T(f : B \to T) = B$$

and a cocone

$$\sigma^T : \Sigma^T \longrightarrow T$$

such that

$$\sigma^T_{(f:B\to T)} = f : \Sigma^T(f : B \to T) = B \longrightarrow T.$$

Lemma 9.6 *The cocone* $\sigma^T : \Sigma^T \stackrel{\cdot}{\longrightarrow} T$ *is a colimiting cocone in* **pOpeCard**. *Such colimiting cocones are called* special colimits. *Any functor from* $\mathbf{pOpeCard}^{op}$ *which preserves special limits preserves special pullbacks as well.*

Proof. To see that $\sigma^T : \Sigma^T \stackrel{\cdot}{\longrightarrow} T$ is a colimiting cocone we proceed by induction on the size of T. If T is a positive opetope, then the category $\mathbf{pOpe} \downarrow T$ has terminal object id_T which is sent by Σ^T to T. Thus in this case T is the colimit of Σ^T. If T is not a positive opetope, then, by Lemma 8.3.5, it can be presented as a special pushout $T = T_2 \oplus_k T_1$

$$\begin{array}{ccc} T_2 \oplus_k T_1 & \xleftarrow{\kappa_1} & T_1 \\ {\scriptstyle \kappa_2}\uparrow & & \uparrow{\scriptstyle \mathbf{c}^{(k)}_{T_1}} \\ T_2 & \xleftarrow[\mathbf{d}^{(k)}_{T_2}]{} & \mathbf{d}^{(k)}T_2 \end{array}$$

with both T_1 and T_2 of dimension larger than k and size smaller than the size of T, for some $k \in \omega$. By inductive assumption, the limits of Σ^T_1, Σ^T_2, and $\Sigma^{\mathbf{d}^{(k)}T_2}$ are T_1, T_2 and $\Sigma^{\mathbf{d}^{(k)}T_2}$, respectively. Each object $f : B \to T$ of $\mathbf{pOpe} \downarrow T$ factorises (as a morphism) via either κ_1 or κ_2.[†] If it factorises by both, it factorises by $\mathbf{d}^{(k)}T_2$. From this description it is easy to see that indeed in this case T is also the colimit of the functor Σ^T. Moreover, if the limit of Σ^T is preserved, then this special pushout is also preserved. $\Box$

Remarks and notation. The full image of the functor $(-)^* : \mathbf{pOpeCard} \longrightarrow \omega Cat$ will be denoted by $\mathbf{pOpeCard}_\omega$. The objects of $\mathbf{pOpeCard}_\omega$ are ω-categories isomorphic to those of form S^* for S being positive opetopic cardinal and the morphism in $\mathbf{pOpeCard}_\omega$ are all ω-functors. In fact, when convenient, we shall think about positive opetopic cardinals S as if they where ω-categories and talk about ω-functors between them. As the above embedding $(-)^*$ is conservative (Corollary 9.4), this will not lead to any confusions.

10 The inner-outer factorization in $\mathbf{pOpeCard}_\omega$

Let $f : S^* \longrightarrow T^*$ be a morphism in $\mathbf{pOpeCard}_\omega$. We say that f is *outer*[1] if there is a map of positive opetopic cardinals $g : S \longrightarrow T$ such that $g^* = f$. We say that f is *inner* iff $f_{dim(S)}(S) = T$. From Corollary 9.3 we have

[†] Here we use the fact that B is principal!

[1] The names 'inner' and 'outer' are introduced in analogy with the morphisms with the same name and role in the category of disks in [J].

Lemma 10.1 *An ω-functor $f : S^* \longrightarrow T^*$ is outer iff it is a polygraph map.* $\Box$

Proposition 10.2 *Let $f : S^* \longrightarrow T^*$ be an inner map, $dim(S) = dim(T) > 0$. The maps $\mathbf{d}(f) : \mathbf{d}(S) \longrightarrow \mathbf{d}(T)$ and $\mathbf{c}(f) : \mathbf{c}(S) \longrightarrow \mathbf{c}(T)$, being the restrictions of f, are well defined, inner and the squares*

$$\begin{array}{ccccc} (\mathbf{d}(T))^* & \xrightarrow{\mathbf{d}^*_T} & T^* & \xleftarrow{\mathbf{c}^*_T} & (\mathbf{c}(T))^* \\ \uparrow{\scriptstyle \mathbf{d}(f)} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \mathbf{c}(f)} \\ (\mathbf{d}(S))^* & \xrightarrow[\mathbf{d}^*_S]{} & S^* & \xleftarrow[\mathbf{c}^*_S]{} & (\mathbf{c}(S))^* \end{array}$$

commute.

Proof. So suppose that $f : S^* \to T^*$ is an inner map. So $f(S) = T$. Since f is an ω-functor, we have

$$f(\mathbf{d}(S)) = \mathbf{d}(f(S)) = \mathbf{d}(T) \quad \text{and} \quad f(\mathbf{c}(S)) = \mathbf{c}(f(S)) = \mathbf{c}(T).$$

This shows the proposition. $\Box$

Proposition 10.3 *The inner and outer morphisms form a factorization system in* **pOpeCard**$_\omega$. *So any ω-functor $f : S^* \longrightarrow T^*$ can be factored essentially uniquely by inner map $\overset{\bullet}{f}$ followed by outer map $\overset{\circ}{f}$:*

$$\begin{array}{ccc} S^* & \xrightarrow{\quad f \quad} & T^* \\ & {\scriptstyle \overset{\bullet}{f}} \searrow \quad \nearrow {\scriptstyle \overset{\circ}{f}} & \\ & f(S)^* & \end{array}$$

Proof. The factorization is almost tautological.[†] $\Box$

The inner maps between positive opetopic cardinals can be further factorized into inner epi and inner mono.

Let P and Q be positive opetopic cardinals, $f : P^* \to Q^*$ an ω-functor between ω-categories generated by them. The *kernel of* f is the set $\ker(f)$ of faces of P sent by f to identities on cells of Q^* of lower dimension. We say that a set I of faces of $P_{>0}$ is an *ideal in* P iff, for any $b \in P_{>0}$:

1. if $\gamma(b) \in I$, then $b \in I$;

2. if $\delta(b) \subseteq I$, then $b \in I$;

3. if $b \in I$, then $|\delta(b) \setminus I| = |\gamma(b)| = 1$.

The proof of the following lemma is left to the reader.

Lemma 10.4 *Let $f : P^* \to Q^*$ be an ω-functor between ω-categories generated by positive opetopic cardinals. Then* $\ker(f)$ *is an ideal.* $\Box$

[†] The essential uniqueness is an easy consequence of Lemma 10.9 below, which implies in particular that outer maps are mono. So, if $g : S^* \to U^*$ and $h : U^* \to T^*$ give another factorisation, without loss of generality we wan assume h to be the inclusion, which forces g to be a corestriction of f, $U = f(S)$ and $g = \overset{\bullet}{f}$.

We shall prove the converse of the above lemma, i.e., that any ideal is a kernel of an ω-functor, in fact an inner epi.

Recall that a face $u \in P_{>0}$ is unary if $\delta(u)$ contains one element. Let $U(P)$ be the set of unary faces in P, $I \subseteq P$ an ideal in P, and $I \neq \emptyset$. The face $u \in P$ is called *safe for* P iff $u \in U(P) - \gamma(P) - \delta(U(P)$, i.e., u is a unary face in P that is not a codomain of any other face in P and it is not in the domain of a unary face in P.

The following lemma says that we can always divide any opetopic cardinal by its set of safe faces.

Lemma 10.5 *Let P be an opetopic cardinal, and u a safe face for P. Then we can divide P by u, i.e., we have a quotient ω-functor $q_u : P^* \longrightarrow (P_{/u})^*$ whose kernel is $\{u\}$. q_u is an inner epi.*

Proof. The opetope $P_{/u}$ is obtained by gluing together $\delta(u)$ and $\gamma(u)$ (to a face $\{\delta(u), \gamma(u)\}$) and dropping u. The map q_u is defined as follows (we describe it on faces of P only)

$$q_u(a) = \begin{cases} \{\delta(u), \gamma(u)\} & \text{if } a = \delta(u),\ \gamma(u),\ u \\ a & \text{otherwise.} \end{cases}$$

i.e., $q_u(\delta(u)) = q_u(\gamma(u)) = q_u(u) = \{\delta(u), \gamma(u)\}$, i.e., the equivalence class containing $\delta(u)$ and $\gamma(u)$. $q_u(a) = a$, for other faces a in P.

Then γ and δ on $P_{/u}$ are so defined to make the quotient map $q_u : P^* \longrightarrow (P_{/u})^*$ preserve both of them.

The only cell sent by q_u to a(n identity on a) cell of a lower dimension is u. Thus $\ker(q_u) = \{u\}$. $\Box$

The following two lemmas show that in any non-empty ideal I in P there is always a safe face for P.

Lemma 10.6 *Let I be a non-empty ideal in P. There is always a unary face $u \in I - \gamma(P)$.*

Proof. Suppose not, and let c be a cell of minimal dimension and $<^+$- minimal in I. If c is not unary then this contradicts condition 3., as c is of minimal dimension in I and hence $\delta(c) \cap I = \emptyset$. If $c \in \gamma(P)$, then this contradicts the choice of c as, if $\gamma(b) = c$, then $\delta(b)$ contains only unary faces, and as c is in I, by 3., $\delta(b) \subseteq I$. Thus $c \in (I \cap U(P)) - \gamma(P)$. $\Box$

Lemma 10.7 *Let I be a non-empty ideal in P. There is always a face $u \in I$ safe for P.*

Proof. Take a unary face u of maximal dimension in I. If $u \in \delta(v)$ with $v \in U(P)$, then, since I satisfies 2., $v \in I \cap U(P)$. This contradicts the choice of u, as $dim(u) < dim(v)$. $\Box$

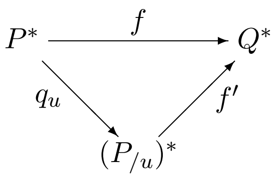
Theorem 10.8 *If $I \subseteq P_{\geq 1}$ is an ideal, then there is an inner epi map $q_I : P^* \to (P_{/I})^*$ such that it has I as its kernel and q_I is moreover a universal map with this property, i.e., whenever there is an ω-functor $f : P^* \to Q^*$ such that $I \subseteq \ker(f)$, then there is a unique map $f' : (P_{/I})^* \to Q^*$ such that $f = f' \circ q_I$.*

Proof. We can divide the opetope P by a unary cell $u \in I - \gamma(I)$ of maximal dimension, getting the map

$$q_u : P^* \to (P_{/u})^*.$$

Then $q_u(I - \{u\})$, the image of $I - \{u\}$ in $P_{/u}$, is an ideal in $P_{/u}$. Thus we can iterate the construction until the resulting ideal will be empty.

To see that q_I has the stated universal property, it is enough to notice that if $u \in \ker(f)$ is a safe face for P, then we have a factorization



Using the description of q_u, this is clear. □

Note that for an ω-functor $f : P^* \to Q^*$ it might seem that to say 'that it is mono (or epi)' is ambiguous, since this can be applied to either just faces of P or all the cells of P^*. However, in both cases the notions of epi and of mono coincide. Thus, in fact, they are not ambiguous no matter how these notions are interpreted.

Lemma 10.9 *Let $f : P^* \to Q^*$ an ω-functor in* $\mathbf{pOpeCard}_\omega$. *Then* $\ker(f) = \emptyset$ *iff f is mono.*

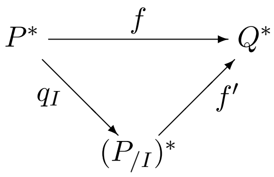
Proof. Suppose that f is not a mono. Let $a, b \in P_m$, $m \in \omega$, be two different cells of minimal dimension such that $f(a) = f(b)$. If $m = 0$, then $a \bowtie^+ b$ by linearity of $<^+$ on P_0, and if $m > 0$, then since

$$f(\gamma(a)) = \gamma(f(a)) = \gamma(f(b)) = f(\gamma(b)),$$

and by minimality of m, we have that $\gamma(a) = \gamma(b)$. Then by pencil linearity we have $a \bowtie^+ b$, as well. Suppose $a <^+ b$. Thus there is an upper path $a, \alpha_1, \ldots, \alpha_k, b$ in P. As $f(a) = f(b)$, we have $f(\alpha_i) = f(a)$ †, and hence, $\alpha_i \in \ker(f)$ for $i =, 1, \ldots, k$, i.e., $\ker(f) \neq \emptyset$. □

Theorem 10.10 *The inner epis and inner monos form a factorization system on the category* $\mathbf{pOpeCard}_{inn}$ *of opetopic cardinals with inner maps.* ‡

Proof. Let $f : P^* \to Q^*$ be an inner map. Then, by Lemma 10.4, $\ker(f) = I$ is an ideal. By the universal property of q_I stated in Theorem 10.8 we have a factorization



with q_I inner epi. Since $\ker(f) = I = \ker(q_I)$, it follows that $\ker(f') = \emptyset$. Moreover, f' is inner since f is. □

† It is a peculiarity of the polygraphs of the form P^* that non-identity cells have non-intersecting domains and codomains.

‡ Putting together the two factorisations, we have thus that an ω-functor f in $\mathbf{pOpeCard}_\omega$ decomposes as $f_3 \circ f_2 \circ f_1$ with f_1 inner epi, f_2 inner mono and f_3 an outer map, and where f_1(resp. f_2) is non trivial iff f_1 maps at least one generator to an identity (resp. f_2 maps at least one generator to a cell that is neither an identity nor a generator).

11 The terminal positive-to-one polygraph

In this section we shall describe the terminal positive-to-one polygraph $\mathcal{T}$ as an ω-category.

The set of n-cells $\mathcal{T}_n$ consists of (isomorphisms classes of) positive opetopic cardinals of dimension less than or equal to n. For $n > 0$, the operations of domain and codomain $d^{\mathcal{T}}, c^{\mathcal{T}} : \mathcal{T}_n \to \mathcal{T}_{n-1}$ are given, for $S \in \mathcal{T}_n$, by [†]

$$d(S) = \begin{cases} S & \text{if } dim(S) < n \\ \mathbf{d}(S) & \text{if } dim(S) = n, \end{cases}$$

and

$$c(S) = \begin{cases} S & \text{if } dim(S) < n \\ \mathbf{c}(S) & \text{if } dim(S) = n. \end{cases}$$

and, for $S, S' \in \mathcal{T}_n$ such that $c^{(k)}(S) = d^{(k)}(S')$, the composition in $\mathcal{T}$ is just the special pushout

$$S' \circ_k S = S' \oplus_k S,$$

i.e.,

$$\begin{array}{ccc} S' \oplus_k S & \longleftarrow & S \\ \uparrow & & \uparrow \mathbf{c}^{(k)} \\ S' & \underset{\mathbf{d}^{(k)}}{\longleftarrow} & \mathbf{c}^{(k)}(S) \end{array}$$

The identity $id_{\mathcal{T}} : T_{n-1} \to \mathcal{T}_n$ is the inclusion map. The n-indeterminates in $\mathcal{T}$ are positive opetopic cardinals of dimension n.

Proposition 11.1 *$\mathcal{T}$ just described is the terminal positive-to-one polygraph.*

Proof. The fact that $\mathcal{T}$ is an ω-category is easy. The fact that $\mathcal{T}$ is free with free n-indeterminates being opetopes of dimension n can be shown much like the freeness of S^* in the proof of Proposition 9.1. The fact that $\mathcal{T}$ is terminal relies on the following observation.

Observation. For every pair of parallel positive opetopic cardinals of dimension n, N and B (i.e., $\mathbf{d}(N) = \mathbf{d}(B)$ and $\mathbf{c}(N) = \mathbf{c}(B)$) such that B is principal, it follows that N is normal and there is a unique (up to an iso) principal positive opetopic cardinal $N^\bullet$ of dimension $n+1$ such that $\mathbf{d}(N)^\bullet = N$ and $\mathbf{c}(N)^\bullet = B$. [‡]

The universal property of $\mathcal{T}$ is then established by induction on the dimension. Let P be a positive-to-one polygraph and let f be a morphism from P to $\mathcal{T}$. Then $f(\alpha)$ has to be a generator, which is uniquely determined by induction and by the observation. □

Lemma 11.2 *Let S be a positive opetopic cardinal and $! : S^* \longrightarrow \mathcal{T}$ the unique map from S^* to $\mathcal{T}$. Then, for $T \in S^*_k$, we have*

$$!_k(T) = T.$$

† An equivalent description consists in setting $d^{\mathcal{T}} = \mathbf{d}^{(n-1)}$ and $c^{\mathcal{T}} = \mathbf{c}^{(n-1)}$ (cf. Section 6).

‡ That N is normal follows from point 1. of Lemma 7.2 and from the fact that the codomain of N is principal, being the codomain of B. Also, following up with the comment on the uniqueness of $N^\bullet$ (Section 7), we also have, for parallel N and B as above, that in fact $B = (\mathbf{d}N)^\bullet$.

Proof. The proof is by induction on $k \in \omega$ and the size of T in S^*_k. For $k = 0, 1$, the lemma is obvious. Let $k > 1$ and assume that the lemma holds for $i < k$.

If $dim(T) = l < k$, then, using the inductive hypothesis and the fact that ! is an ω-functor, we have

$$!_k(T) = !_k(1^{(k)}_T) = 1^{(k)}_{!_l(T)} = 1^{(k)}_T = T.$$

Suppose that $dim(T) = k$ and T is principal. As ! is a polygraph map, $!_k(T)$ is an indeterminate, and thus principal. Using again the inductive hypothesis and the fact that ! is an ω-functor, we obtain

$$d(!_k(T)) = !_{k-1}(\mathbf{d}(T)) = \mathbf{d}(T)$$

$$c(!_k(T)) = !_{k-1}(\mathbf{c}(T)) = \mathbf{c}(T).$$

As T is the only (up to a unique iso) positive opetopic cardinal with the domain $\mathbf{d}(T)$ and the codomain $\mathbf{c}(T)$, it follows that $!_k(T) = T$, as required.

Finally, suppose that $dim(T) = k$, T is not principal, and that the lemma holds for all positive opetopic cardinals of size smaller than the size of T. Thus there are $l \in \omega$ and $a \in Sd(T)_l$ so that

$$!_k(T) = !_k(T^{\downarrow a} \oplus_l T^{\uparrow a}) = !_k(T^{\downarrow a}) \oplus_l !_k(T^{\uparrow a}) = T^{\downarrow a} \oplus_l T^{\uparrow a} = T,$$

as required □

12 A description of the positive-to-one polygraphs

In this section we shall describe all the cells in positive-to-one polygraphs using positive opetopic cardinals, in other words we shall describe in concrete terms the functor:

$$\overline{(-)}^n : Set \downarrow D_{n-1} \longrightarrow \mathbf{pPoly}_n.$$

More precisely, the positive-to-one polygraphs of dimension 1 (and all polygraphs, as well) are free polygraphs over graphs and are well understood. So suppose that $n > 1$, and we are given an object of $Set \downarrow D_{n-1}$, i.e., a quadruple $(|P|_n, P, d, c)$ such that

1. a positive-to-one $(n-1)$-polygraph P;
2. a set $|P|_n$ with two functions $c : |P|_n \longrightarrow |P|_{n-1}$ and $d : |P|_n \longrightarrow P_{n-1}$ such that, for $x \in |P|_n$, $cc(x) = cd(x)$ and $dc(x) = dd(x)$; we assume that $d(x)$ is not an identity, for any $x \in |P|_n$.

If the maps d and c in the object $(|P|_n, P, d, c)$ are understood from the context, we can abbreviate the notation to $(|P|_n, P)$.

Recall that for a positive opetopic cardinal S, with $dim(S) \leq n$, we denote by $S^{\sharp,n}$ the object $(S_n, (S_{<n})^*, [\delta], [\gamma])$ in $Set \downarrow D_{n-1}$. In fact, we have an obvious functor

$$(-)^{\sharp,n} : \mathbf{pOpeCard} \longrightarrow Set \downarrow D_{n-1}$$

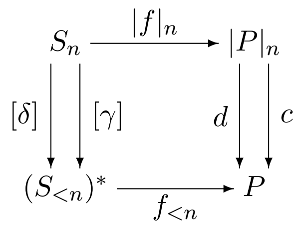
$$S \mapsto (S_n, (S_{<n})^*, [\delta], [\gamma]).$$

We shall describe the positive-to-one n-polygraph $\overline{P} = \overline{(|P|_n, P, d, c)}^n$ whose $(n-1)$-truncation is P and whose n-indeterminates are $|P|_n$ with the domains and codomains given by maps c and d.

n-cells of $\overline{P}$. An n-cell in $\overline{P}_n$ is a(n equivalence class of) pair(s) (S, f), which we shall call *auxiliary descriptions*, where

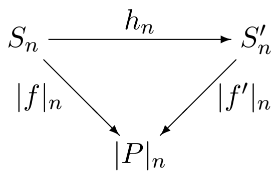
1. S is a positive opetopic cardinal, $dim(S) \leq n$;

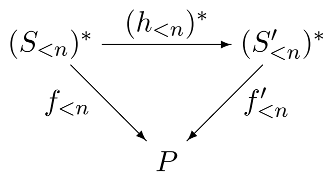
2. $f : (S_n, (S_{<n})^*, [\delta], [\gamma]) \longrightarrow (|P|_n, P, d, c)$ is a morphism in $Set \downarrow D_{n-1}$, i.e., $f = (|f|_n, f_{<n})$, and



commutes.

We identify two pairs (S, f), (S', f') if there is an isomorphism $h : S \longrightarrow S'$ such that the triangles of sets and of $(n-1)$-polygraphs





commute. Clearly, if such an h exists, it is unique. Even if formally cells in P_n are equivalence classes of triples, we will work on triples themselves as if they were cells understanding that equality between such cells is an isomorphism in the sense defined above.

Domains and codomains. The domain and codomain functions

$$d^{(k)}, c^{(k)} : \overline{P}_n \longrightarrow \overline{P}_k$$

are defined for an n-cell (S, f) as follows:

$$d^{(k)}(S, f) = (\mathbf{d}^{(k)}(S), \mathbf{d}^{(k)} f)$$

where, for $x \in (\mathbf{d}^{(k)}(S))_k$

$$(\mathbf{d}^{(k)} f)_k(x) = f_k([x])(x)$$

(i.e., we take the positive opetopic cardinals $[x]$ contained in S, then the value of f on it, and then we evaluate the map in $Set \downarrow D_{n-1}$ on x, the only element of $[x]_k$),

$$(\mathbf{d}^{(k)} f)_l = f_l$$

for $l < k$;

$$c^{(k)}(S, f) = (\mathbf{c}^{(k)}(S), \mathbf{c}^{(k)}(f))$$

where, for $x \in (\mathbf{c}^{(k)}(S))_k$

$$(\mathbf{c}^{(k)}(f))_k(x) = f_k([x])(x)$$

and

$$(\mathbf{d}^{(k)}(f))_l = f_l$$

for $l < k$, i.e., we calculate the k-th domain and k-th codomain of an n-cell (S, f) by taking $\mathbf{d}^{(k)}$ and $\mathbf{c}^{(k)}$ of the domain S of the cell f, respectively, and by restricting the maps f accordingly.

Identities. The identity function

$$\mathbf{i} : \overline{P}_{n-1} \longrightarrow \overline{P}_n$$

is defined, for an $(n-1)$-cell $((S,f)$ in P_{n-1}, as follows:

$$\mathbf{i}(S,f) = \begin{cases} (S,f) & \text{if } dim(S) < n-1 \\ (S,\overline{f}) & \text{if } dim(S) = n-1. \end{cases}$$

Note that $\overline{f}$ is the map $\mathbf{pPoly}_{n-1}$ which is the value of the functor $\overline{(-)}$ on a map f from $Set \downarrow D_{n+1}$. So it is in fact defined as 'the same $(n-1)$-cell' but considered as an n-cell.

Compositions. Suppose that (S^i, f^i) are n-cells for $i = 0, 1$ such that

$$c^{(k)}(S^0, f^0) = d^{(k)}(S^1, f^1).$$

Then their composition is defined, via pushout in $Set \downarrow D_{n-1}$, as

$$(S^1, f^1) \circ_k (S^0, f^0) = (S^1 \oplus_k S^0, [f^1, f^0]),$$

i.e.,

$$\begin{array}{ccc} S^0_n \sqcup S^1_n & \xrightarrow{\quad [f^0_n, f^1_n] \quad} & |P|_n \\ {\scriptstyle [\delta]}\downarrow\,\downarrow{\scriptstyle [\gamma]} & & {\scriptstyle d}\downarrow\,\downarrow{\scriptstyle c} \\ ((S^0 \oplus_k S^1)_{\leq n-1})^*_{n-1} & \xrightarrow[\quad [f^0_{n-1}, f^1_{n-1}] \quad]{} & P_{n-1} \end{array}$$

This ends the description of the polygraph $\overline{P}$.

Now let $h : P \to Q$ be a morphism in $Set \downarrow D_{n-1}$, i.e., a function $h_n : |P|_n \longrightarrow |Q|_n$ and a $(n-1)$-polygraph morphism $h_{<n} : P_{<n} \longrightarrow Q_{<n}$ such that the square

$$\begin{array}{ccc} |P|_n & \xrightarrow{\quad h_n \quad} & |Q|_n \\ {\scriptstyle d}\downarrow\,\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\,\downarrow{\scriptstyle c} \\ P_{n-1} & \xrightarrow[\quad h_{n-1} \quad]{} & Q_{n-1} \end{array}$$

commutes serially. We define

$$\bar{h} : \bar{P} \longrightarrow \bar{Q}$$

by putting $\bar{h}_k = h_k$, for $k < n$, and, for $(S,f) \in \bar{P}_n$, we put

$$\bar{h}(S,f) = (S, h \circ f).$$

Notation. Let $x = (S,f)$ be a cell in $\bar{P}_n$ as above, and $a \in Sd(S)$. Then we denote by $x^{\downarrow a} = (S^{\downarrow a}, f^{\downarrow a})$ and $x^{\uparrow a} = (S^{\uparrow a}, f^{\uparrow a})$ the cells in $\overline{P}_n$ that are the obvious restrictions of x. Clearly, we have $c^{(k)}(x^{\downarrow a}) = d^{(k)}((x^{\uparrow a})$ and $x = x^{\uparrow a} \circ_k x^{\downarrow a}$, where $k = dim(a)$.

In the following proposition, we collect several statements concerning the above construction. This includes that the above construction is correct. We need to prove them together, that is, by simultaneous induction.

Proposition 12.1 *Let $n \in \omega$. We have*

1. *Let P be an object of Set $\downarrow D_n$. We define the function*

$$\eta_P : |P|_n \longrightarrow \overline{P}_n$$

as follows. Let $x \in |P|_n$. By induction, there is a unique description (as defined in Section 9)

$$< T_{d(x)}, \tau_{d(x)} : T^*_{d(x)} \longrightarrow P_{<n} >$$

of the cell $d(x)$, where $T_{d(x)}$ is a normal positive opetopic cardinal.[†]

Then we have a unique auxiliary description in $\overline{P}$

$$\bar{x} = ((T_{d(x)})^\bullet,\ (|\overline{\tau}_x|_n : \{(T_{d(x)})^\bullet\} \to |P|_n,\ (\overline{\tau}_x)_{<n} : ((T_{d(x)})^\bullet)^*_{<n} \to P_{<n}))$$

(note: $|(T_{d(x)})^\bullet|_n = \{(T_{d(x)})^\bullet\}$) such that

$$|\overline{\tau}_x|_n((T_{d(x)})^\bullet) = x$$

and

$$(\overline{\tau}_x)_{n-1}(S) = \begin{cases} c(x) & \text{if } S = \mathbf{c}((T_{d(x)})^\bullet) \\ (\tau_{dx})_{n-1}(S) & \text{if } S \subseteq T_{dx} \end{cases}$$

and $(\overline{\tau}_x)_{<(n-1)} = (\tau_{dx})_{<(n-1)}$. We put $\eta_P(x) = \bar{x}$.

Then $\overline{P}$ is a positive-to-one polygraph with η_P the inclusion of n-indeterminates. Then any positive-to-one n-polygraph Q is equivalent to a polygraph $\overline{P}$, for some P in Set $\downarrow D_{n-1}$.

2. *Let P be an object of Set $\downarrow D_{n-1}$, $! : \overline{P} \longrightarrow \mathcal{T}$ the unique morphism into the terminal object $\mathcal{T}$ and $f : S^{\sharp,n} \to P$ a cell in $\overline{P}_n$. Then*

$$!_n(f : S^{\sharp,n} \to P) = S.$$

3. *Let $h : P \to Q$ be an object of Set $\downarrow D_{n-1}$. Then $\bar{h} : \overline{P} \longrightarrow \overline{Q}$ is a polygraph morphism.*

4. *Let S be a positive opetopic cardinal of dimension at most n. For a morphism $f : S^{\sharp,n} \longrightarrow P$ in Set $\downarrow D_{n-1}$, we have that*

$$\overline{f}_k(T) = f \circ (i_T)^{\sharp,n}$$

where $k \leq n$, $T \in S^*_k$ and $i_T : T \longrightarrow S$ is the inclusion.

5. *Let S be a positive opetopic cardinal of dimension n, P a positive-to-one polygraph, $g, h : S^* \longrightarrow P$ polygraph maps. Then*

$$g = h \qquad \text{iff} \qquad g_n(S) = h_n(S).$$

6. *Let S be a positive opetopic cardinal of dimension at most n, P be an object in Set $\downarrow D_{n-1}$. Then we have a bijective correspondence*

$$\frac{f : S^{\sharp,n} \longrightarrow P \quad \in \text{Set} \downarrow D_{n-1}}{\overline{f} : S^* \longrightarrow \overline{P} \quad \in \mathbf{pPoly}_n}$$

such that $\overline{f}_n(S) = f$, and, for $g : S^* \longrightarrow \overline{P}$, we have $g = \overline{g_n(S)}$.[‡]

[†] This follows from the observation in the proof of Proposition 11.1 and from the following one: since by definition of positive-to-one polygraphs $d(x)$ is not an identity cell, it follows that $T_{d(x)}$ has the same dimension as $T_{c(x)}$ which is principal and parallel to it.

[‡] The correspondence $f \mapsto \overline{f}$ shows the equivalence between descriptions and auxiliary descriptions (a terminology introduced in the revision). Systematically unfolding an auxiliary description yields the following informal third description: a cell of a positive-to-one polygraph is a positive opetopic cardinal all of whose faces are (consistently) decorated by generators.

7. *We have a bijection*

$$\kappa^P_n : \coprod_S \mathbf{pPoly}(S^*, \overline{P}) \longrightarrow \overline{P}_n$$

$$g : S^* \to \overline{P} \quad \mapsto \quad g_n(S)$$

where the coproduct is taken over all (up to iso) positive opetopic cardinals S of dimension at most n. In other words, any cell in $\bar{P}$ has a unique description.

Proof. Ad 1. We have to verify that $\overline{P}$ satisfies the laws of ω-categories and that it is free in the appropriate sense.

The laws for ω-categories are left for the reader, as they easily follow from the fact that S^* is a positive to one-polygraph for any positive opetopic cardinal S. We shall show that $\overline{P}$ is free in the appropriate sense.

Let C be an ω-category, $g_{<n} : P_{<n} \to C_{<n}$ and $(n-1)$-functor and $g_n : |P|_n \to C_n$ a function so that the diagram

$$\begin{array}{ccc} |P|_n & \xrightarrow{\quad g_n \quad} & C_n \\ d \Big\downarrow \Big\downarrow c & & d \Big\downarrow \Big\downarrow c \\ P_{n-1} & \xrightarrow[\quad g_{n-1} \quad]{} & C_{n-1} \end{array}$$

commutes serially. We shall define an n-functor $\overline{g} : \overline{P} \to C$ extending $g_{<n}$ and g_n. For $x = (S, f) \in \overline{P}_n$, we put

$$\overline{g}_n(x) = \begin{cases} 1_{g_{n-1} \circ f_{n-1}(S)} & \text{if } dim(S) < n \\ g_n \circ f_n(m_S) & \text{if } dim(S) = n,\ S \text{ is principal, } S_n = \{m_S\} \\ \overline{g}_n(x^{\uparrow a}) \circ_k \overline{g}_n(x^{\downarrow a}) & \text{if } dim(S) = n,\ a \in Sd(S)_k. \end{cases}$$

We need to check that $\overline{g}$ is well defined, that it is unique extending g, and that it preserves domains, codomains, compositions and identities.

All these calculations are similar, and they are very much like those in the proof of Proposition 9.1 and use facts from Section 8. We shall check, assuming that we already know that $\overline{g}$ is well defined and preserves identities, that compositions are preserved. The proof is by induction on the size of the composition and uses Lemma 8.7. So let T, T_1, T_2 be positive opetopic cardinals such that $T = T_2 \oplus_k T_1$. Since $\overline{g}$ preserves identities, we can restrict our attention to the case $dim(T_1), dim(T_2) > k$.

Fix $a \in \mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1)$. So $a \in Sd(T)_k$. If $T_1 = T^{\downarrow a}$ and $T_2 = T^{\uparrow a}$, then we have

$$\overline{g}(T) = \overline{g}(T^{\uparrow a}) \circ_k \overline{g}(T^{\downarrow a}) = \overline{g}(T_2) \circ_k \overline{g}(T_1).$$

If $a \in Sd(T_1)_k$, then

$$\begin{aligned} \overline{g}(T) = \overline{g}(T^{\uparrow a}) \circ_k \overline{g}(T^{\downarrow a}) = \\ = (\overline{g}(T_2) \circ_k \overline{g}(T_1^{\uparrow a})) \circ_k \overline{g}(T^{\downarrow a}) = \\ = \overline{g}(T_2) \circ_k (\overline{g}(T_1^{\uparrow a}) \circ_k \overline{g}(T_1^{\downarrow a})) = \\ \overline{g}(T_2) \circ_k \overline{g}(T_1). \end{aligned}$$

The remaining verifications are similar.

Ad 2. Let $! : \overline{P} \longrightarrow \mathcal{T}$ be the unique polygraph map into the terminal object, S a positive opetopic cardinal such that $dim(S) = l \leq n$, $f : S^{\sharp,n} \longrightarrow P$ a cell in $\overline{P}_n$.

If $l < n$, then by induction we have $!_n(f) = S$. If $l = n$ and S is principal, then we have, by induction

$$!_n(d(f) : (\mathbf{d}(S))^{\sharp,n} \to P) = \mathbf{d}(S) \qquad !_n(c(f) : (\mathbf{c}(S))^{\sharp,n} \to P) = \mathbf{c}(S).$$

As f is an indeterminate in $\overline{P}$, $!_n(f)$ is a positive opetope. But the only (up to an iso) positive opetope B such that

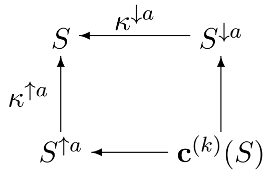
$$\mathbf{d}(B) = \mathbf{d}(S) \qquad \mathbf{d}(B) = \mathbf{d}(S)$$

is S itself. Thus, in this case, $!_n(f) = S$.

Now assume that $l = n$, and S is not principal, and that for positive opetopic cardinals T of smaller size than S the statement holds. Let $a \in Sd(S)_k$. We have

$$!_n(f) = f^{\uparrow a}) \circ_k !_n(f^{\downarrow a} = !_n(f^{\uparrow a}) \oplus_k !_n(f^{\downarrow a}) = S^{\uparrow a} \oplus_k S^{\downarrow a} = S$$

where $f^{\downarrow a} = f \circ (\kappa^{\downarrow a})^{\sharp,n}$ and $f^{\uparrow a} = f \circ (\kappa^{\uparrow a})^{\sharp,n}$ and $\kappa^{\downarrow a}$ and $\kappa^{\uparrow a}$ are the maps as in the following pushout:



Ad 3. The main thing is to show that $\overline{h}$ preserves compositions. This follows from the fact that the functor

$$(-)^{\sharp,n} : \mathbf{pOpeSet}_n \longrightarrow Set \downarrow D_{n-1}$$

preserves special pullbacks.

Ad 4. This is an immediate consequence of 3.

Ad 5. Let S be a positive opetopic cardinal S of dimension at most n. To prove 5., we are going to use the auxiliary description of the n-cells in positive-to-one polygraphs given in 1. Moreover, note that by 3. and Lemma 11.2 we have that for $T \in S^*_k$, the value of g at T is a map in $Set \downarrow D_k$ such that $g_k(T) : T^{\sharp,k} \longrightarrow U_k(P)$, i.e., the domain of $g_k(T)$ is necessarily $T^{\sharp,k}$.

The implication $\Rightarrow$ is obvious. So assume that $g, h : S^* \longrightarrow P$ are different polygraph maps. Then there is $k \leq n$ and $x \in S_k$ such that $g_k([x]) \neq h_k([x])$. We shall show, by induction on the size of T, that for any $T \in S^*_l$ such that $x \in T$, we have

$$g_k(T) \neq h_k(T). \tag{6}$$

If $T = [x]$, then T has the least size among those positive opetopic cardinals that contain x, and (6) holds in this case by assumption.

Suppose that (6) holds for all $U \in S^*_{l'}$ whenever $l' < l$ and $x \in U$. Suppose that $T = [y]$, for some $y \in S_l$, and $x \in [y]$. Then either $x \in \mathbf{d}[y]$ or $x \in \mathbf{c}([y])$. In the former case we have, by inductive hypothesis, that $g_k(\mathbf{d}T) \neq h_k(\mathbf{d}T)$. Thus

$$d(g_k(T)) = g_k(\mathbf{d}T) \neq h_k(\mathbf{d}T) = d(g_k(T)).$$

But then (6) holds as well. The latter case ($x \in \mathbf{c}([y])$) is similar.

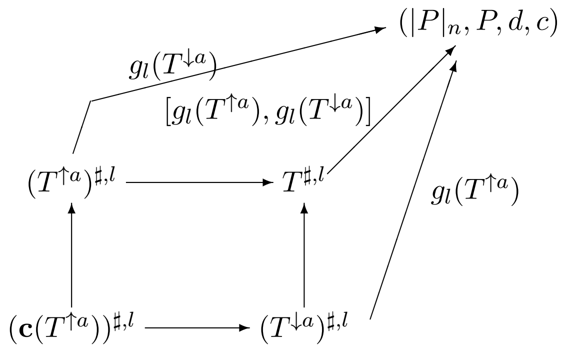
Now suppose that T is not principal $x \in T$ and that, for U of a smaller size with $x \in U$, the condition (6) holds. Let $a \in Sd(T)_r$. Then either $x \in T^{\downarrow a}$ or $x \in T^{\uparrow a}$. Both cases are similar, so we will consider the first one only. Thus, as $T^{\downarrow a}$ has a smaller size than T, by inductive hypothesis we have

$$g_k(T^{\downarrow a}) \neq h_k(T^{\downarrow a}). \tag{7}$$

As the compositions in P are calculated via pushouts, we have that

$$g_l(T^{\uparrow a}) \circ_r g_l(T^{\downarrow a}) = [g_l(T^{\uparrow a}), g_l(T^{\downarrow a})],$$

where $[g_l(T^{\downarrow a}), g_l(T^{\uparrow a})]$ is the unique morphism from the pushout as in the following diagram:



Similarly

$$h_l(T^{\uparrow a}) \circ_r h_l(T^{\downarrow a}) = [h_l(T^{\uparrow a}), h_l(T^{\downarrow a})].$$

As morphisms from the pushout are equal if and only if both of their components are equal we have

$$g_l(T) = g_l(T^{\uparrow a} \oplus_r T^{\downarrow a}) = g_l(T^{\uparrow a}) \circ_r g_l(T^{\downarrow a}) =$$

$$[g_l(T^{\uparrow a}), g_l(T^{\downarrow a})] \neq [h_l(T^{\uparrow a}), h_l(T^{\downarrow a})] =$$

$$= h_l(T^{\uparrow a}) \circ_r h_l(T^{\downarrow a}) = h_l(T^{\uparrow a} \oplus_r T^{\downarrow a}) = h_l(T).$$

Thus (6) holds for all $T \in S^*$ such that $x \in T$. As $x \in S$, we get that

$$g_n(S) \neq h_n(S),$$

as required.

Ad 6. Fix a positive opetopic cardinal S of dimension at most n.

Let $f : S^{\sharp,n} \longrightarrow P$ be a cell in $\overline{P}_n$. By 4., we have

$$\overline{f}_n(S) = f \circ (i_S)^{\sharp,n} = f \circ (1_S)^{\sharp,n} = f \circ (1_S^{\sharp,n}) = f.$$

Let $g : S^* \longrightarrow \overline{P}$ be a polygraph map. To show that $g = \overline{g_n(S)}$, by 5., it is enough to show that

$$(\overline{g_n(S)})_n(S) = g_n(S).$$

Using 4. again, we have

$$(\overline{g_n(S)})_n(S) = g_n(S) \circ (i_S)^{\sharp,n} =$$

$$= g_n(S) \circ i_{S^{\sharp,n}} = g_n(S) \circ 1_{S^{\sharp,n}} = g_n(S).$$

Thus, by 5., $(\overline{g_n(S)}) = g$.

Ad 7. It follows immediately from 6. $\Box$

From Proposition 12.1.7 we know that each cell in a positive-to-one polygraph has (up to an isomorphism) a unique description. The following proposition is a bit more specific.

Proposition 12.2 *Let P be a positive-to-one polygraph, $n \in \omega$, and $a \in P_n$. Let T_a be $!^P_n(a)$ (where $!^P : P \longrightarrow \mathcal{T}$ is the unique morphism into the terminal polygraph). Then there is a unique polygraph map $\tau_a : T^*_a \longrightarrow P$ such that $(\tau_a)_n(T_a) = a$, i.e., each cell has an essentially unique description. Moreover:*

1. *For any $a \in P$, we have*

$$\tau_{da} = d(\tau_a) = \tau_{da} = \tau_a \circ (\mathbf{d}_{T_a})^* \qquad \tau_{c(a)} = c(\tau_a) = \tau_{c(a)} = \tau_a \circ (\mathbf{c}_{T_a})^*$$

$$\tau_{1_a} = \tau_a.$$

2. *For any $a, b \in P$ such that $c^{(k)}(a) = d^{(k)}(b)$, we have*

$$\tau_{a;_k b} = [\tau_a, \tau_b] : T^*_a +_{\mathbf{c}^{(k)} T^*_a} T^*_b \longrightarrow P.$$

3. *For any positive opetopic cardinal S, for any polygraph map $f : S^* \longrightarrow P$,*

$$\overline{\tau_{f_n(S)}} = f.$$

4. *For any positive opetopic cardinal S, any ω-functor $f : S^* \longrightarrow P$ can be essentially uniquely factorized as*

$$\begin{array}{ccc} S^* & \xrightarrow{\quad f \quad} & P \\ & {\scriptstyle f^{in}}\searrow \qquad \nearrow{\scriptstyle \tau_{f(S)}} & \\ & T^*_{f(S)} & \end{array}$$

where f^{in} is an inner map and $(\tau_{f(S)}, T_{f(S)})$ is the description of the cell $f(S)$.

Proof. Using the above description of the positive-to-one polygraph P, we have that $a : (T_a)^{\sharp,n} \longrightarrow (|P|_n, P, d, c)$. We put $\tau_a = \overline{a}$. By Proposition 12.1.6, we have that $(\tau_a)_n(T_a) = \overline{a}_n(T_a) = a$, as required.

The uniqueness of (T_a, τ_a) follows from Proposition 12.1 point 5.

The remaining part is left for the reader. □

13 Positive-to-one polygraphs form a presheaf category

In this section we want to prove that the category **pPoly** is equivalent to the presheaf category $\widehat{\mathbf{pOpe}}$. In fact, we will show that both categories are equivalent to the category $sPb((\mathbf{pOpeCard})^{op}, Set)$ of special pullbacks preserving functors from $(\mathbf{pOpeCard})^{op}$ to Set.

First note that the inclusion functor $\mathbf{i} : \mathbf{pOpe} \longrightarrow \mathbf{pOpeCard}$ induces the adjunction

$$\widehat{\mathbf{pOpe}} \quad \underset{\mathbf{i}^*}{\overset{Ran_{\mathbf{i}}}{\rightleftarrows}} \quad \widehat{\mathbf{pOpeCard}}$$

where $\mathbf{i}^*$ is the functor of composing with $\mathbf{i}$ and $Ran_{\mathbf{i}}$ is the right Kan extension along $\mathbf{i}$. Recall that for F in $\widehat{\mathbf{pOpe}}$, S in $\mathbf{pOpeCard}$, it is defined as the following limit

$$(Ran_{\mathbf{i}} F)(S) = Lim(F \circ \Sigma^{S,op})$$

where $\Sigma^{S,op}$ is the dual of the functor Σ^S defined before Lemma 9.6. Note that, as $(\mathbf{pOpe} \downarrow S)^{op} = S \downarrow (\mathbf{pOpe})^{op}$, we have

$$\Sigma^{S,op} : S \downarrow (\mathbf{pOpe})^{op} \longrightarrow (\mathbf{pOpe})^{op}.$$

As $\mathbf{i}$ is full and faithful, the right Kan extension $Ran_{\mathbf{i}}(F)$ is an extension. Therefore the counit of this adjunction

$$\varepsilon_F : (Ran_{\mathbf{i}}\, F) \circ \mathbf{i} \longrightarrow F$$

is an isomorphism. The functor $Ran_{\mathbf{i}}F$ is so defined that it preserves special limits. Hence, by Lemma 9.6, it preserves special pullbacks. As any positive opetopic cardinal can be constructed from positive opetopes via special pushouts, for G in $\widehat{\mathbf{pOpeCard}}$, the unit

$$\eta_G : G \longrightarrow Ran_{\mathbf{i}}(G \circ \mathbf{i})$$

is an isomorphism iff G preserves special pullbacks. This establishes the following proposition.

Proposition 13.1 *The above adjunction restricts to the following equivalence of categories*

$$\widehat{\mathbf{pOpe}} \underset{\mathbf{i}^*}{\overset{Ran_{\mathbf{i}}}{\rightleftarrows}} sPb((\mathbf{pOpeCard})^{op}, Set).$$

□

Now we will set up the adjunction

$$sPb((\mathbf{pOpeCard})^{op}, Set) \underset{\widehat{(-)} = \mathbf{pPoly}((\simeq)^*, -)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathbf{pPoly}$$

which will be later proved to be an equivalence of categories. The functor $\widehat{(-)}$ sends a positive-to-one polygraph P to the functor

$$\widehat{P} = \mathbf{pPoly}((-)^*, P) : (\mathbf{pOpeCard})^{op} \longrightarrow Set.$$

The action of $\widehat{(-)}$ on morphisms is defined in the obvious way, by composition.

Lemma 13.2 *Let P be a positive-to-one polygraph. Then $\widehat{P}$ defined above is a special pullbacks preserving functor.*

Proof. This is an immediate consequence of the fact that the functor $(-)^*$ preserves special pushouts. □

Now suppose we have a special pullbacks preserving functor F : $(\mathbf{pOpeCard})^{op} \longrightarrow Set$. We shall define a positive-to-one polygraph $\widetilde{F}$.

As n-cells of $\widetilde{F}$ we put

$$\widetilde{F}_n = \coprod_S F(S)$$

where the coproduct is taken over all[2] (up to iso) positive opetopic cardinals S of dimension at most n.

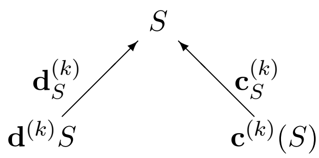
If $k \leq n$, the identity map

$$1^{(n)} : \widetilde{F}_k \longrightarrow \widetilde{F}_n$$

[2]In fact, we think about such a coproduct $\coprod_S F(S)$ as if it were to be taken over a sufficiently large (so that each isomorphism type of positive opetopic cardinals is represented) set of positive opetopic cardinals S of dimension at most n. Then, if positive opetopic cardinals S and S' are isomorphic via (necessarily unique) isomorphism h, then the cells $x \in F(S)$ and $x' \in F(S')$ are considered equal iff $F(h)(x) = x'$.

is the obvious embedding induced by identity maps on the components of the coproducts.

Now we shall describe the domains and codomains in $\widetilde{F}$. Let S be a positive opetopic cardinal of dimension at most n, $a \in F(S) \hookrightarrow \widetilde{F}_n$. We have in **pOpeCard** the k-th domain and the k-th codomain morphisms:



We put

$$\mathbf{d}^{(k)}(a) = F(\mathbf{d}^{(k)}_S)(a) \in F(\mathbf{d}^{(k)}(S)) \hookrightarrow \widetilde{F}_k$$
$$\mathbf{c}^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) \in F(\mathbf{c}^{(k)}(S)) \hookrightarrow \widetilde{F}_k.$$

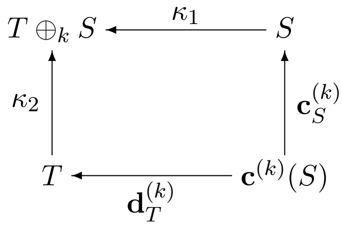
Finally, we define the compositions in $\widetilde{F}$. Let $n_1, n_2 \in \omega$, $n = max(n_1, n_2)$, $k < min(n_1, n_2)$, and

$$a \in F(S) \hookrightarrow \widetilde{F}_{n_1} \qquad b \in F(T) \hookrightarrow \widetilde{F}_{n_2},$$

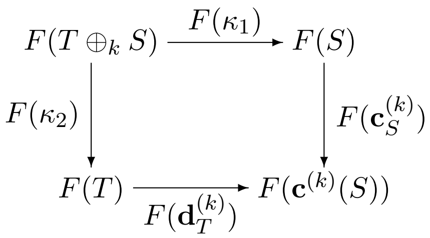
such that

$$\mathbf{c}^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) = F(\mathbf{d}^{(k)}_T)(b) = \mathbf{d}^{(k)}(b).$$

We shall define the cell $b \circ_k a \in \widetilde{F}_n$. We take a special pushout in **pOpeCard**:



As F preserves special pullbacks (in $(\mathbf{pOpeCard})^{op}$), it follows that the square



is a pullback in Set. Thus there is a unique element

$$x \in F(S \oplus_k T) \hookrightarrow \widetilde{F}_n$$

such that

$$F(\kappa_1)(x) = a \qquad F(\kappa_2)(x) = b.$$

We put

$$b \circ_k a = x.$$

This ends the definition of $\widetilde{F}$.

For a morphism $\alpha : F \longrightarrow G$ in $sPb((\mathbf{pOpeCard})^{op}, Set)$, we put

$$\widetilde{\alpha} = \{\widetilde{\alpha}_n : \widetilde{F}_n \longrightarrow \widetilde{G}_n\}_{n \in \omega}$$

such that

$$\widetilde{\alpha}_n = \coprod_S \alpha_S : \widetilde{F}_n \longrightarrow \widetilde{G}_n,$$

where the coproduct is taken over all (up to iso) positive opetopic cardinals S of dimension at most n. This ends the definition of the functor $\widetilde{(-)}$.

Proposition 13.3 *The functor*

$$\widetilde{(-)} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow \mathbf{pPoly}$$

is well defined.

Proof. The verification that $\widetilde{(-)}$ is a functor into ωCat is left for the reader. We shall verify that, for any special pullbacks preserving functor $F : \mathbf{pOpeCard}^{op} \longrightarrow Set$, $\widetilde{F}$ is a positive-to-one polygraph whose n-indeterminates are

$$|\widetilde{F}|_n = \coprod_{B \in \mathbf{pOpe}, dim(B)=n} F(B) \hookrightarrow \coprod_{S \in \mathbf{pOpeCard}, dim(S) \leq n} F(S) = \widetilde{F}_n.$$

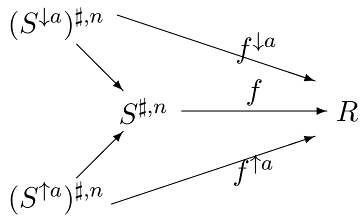
We call R the object of $Set \downarrow D_{n-1}$ defined by $R = (|\widetilde{F}|_n, \widetilde{F}_{<n}, \mathbf{d}^{(n-1)}, \mathbf{c}^{(n-1)})$. We shall show that $\widetilde{F}_{\leq n}$ is in a bijective correspondence with $\overline{R}$ described in the previous section. We define a function

$$\varphi : \overline{R}_n \longrightarrow \widetilde{F}_n$$

as follows: for a cell $f : S^{\sharp,n} \longrightarrow R$ in $\overline{R}_n$, we put

$$\varphi(f) = \begin{cases} 1_{f_{n-1}(S)} & \text{if } dim(S) < n \\ f_n(m_S) & \text{if } dim(S) = n,\ S \text{ principal},\ S_n = \{m_S\} \\ \varphi(f^{\uparrow a}) \circ_k \varphi(f^{\downarrow a}) & \text{if } dim(S) = n,\ a \in Sd(S)_k. \end{cases}$$

and the morphisms in $\varphi(f^{\downarrow a})$ and $\varphi(f^{\uparrow a})$ in $Set \downarrow D_{n-1}$ are obtained by compositions so that the diagram



commutes. We need to verify, by induction on n, that φ is well defined, bijective and that it preserves compositions, domains, and codomains.

We shall only verify (partially) that φ is well defined, i.e., the definition φ for any positive opetopic cardinal S of dimension n that is not a positive opetope does not depend on the choice of the saddle point of S. Let $a, x \in Sd(S)$ so that $k = dim(x) < dim(a) = m$. Using Lemma 8.4 and the fact that $(-)^{\sharp,n}$ preserves special pushouts, we have

$$\begin{aligned} &\varphi(f^{\uparrow a}) \circ_m \varphi(f^{\downarrow a}) = \\ &= (\varphi(f^{\uparrow a \uparrow x}) \circ_k \varphi(f^{\uparrow a \downarrow x})) \circ_m (\varphi(f^{\downarrow a \uparrow x}) \circ_k \varphi(f^{\downarrow a \downarrow x})) = \\ &= (\varphi(f^{\uparrow a \uparrow x}) \circ_m \varphi(f^{\downarrow a \uparrow x})) \circ_k (\varphi(f^{\uparrow a \downarrow x}) \circ_m \varphi(f^{\downarrow a \downarrow x})) = \\ &= (\varphi(f^{\uparrow x \uparrow a}) \circ_m \varphi(f^{\uparrow x \downarrow a})) \circ_k (\varphi(f^{\downarrow x \uparrow a}) \circ_m \varphi(f^{\downarrow x \downarrow a})) = \\ &= \varphi(f^{\uparrow x}) \circ_m \varphi(f^{\downarrow x}), \end{aligned}$$

as required in this case. The reader can compare these calculations with those, in the same case, of Proposition 9.1 (F is replaced by φ and T is replaced by f). So there is no point to repeat the other calculations. □

For P in $\mathbf{pPoly}$, we define a polygraph map

$$\eta_P : P \longrightarrow \widetilde{\widetilde{P}}$$

so that, for $x \in P_n$, we put

$$\eta_{P,n}(x) = \tau_x : T_x^* \to P.$$

For F in $sPb((\mathbf{pOpeCard})^{op}, Set)$, we define a natural transformation

$$\varepsilon_F : \widehat{\widetilde{F}} \longrightarrow F$$

such that, for a positive opetopic cardinal S of dimension n,

$$(\varepsilon_F)_S : \widehat{\widetilde{F}}(S) \longrightarrow F(S)$$

and $g : S^* \to \widetilde{F} \in \widehat{\widetilde{F}}(S)$, we put

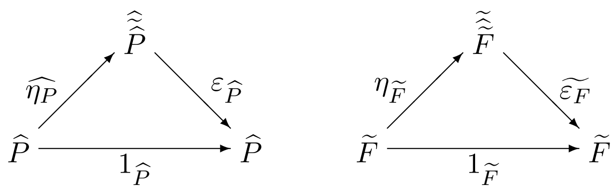
$$(\varepsilon_F)_S(g) = g_n(S).$$

Proposition 13.4 *The functors*

$$sPb((\mathbf{pOpeCard})^{op}, Set) \underset{\widehat{(-)} = \mathbf{pPoly}((\simeq)^*, -)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathbf{pPoly}$$

together with the natural transformations η and ε defined above form an adjunction $(\widehat{(-)} \dashv \widetilde{(-)})$, which is an equivalence of categories.

Proof. The fact that both η and ε are bijective on each component follows immediately from Proposition 12.1.6. So we shall verify the triangular equalities only. Let P be a polygraph, and F be a functor in $sPb((\mathbf{pOpeCard})^{op}, Set)$. We need to show that the triangles



commute. So let $f : S^* \to P \in \widehat{P}(S)$, $dim(S) = n$. Then we have

$$\varepsilon_{\widehat{P}} \circ \widehat{\eta_P}(f) = \varepsilon_{\widehat{P}}(\eta_P \circ f) = (\eta_P \circ f)_n(S) =$$

$$= (\eta_P)_n(f_n(S)) = \tau_{f_n(S)} = f.$$

So let $x \in F(S) \longrightarrow \widetilde{F}_n$. Then we have

$$\widetilde{\varepsilon_F} \circ \eta_{\widetilde{F}}(x) = \widetilde{\varepsilon_F}(\tau_x) = (\tau_x)_n(1_{T_x}) = x.$$

So both triangles commutes, as required. $\Box$

From Propositions 13.1 and 13.4 we get immediately the following corollary.

Corollary 13.5 *The functor*

$$\widehat{(-)} : \mathbf{pPoly} \longrightarrow \widehat{\mathbf{pOpe}}$$

such that, for a positive-to-one polygraph X,

$$\widehat{X} = \mathbf{pPoly}((-)^*, X) : (\mathbf{pOpe})^{op} \longrightarrow Set$$

is an equivalence of categories.

14 The principal pushouts

Recall the positive opetopic cardinals α^n from section 9. A *total composition map* is an inner ω-functor whose domain is of form $(\alpha^n)^*$ (which we also write as $\alpha^{n,*}$), for some $n \in \omega$. If S is a positive opetopic cardinal of dimension n, then the total composition of S (in fact S^*) is denoted by

$$\boldsymbol{\mu}^{S^*} : \alpha^{n,*} \longrightarrow S^*.$$

It is uniquely determined by the condition $\boldsymbol{\mu}^{S^*}_n(\alpha^n) = S$. We have the following

Proposition 14.1 *Let N be a normal positive opetopic cardinal. With the notation as above, the square*

$$\begin{array}{ccc} N^* & \xrightarrow{\mathbf{d}^*_{N^\bullet}} & N^{\bullet,*} \\ {\scriptstyle \boldsymbol{\mu}^{N^*}} \uparrow & & \uparrow {\scriptstyle \boldsymbol{\mu}^{N^{\bullet,*}}} \\ \alpha^{n,*} & \xrightarrow[\mathbf{d}^*_{\alpha^{n+1}}]{} & \alpha^{n+1,*} \end{array}$$

is a pushout in $\mathbf{pOpeCard}_\omega$.

Proof. This is an easy consequence of Proposition 7.3, particularly point 4. □

Pushouts described in the above Proposition are called *principal pushouts*.
From the above proposition we immediately get

Corollary 14.2 *If $n > 0$ and P is a positive opetope of dimension n, then the square*

$$\begin{array}{ccc} \mathbf{d}P^* & \xrightarrow{\mathbf{d}^*_{P}} & P^* \\ {\scriptstyle \boldsymbol{\mu}^{\mathbf{d}P}} \uparrow & & \uparrow {\scriptstyle \boldsymbol{\mu}^{P}} \\ \alpha^{n-1,*} & \xrightarrow[\mathbf{d}_{\alpha^{n,*}}]{} & \alpha^{n,*} \end{array}$$

is a (principal) pushout in $\mathbf{pOpeCard}_\omega$. □

Theorem 14.3 (V.Harnik) [3] *Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor. Then F preserves the principal pullbacks, as well.*

The proof of the above theorem will be divided into three Lemmas. Theorem 14.3 is a special case of Lemma 14.6, for $k = n - 1$.

Before we even formulate the next three Lemmas, we need to introduce some constructions on positive opetopic cardinals and define some ω-functors between positive opetopic cardinals. Along the way, we whall introduce some notations for them, and we shall make some comments on how they are going to be interpreted by special pullback preserving morphisms from $(\mathbf{pOpeCard}_\omega)^{op}$ to Set.

[3] The original statement of V. Harnik is saying that the nerve functor from ω-categories to (all) polygraphs is monadic. However, in the present context the argument given by V.Harnik, c.f. [H], is directly proving the present statement, i.e., that the principal pullbacks are preserved whenever the special ones are. This statement is used to show that the category of ω-categories is equivalent to the category of special pullback preserving functors from $(\mathbf{pOpeCard}_\omega)^{op}$ to Set, c.f. Corollary 15.2. From that statement, the monadicity of the nerve functor is an easy corollary, c.f. Theorem 16.5. In the remainder of this section, Harnik's argument, adopted to the present context, is presented.

Notation for presheaves. To simplify the notation concerning presheaf $F : \mathbf{pOpeCard}_\omega{}^{op} \to Set$, for a morphism $g : P \to Q \in \mathbf{pOpeCard}_\omega$, and an element $a \in F(Q)$, we will write $a \cdot g$ for $F(g)(a)$, i.e., we treat F as a family of sets with a right action of the category $\mathbf{pOpeCard}_\omega$. This convention is explained, for example, in [SGL] page 121. Such notation suppresses the name of the presheaf F but it will be always clear from the context. In this section we deal only with one presheaf named F that preserves special pullbacks. We also have $(a \cdot g) \cdot f = a \cdot (g \circ f)$ and $a \cdot id_Q = a$ whenever these formulas are well defined.

Fix $0 < k \leq n$, and a P positive opetope of dimension n. We say that P is *k-globular* iff $\mathbf{d}^{(l)}P$ is a positive opetope, for $k \leq l \leq n$, i.e., $\delta^{(l)}(\mathbf{p}_n)$ is a singleton, for $k \leq l \leq n$, where $P_n = \{\mathbf{p}_n\}$. The *k-globularization* $\boxed{k}P$ of P is the k-globular positive opetope of dimension n defined as follows. We put

$$\boxed{k}P_l = \begin{cases} \{\mathbf{p}_n\} & \text{for } l = n \\ \{\mathbf{q}_l, \mathbf{p}_l\} & \text{for } k \leq l < n \\ \delta^{(k-1)}(\mathbf{p}_n) \cup \{\mathbf{p}_{k-1}\} & \text{for } l = k-1 \\ P_l & \text{otherwise.} \end{cases}$$

For $x \in \boxed{k}P$,

$$\gamma^{\boxed{k}P}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l, \text{ for some } k \leq l < n \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

and

$$\delta^{\boxed{k}P}(x) = \begin{cases} \mathbf{q}_l & \text{if } x \in \boxed{k}P_{l+1}, \text{ for some } k < l < n \\ \delta^P(\mathbf{p}_k) & \text{if } x \in \boxed{k}P_k \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

Note that $\boxed{n}P$ is P itself and $\boxed{n}P$ is α^n. Thus the elements of the shape $\boxed{k}P^*$ are k-globularized versions of the elements of the shape P^*. As the following positive opetopic cardinals

$$\mathbf{c}^{(k)}P \;\cong\; \mathbf{c}^{(k)}\boxed{k}P \;\cong\; \mathbf{cc}^{(k+1)}\boxed{k}P \;\cong\; \mathbf{dc}^{(k+1)}\boxed{k}P \;\cong\; \mathbf{d}^{(k)}\boxed{k}P$$

are isomorphic, we can form the following special pushouts

$$\begin{array}{ccc} \mathbf{c}^{(k+1)}P^* & \xrightarrow{\kappa_1} & \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \\ {\scriptstyle \mathbf{c}^*_{\mathbf{c}^{(k+1)}P}} \uparrow & & \uparrow {\scriptstyle \kappa_2} \\ \mathbf{c}^{(k)}P^* & \xrightarrow[\mathbf{d}^{(k),*}_{\boxed{k}P}]{} & \boxed{k}P^* \end{array}$$

and

$$\begin{array}{ccc} \mathbf{c}^{(k+1)}\boxed{k}P^* & \xrightarrow{\kappa'_1} & \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \\ {\scriptstyle \mathbf{c}^*_{\mathbf{c}^{(k+1)}\boxed{k}P}} \uparrow & & \uparrow {\scriptstyle \kappa'_2} \\ \mathbf{c}^{(k)}P^* & \xrightarrow[\mathbf{d}^{(k),*}_{\boxed{k}P}]{} & \boxed{k}P^* \end{array}$$

We describe in detail the positive opetopic cardinals we have just defined. Their faces are described in the following table:

dim	$\boxed{k{+}1}P$	$\boxed{k}P$	$P' = \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$	$P'' = \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$
n	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$
$n{-}1$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k{+}1$	$\{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$	$\{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$	$\{\mathbf{r}_{k+1}, \mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$	$\{\mathbf{r}_{k+1}, \mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$
k	$\partial(\mathbf{p}_{k+1})$	$\{\mathbf{q}_k, \mathbf{p}_k\}$	$\delta(\mathbf{p}_{k+1}) \cup \{\mathbf{q}_k, \mathbf{p}_k\}$	$\{\mathbf{r}_k, \mathbf{q}_k, \mathbf{p}_k\}$
$k{-}1$	P_{k-1}	$\partial(\mathbf{p}_k)$	P_{k-1}	$\partial(\mathbf{p}_k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l	P_l	P_l	P_l	P_l

$0 \leq l < k$. The functions γ and δ in $P' = \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$ are given by the following formulas

$$\gamma^{P'}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l \text{ and } k \leq l < n \\ \mathbf{q}_k & \text{if } x = \mathbf{r}_{k+1} \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

$$\delta^{P'}(x) = \begin{cases} \{\mathbf{q}_{l-1}\} & \text{if } x = \mathbf{q}_l \text{ and } k < l < n \\ & \text{or } x = \mathbf{p}_l \text{ and } k < l \leq n \\ \delta^P(\mathbf{p}_{k+1}) & \text{if } x = \mathbf{r}_{k+1} \\ \delta^P(\mathbf{p}_k) & \text{if } x = \mathbf{q}_k \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

The functions γ and δ in $P'' = \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$ are given by the following formulas

$$\gamma^{P''}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l \text{ and } k \leq l < n \\ \mathbf{q}_k & \text{or } x = \mathbf{r}_{k+1} \\ \mathbf{p}_{k-1} & \text{or } x = \mathbf{r}_k \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

$$\delta^{P''}(x) = \begin{cases} \{\mathbf{q}_{l-1}\} & \text{if } x = \mathbf{q}_l \text{ and } k < l < n \\ & \text{or } x = \mathbf{p}_l \text{ and } k < l \leq n \\ \{\mathbf{r}_k\} & \text{if } x = \mathbf{r}_{k+1} \\ \delta^P(\mathbf{p}_k) & \text{if } x \in \{\mathbf{r}_k, \mathbf{q}_k, \mathbf{p}_k\} \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

Now we shall define some ω-functors between some positive opetopic cardinals just defined. To describe their meaning, let us fix a special pullback preserving functors from $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$.

The ω-functors denoted by letter $\boldsymbol{\mu}$ are interpreted as operation that 'globularize' cells. We have two of them. The first one

$$\boldsymbol{\mu}^{S^*} : \alpha^{n,*} \longrightarrow S^*$$

was already introduced at the beginning of this section for any positive opetopic cardinal S. The second is the ω-functor

$$\boldsymbol{\mu}_{\boxed{k}} : \boxed{k}P^* \longrightarrow \boxed{k{+}1}P^*$$

such that

$$\boldsymbol{\mu}_{\boxed{k}}(X) = \begin{cases} (X - \{\mathbf{q}_k\}) \cup \delta(\mathbf{p}_{k+1}) & \text{if } \mathbf{q}_k \in X \\ X & \text{otherwise.} \end{cases}$$

for $X \in \boxed{k}P^*$.

The fact that these operations are interpreted as globularization of cells can be explained as follows. The function

$$F(\boldsymbol{\mu}_{\boxed{k}}) : F(\boxed{k+1}P^*) \longrightarrow F(\boxed{k}P^*),$$

takes a $(k+1)$-globular n-cell $a \in F(\boxed{k+1}P^*)$ and returns a k-globular n-cell $a \cdot \boldsymbol{\mu}_{\boxed{k}} = F(\boldsymbol{\mu}_{\boxed{k}})(a) \in F(\boxed{k}P^*)$. Intuitively, $F(\boldsymbol{\mu}_{\boxed{k}})$ is composing the k-domain of a leaving the rest 'untouched'. So it is a 'one-step globularization'. On the other hand, the function

$$F(\boldsymbol{\mu}^{S^*}) : F(S^*) \longrightarrow F(\alpha^{n,*})$$

is taking an n-cell $b \in F(S^*)$ of an arbitrary shape S^* of dimension n, and it is returning a globular n-cell $b \cdot \boldsymbol{\mu}^{S^*} \in F(\alpha^{n,*})$. This time $F(\boldsymbol{\mu}^{S^*})$ is composing all the domains and codomains in the cell b as much as possible, so that there is nothing left to be composed. This is the 'full globularization'.

We need a separate notation for the ω-functor $\boldsymbol{\mu}_k : \mathbf{c}^{(k+1)}\boxed{k}P^* \longrightarrow \mathbf{c}^{(k+1)}P^*$ such that

$$\boldsymbol{\mu}_k(X) = \begin{cases} \mathbf{c}^{(k+1)}P & \text{if } X = \mathbf{c}^{(k+1)}\boxed{k}P \\ \mathbf{d}^{(k)}P & \text{if } X = \mathbf{d}^{(k)}\boxed{k}P \\ X & \text{otherwise,} \end{cases}$$

for $X \in \mathbf{c}^{(k+1)}\boxed{k}P^*$. It is a version of $\boldsymbol{\mu}_{\boxed{k}}$. The ω-functor

$$\boldsymbol{\nu}_P : P^* \longrightarrow \mathbf{d}P^*$$

is given by

$$\boldsymbol{\nu}_P(X) = \begin{cases} \mathbf{d}P & \text{if } \mathbf{c}P \subseteq X \\ X & \text{otherwise,} \end{cases}$$

for $X \in P^*$. $\boldsymbol{\nu}_P$ is a kind of degeneracy map and it is interpreted as 'a kind of identity'. For a cell $t \in F(\mathbf{d}P^*)$, $t \cdot \boldsymbol{\nu}_P \in F(P^*)$ is 'identity on t' but with the codomain composed. The ω-functor

$$\boldsymbol{\beta}_k : \mathbf{c}^{(k)}P^* \longrightarrow \mathbf{d}^{(k)}P^*$$

such that

$$\boldsymbol{\beta}_k(X) = \begin{cases} \mathbf{d}^{(k)}P & \text{if } X = \mathbf{c}^{(k)}P \\ X & \text{otherwise.} \end{cases}$$

for $X \in P^*$, is the operation of 'composition of all the cells at the top' leaving the rest untouched. The map $\boldsymbol{\beta}_{n-1}$ is equal to the composition

$$\mathbf{c}P^* \xrightarrow{\ \mathbf{c}^*_P\ } P^* \xrightarrow{\ \boldsymbol{\nu}_P\ } \mathbf{d}P^*.$$

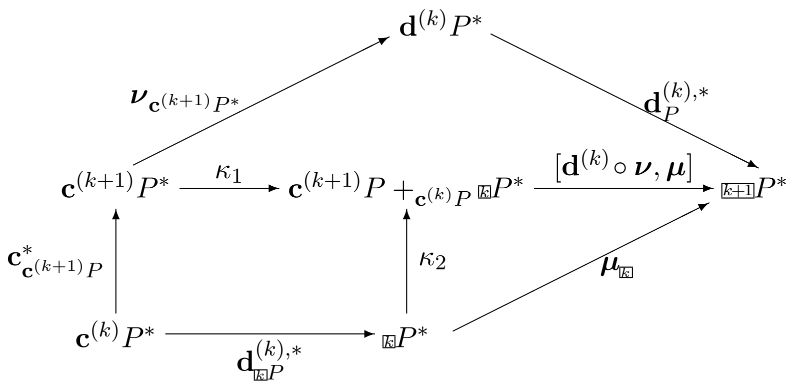
The following two ω-functors

$$[\mathbf{d}^{(k),*}_P \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P}, \boldsymbol{\mu}_P] : \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \longrightarrow P^*$$

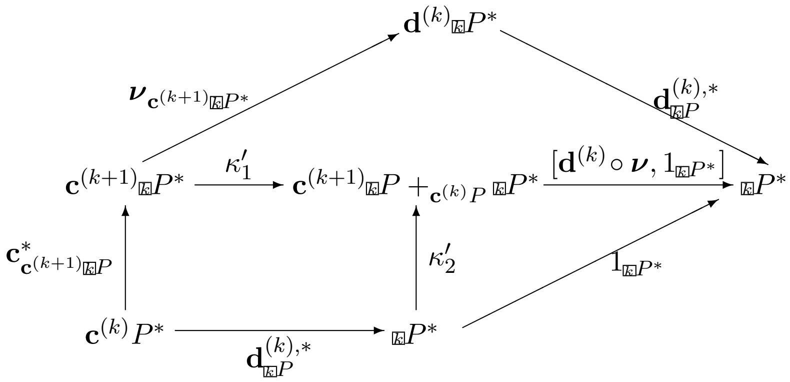
and

$$[\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P}, 1_{\boxed{k}P}] : \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \longrightarrow \boxed{k}P^*$$

are defined as the unique ω-functors making the following diagrams



and



commute in $\mathbf{pOpeCard}_\omega$.

Finally, we introduce two maps that are a kind of binary composition combined with wiskering. The first

$$\diamond_{\boxed{k+1}} : \boxed{k+1}P^* \longrightarrow \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$$

is given by

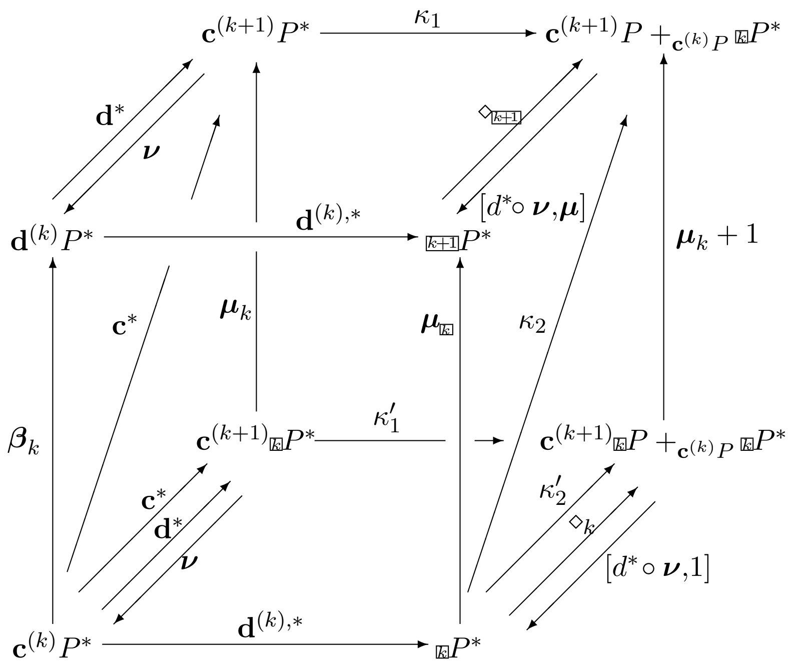
$$\diamond_{\boxed{k+1}}(X) = \begin{cases} X \cup \{\mathbf{r}_{k+1}, \mathbf{q}_k\} & \text{if } X \cap \{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\} \neq \emptyset \\ X & \text{otherwise,} \end{cases}$$

for $X \in \boxed{k+1}P^*$. The other composition map is

$$\diamond_k : \boxed{k}P^* \longrightarrow \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P,$$

given by the same defining formula as $\diamond_{\boxed{k+1}}$, for $X \in \boxed{k}P^*$.

In the following diagram all the morphisms that we introduced above are displayed. Most of the subscripts of the morphisms are suppressed for clarity of the picture.



The above cube contains two special pushouts mentioned above. The following lemma describes some other commutations.

Lemma 14.4 *With the notation as above we have, for $k \geq 1$,*

1. $\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \mathbf{c}^*_{\mathbf{c}^{(k+1)}P} = \boldsymbol{\beta}_k$,
2. $\kappa_1 \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = (\diamond_{\boxed{k+1}}) \circ \mathbf{d}^{(k),*}_P$,
3. $\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = 1_{\mathbf{d}^{(k)}P^*}$,
4. $(\diamond_{\boxed{k+1}}) \circ \boldsymbol{\mu}_{\boxed{k}} = (\boldsymbol{\mu}_k + 1_{\boxed{k}P}) \circ (\diamond_k)$,
5. $\boldsymbol{\beta}_k \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*} = \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \boldsymbol{\mu}_k$,
6. $[\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, 1_{\boxed{k}P^*}] \circ (\diamond_k) = 1_{\boxed{k}P^*}$,
7. $[\mathbf{d}^{(k),*}_P \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, \boldsymbol{\mu}_{\boxed{k}}] \circ (\diamond_{\boxed{k+1}}) = 1_{\boxed{k+1}P^*}$.

Proof. Routine check. □

Lemma 14.5 *Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a positive opetope of dimension n. Then, for any $0 \leq k < n$, F preserves the pullback in* $(\mathbf{pOpeCard}_\omega)^{op}$

$$\begin{array}{ccc} \mathbf{d}^{(k)}P^* & \xrightarrow{\mathbf{d}^{(k),*}_{\boxed{k+1}P}} & \boxed{k+1}P^* \\ \uparrow{\scriptstyle \boldsymbol{\beta}_k} & & \uparrow{\scriptstyle \boldsymbol{\mu}_{\boxed{k}}} \\ \mathbf{c}^{(k)}P^* & \xrightarrow[\mathbf{d}^{(k),*}_{\boxed{k}P}]{} & \boxed{k}P^* \end{array}$$

Proof. Let $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a positive opetope of dimension n. We need to show that the square

$$
\begin{array}{ccc}
F(\mathbf{d}^{(k)}P^*) & \xleftarrow{F(\mathbf{d}^{(k),*}_{\boxed{k+1}P})} & F(\boxed{k+1}P^*) \\
\Big\downarrow F(\boldsymbol{\beta}_k) & & \Big\downarrow F(\boldsymbol{\mu}_{\boxed{k}}) \\
F(\mathbf{c}^{(k)}P^*) & \xleftarrow[F(\mathbf{d}^{(k),*}_{\boxed{k}P})]{} & F(\boxed{k}P^*)
\end{array}
$$

is a pullback in Set. Let us fix $t \in F(\mathbf{d}^{(k)}P^*)$ and $a \in F(\boxed{k}P^*)$ such that

$$t \cdot \boldsymbol{\beta}_k = a \cdot \mathbf{d}^{(k),*}_{\boxed{k}P}.$$

We will check that it is a pullback, by showing existence and uniqueness of an element $b \in F(\boxed{k+1}P^*)$ such that

$$b = t \cdot \mathbf{d}^{(k),*}_{\boxed{k+1}P} \quad \text{and} \quad b = a \cdot \boldsymbol{\mu}_{\boxed{k}}.$$

Existence. Put $t' = t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}$. By Lemma 14.4.1, we have

$$t' \cdot \mathbf{c}^*_{\mathbf{c}^{(k+1)}P} = t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \cdot \mathbf{c}^*_{\mathbf{c}^{(k+1)}P} = t \cdot \boldsymbol{\beta}_k = a \cdot \mathbf{d}^{(k),*}_{\boxed{k}P}.$$

Since F preserves special pullbacks and $\mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$ is a special pullback in $(\mathbf{pOpeCard}_\omega)^{op}$, we have an element

$$\langle t', a\rangle \in F(\mathbf{c}^{(k+1)}P) \times_{F(\mathbf{c}^{(k)}P)} F(\boxed{k}P) \cong F(\mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P)$$

such that

$$\langle t', a\rangle \cdot \kappa_1 = t' \quad \text{and} \quad \langle t, a\rangle \cdot \kappa_2 = a.$$

We put $b = \langle t', a\rangle \cdot \diamond_{\boxed{k+1}} \in F(\boxed{k+1}P^*)$. We have

$$
\begin{aligned}
b \cdot \mathbf{d}^{(k),*}_{\boxed{k+1}P} &= (\text{def of } b) \\
&= (\langle t', a\rangle \cdot \diamond_{\boxed{k+1}}) \cdot \mathbf{d}^{(k),*}_{\boxed{k+1}P} = (F \text{ presheaf}) \\
&= \langle t', a\rangle \cdot (\diamond_{\boxed{k+1}} \circ \mathbf{d}^{(k),*}_{\boxed{k+1}P}) = (\text{Lemma 14.4.2}) \\
&= \langle t', a\rangle \cdot (\kappa_1 \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) = (F \text{ presheaf}) \\
&= (\langle t', a\rangle \cdot \kappa_1) \cdot \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = (F \text{ pres. special pb's}) \\
&= t' \cdot \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = (F \text{ pres. special pb's, def } t') \\
&= (t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}) \cdot \mathbf{d}^*_{\mathbf{c}^{(k+1)}P} = (F \text{ presheaf}) \\
&= t \cdot (\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) = (\text{Lemma 14.4.3}) \\
&= t \cdot 1_{\mathbf{d}^{(k)}P^*} = (F \text{ presheaf}) \\
&= t
\end{aligned}
$$

and

$$
\begin{aligned}
b \cdot \boldsymbol{\mu}_{\boxed{k}} &= (\text{def of } b) \\
&= (\langle t', a\rangle \cdot \diamond_{\boxed{k+1}}) \cdot \boldsymbol{\mu}_{\boxed{k}} = (F \text{ presheaf}) \\
&= \langle t', a\rangle \cdot (\diamond_{\boxed{k+1}} \circ \boldsymbol{\mu}_{\boxed{k}}) = (\text{Lemma 14.4.4}) \\
&= \langle t', a\rangle \cdot ((\boldsymbol{\mu}_k + 1) \circ \diamond_k) = (F \text{ presheaf}) \\
&= (\langle t', a\rangle \cdot (\boldsymbol{\mu}_k + 1)) \cdot \diamond_k = (F \text{ pres. special pb's}) \\
&= \langle t' \cdot \boldsymbol{\mu}_k, a\rangle \cdot \diamond_k = (\text{def } t')
\end{aligned}
$$

$$= \langle (t \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}) \cdot \boldsymbol{\mu}_k, a \rangle \cdot \diamond_k = (F \text{ presheaf})$$

$$= \langle (t \cdot (\boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*} \circ \boldsymbol{\mu}_k), a \rangle \cdot \diamond_k = (\text{Lemma 14.4.5})$$

$$= \langle (t \cdot (\boldsymbol{\beta}_k \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}), a \rangle \cdot \diamond_k = (F \text{ presheaf})$$

$$= \langle (t \cdot \boldsymbol{\beta}_k) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, a \rangle \cdot \diamond_k = (\text{assumption on } a \text{ and } t)$$

$$= \langle (a \cdot \mathbf{d}^{(k),*}_{\boxed{k}P}) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, a \rangle \cdot \diamond_k = (F \text{ presheaf})$$

$$= \langle a \cdot (\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}), a \rangle \cdot \diamond_k = (F \text{ pres. special pb's})$$

$$= (a \cdot [\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, 1_{\boxed{k}P^*}]) \cdot \diamond_k = (F \text{ presheaf})$$

$$= a \cdot ([\mathbf{d}^{(k),*}_{\boxed{k}P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}\boxed{k}P^*}, 1_{\boxed{k}P^*}] \circ \diamond_k) = (\text{Lemma 14.4.6})$$

$$= a \cdot 1_{\boxed{k}P^*} = (F \text{ presheaf})$$

$$= a.$$

Uniqueness. Now suppose that we have two elements $b, b' \in F(\boxed{k+1}P^*)$ such that $\mathbf{d}^{(k)}(b) = t = \mathbf{d}^{(k)}(b')$ and $\boldsymbol{\mu}(b) = a = \boldsymbol{\mu}(b')$. Then, using Lemma 14.4 7 and the assumption, we have (we won't mention that we use the fact that F is a sheaf anymore)

$$b = (\text{Lemma 14.4.7})$$

$$= b \cdot ([\mathbf{d}^{(k),*}_{P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, \boldsymbol{\mu}_{\boxed{k}}] \circ (\ \diamond_{\boxed{k+1}})) =$$

$$= \langle (b \cdot \mathbf{d}^{(k),*}_{P}) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, b \cdot \boldsymbol{\mu}_{\boxed{k}} \rangle \cdot (\ \diamond_{\boxed{k+1}}) = (\text{assumption on } b \text{ and } b')$$

$$= \langle (b' \cdot \mathbf{d}^{(k),*}_{P}) \cdot \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, b' \cdot \boldsymbol{\mu}_{\boxed{k}} \rangle \cdot (\ \diamond_{\boxed{k+1}}) =$$

$$= b' \cdot ([\mathbf{d}^{(k),*}_{P} \circ \boldsymbol{\nu}_{\mathbf{c}^{(k+1)}P^*}, \boldsymbol{\mu}_{\boxed{k}}] \circ (\ \diamond_{\boxed{k+1}})) = (\text{Lemma 14.4.7})$$

$$= b'.$$

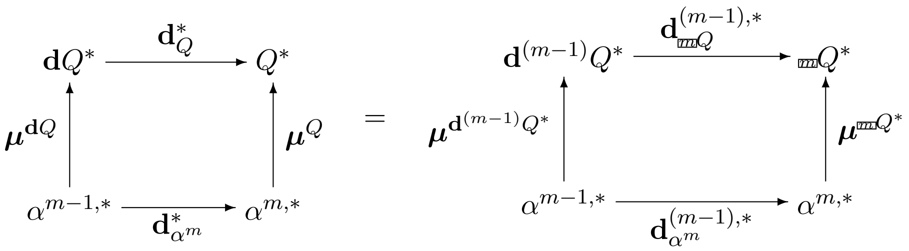
So the element with these properties is unique. □

Lemma 14.6 *Let* $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ *be a special pullback preserving functor,* P *a positive opetope of dimension* n*. Then, for any* $0 \leq k < n$*,* F *preserves the following pullback in* $(\mathbf{pOpeCard}_\omega)^{op}$

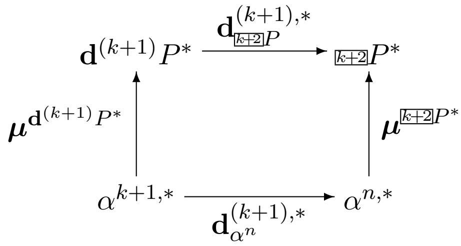
$$\begin{array}{ccc} \mathbf{d}^{(k)}P^* & \xrightarrow{\ \mathbf{d}^{(k),*}_{\boxed{k+1}P}\ } & \boxed{k+1}P^* \\ {\scriptstyle \boldsymbol{\mu}^{\mathbf{d}^{(k)}P^*}} \uparrow & & \uparrow {\scriptstyle \boldsymbol{\mu}^{\boxed{k+1}P^*}} \\ \alpha^{k,*} & \xrightarrow[\ \mathbf{d}^{(k),*}_{\alpha^n}\ ]{} & \alpha^{n,*} \end{array} \qquad (8)$$

Proof. The proof is by double induction, on the dimension n of the positive opetopic P, and $k < n$. Note that if $k = 0$, then, for any $n > 0$, the vertical arrows in (8) are isomorphisms, so any functor from $(\mathbf{pOpeCard}_\omega)^{op}$ sends (8) to a pullback. This shows in particular that the Lemma holds for $n = 1$. As we already mentioned, if $k = n - 1$, the square (8) is an arbitrary special pushout.

Thus, we assume that $F : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$ is a special pullback preserving functor, and that P is a positive opetope of dimension n, $0 \leq k < n$, F preserves the pullback (8). Moreover, for $m < n$ and the positive opetope Q of dimension m, F preserves the principal pullback in $(\mathbf{pOpeCard}_\omega)^{op}$:

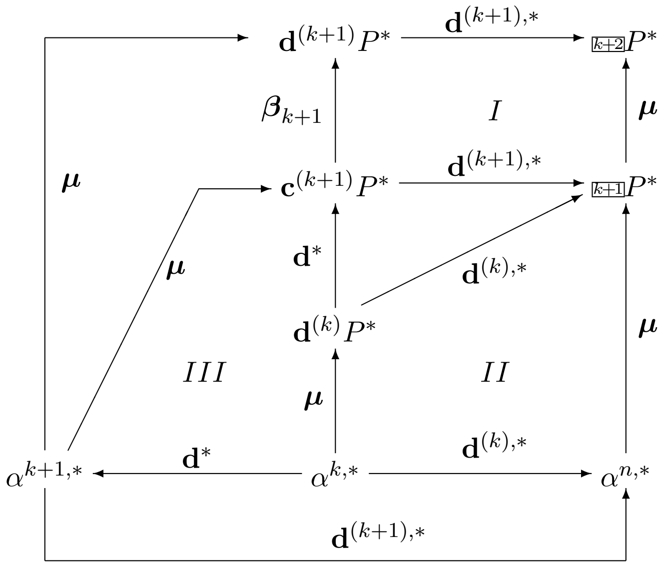


We shall show that F preserves the pullback



(9)

in $(\mathbf{pOpeCard}_\omega)^{op}$, as well. In the following diagram (most of the subscripts and some superscripts were suppressed for clarity):



all the squares and triangles commute. Moreover, F sends the squares I, II, III to pullbacks in Set: I by Lemma 14.5, II by inductive hypothesis for k, III by inductive hypothesis since $dim(\mathbf{c}^{(k+1)}P) < n$.

Let $f : X \longrightarrow F(\mathbf{d}^{(k+1)}P^*)$ and $g : X \longrightarrow F(\alpha^{n,*})$ be functions such that

$$F(\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}) \circ f = F(\mathbf{d}^{(k+1),*}_{\alpha^n}) \circ g.$$

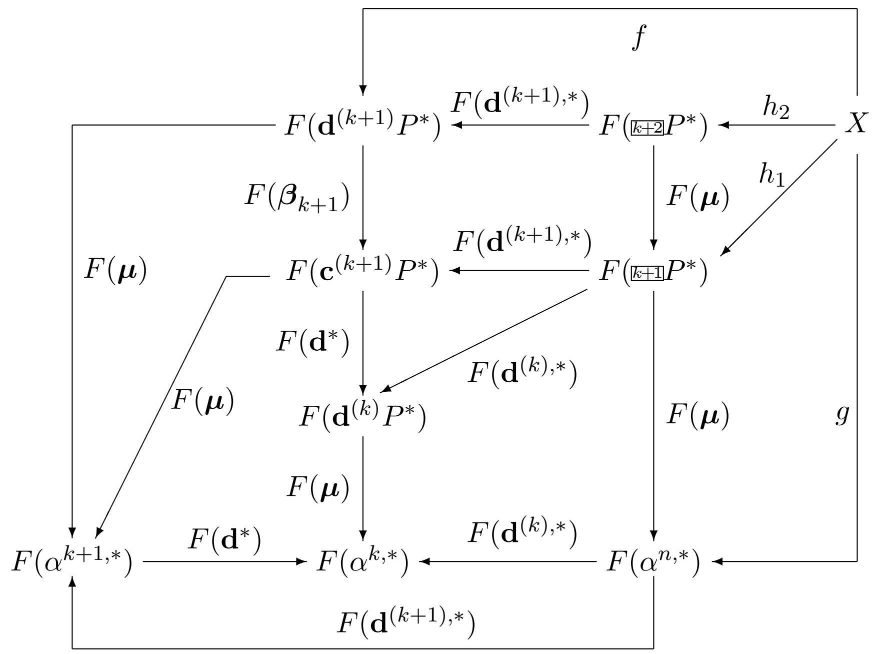
Since F applied to II is a pullback in Set, and all squares and triangles in the above diagram commute, there is a unique function $h_1 : X \longrightarrow F(\boxed{k+1}P^*)$ such that

$$F(d^{(k),*}_{\boxed{k+1}P}) \circ h_1 = F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f \quad \text{and} \quad F(\boldsymbol{\mu}^{\boxed{k+1}P^*}) \circ h_1 = g. \qquad (10)$$

To get a unique function $h_2 : X \longrightarrow F(\boxed{k+2}P^*)$ such that

$$F(\mathbf{d}^{k+1,*}_{\boxed{k+2}P}) \circ h_2 = f \quad \text{and} \quad F(\boldsymbol{\mu}_{\boxed{k+1}}) \circ h_2 = h_1, \qquad (11)$$

we use the fact that F sends III to a pullback in Set. The application of F to the diagram above will give the following diagram in Set, where we added the additional functions f, g, h_1, and h_2:



Thus in order to verify that

$$F(\boldsymbol{\beta}_{k+1}) \circ f = F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1$$

and to get h_2 satisfying (11), it is enough to verify that

$$F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f = F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1 \tag{12}$$

and

$$F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f = F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1. \tag{13}$$

For (12), we have

$$\begin{aligned} &F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f = \\ &= F(\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}) \circ f = \\ &= F(\mathbf{d}^{(k+1),*}_{\alpha^n}) \circ g = \\ &= F(\mathbf{d}^{(k+1),*}_{\alpha^n}) \circ F(\boldsymbol{\mu}^{\boxed{k+1}P^*}) \circ h_1 = \\ &= F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \circ F(\mathbf{d}^{(k+1)^*}_{\boxed{k+1}P}) \circ h_1, \end{aligned}$$

and for (13), we have

$$\begin{aligned} &F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\boldsymbol{\beta}_{k+1}) \circ f = \\ &= F(d^{(k),*}_{\boxed{k+1}P}) \circ h_1 = \\ &= F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \circ F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}) \circ h_1. \end{aligned}$$

By uniqueness of both h_1 and h_2, h_2 is the unique function such that

$$\mathbf{d}^{(k+1),*}_{\boxed{k+2}P} \circ h_2 = f \quad \text{and} \quad \boldsymbol{\mu}^{\boxed{k+2}\mathrm{P}^*} \circ \mathrm{h}_2 = \mathrm{g},$$

i.e., F sends (9) to a pullback in Set, as required. □

15 A full nerve of ω-categories

Let $\mathcal{S}$ denote the category of simple ω-categories, described in [MZ]. It was proved there that any simple ω-category is isomorphic to one of form $(\alpha^{\vec{u}})^*$ for some ud-vector $\vec{u}$. In fact what we need here is that any simple ω-category can be obtained from those of form $(\alpha^n)^*$, with $n \in \omega$, via special pushouts. For more details the reader should consult [MZ].

As every simple ω-category is a positive opetopic cardinal (considered as an ω-category), we have a full inclusion functor

$$\mathbf{k} : \mathcal{S} \longrightarrow \mathbf{pOpeCard}_\omega$$

whose essential image is spanned by the opetopic cardinals all of whose faces are globular.

In [MZ] we have shown that $sPb(\mathcal{S}^{op}, Set)$, the category of special pullbacks preserving functors from the dual of $\mathcal{S}$ to Set, is equivalent to the category ω-categories. We have in fact an adjoint equivalence

$$\omega Cat \underset{\widehat{(-)} = \omega Cat(\simeq, -)}{\overset{\widetilde{(-)}}{\rightleftarrows}} sPb(\mathcal{S}^{op}, Set)$$

where

$$\widehat{C} : \mathcal{S}^{op} \longrightarrow Set$$

is given by

$$\widehat{C}(A) = \omega Cat(A, C),$$

where A is a simple ω-category.

Proposition 15.1 *The adjunction*

$$\widehat{\mathcal{S}} \underset{\mathbf{k}^*}{\overset{Ran_{\mathbf{k}}}{\rightleftarrows}} \widehat{\mathbf{pOpeCard}_\omega}$$

restricts to an equivalence of categories.

$$sPb(\mathcal{S}^{op}, Set) \underset{\mathbf{k}^*}{\overset{Ran_{\mathbf{k}}}{\rightleftarrows}} sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

where $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ is the category of the special pullbacks preserving functors.

Proof. First we shall describe the adjunction in details.

The counit. Let G be a functor in $sPb(\mathcal{S}^{op}, Set)$ and A be a simple ω-category. We have a functor

$$(\mathbf{k} \downarrow A)^{op} \xrightarrow{\pi^A} \mathcal{S}^{op} \xrightarrow{G} Set$$

with the limit, say $(Lim(G \circ \pi^A), \sigma^A)$. Then the counit $(\varepsilon_G)_A$ is

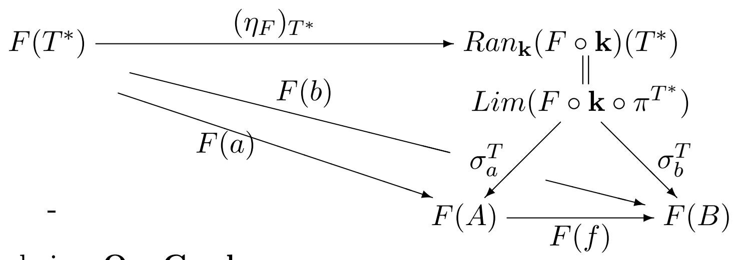
$$(\varepsilon_G)_A : (Ran_{\mathbf{k}}(G) \circ \mathbf{k})(A) = Lim(G \circ \pi^A) \xrightarrow{\sigma^A_{1_A}} G(A)$$

As $\mathbf{k}$ is full and faithful[4], for any G, ε_G is an iso. Thus ε is an iso.

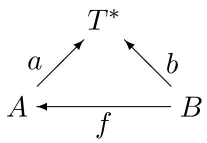
The unit. Let F be a functor in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$, T a positive opetopic cardinal. We have a functor

$$(\mathbf{k} \downarrow T^*)^{op} \xrightarrow{\pi^{T^*}} \mathcal{S}^{op} \xrightarrow{\mathbf{k}} \mathbf{pOpeCard}_\omega \xrightarrow{F} Set$$

with the limit, say $(Lim(F \circ \mathbf{k} \circ \pi^{T^*}), \sigma^{T^*})$. Then the unit $(\eta_F)_{T^*}$ is the unique morphism into the limit:



where the triangle in $\mathbf{pOpeCard}_\omega$



commutes.

Note that, as F preserves special pullbacks, and any simple ω-category can be obtained from those of form α^n with $n \in \omega$, we can restrict the limiting cone $(Lim(F \circ \mathbf{k} \circ \pi^{T^*}), \sigma^{T^*})$ to the objects of form α^n, with $n \in \omega$.

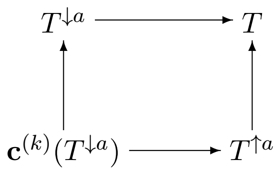
After this observation we shall prove, by induction on the size of a positive opetopic cardinal T, that $(\eta_F)_{T^*}$ is an iso.

If $dim(T) \leq 1$, then $(\eta_F)_{T^*}$ is obviously an iso.

Suppose T is not principal, i.e., we have $a \in Sd(T)$, for some $k \in \omega$. By inductive hypothesis the morphisms

$$(\eta_F)_{(T^{\downarrow a})^*}, \quad (\eta_F)_{\mathbf{c}^{(k)}(T^{\downarrow a})^*}, \quad (\eta_F)_{(T^{\uparrow a})^*}$$

are isos, and the square



is a special pushout (see Proposition 6.2) which is sent by F to a pullback. Hence the morphism

$$(\eta_F)_{T^*} = (\eta_F)_{(T^{\downarrow a})^*} \times (\eta_F)_{(T^{\uparrow a})^*}$$

is indeed an iso in this case.

If T is principal and $T = (\alpha^n)^*$, then the category $(\mathbf{k} \downarrow (\alpha^n)^*)^{op}$ has the initial object $1_{(\alpha^n)^*}$, so the morphism

$$(\eta_F)_{(\alpha^n)^*} : F((\alpha^n)^*) \longrightarrow Ran_{\mathbf{k}}(F \circ \mathbf{k})((\alpha^n)^*)$$

is an iso.

Finally, let us assume that $T (= P)$ is any positive opetope of dimension n. Thus, by Corollary 14.2, we have a principal pushout

[4]This condition translates to the fact that 1_A is the initial object in $(\mathbf{k} \downarrow A)^{op}$ and therefore that we have an iso $\sigma^A_{1_A} : Lim(G \circ \pi^A) \cong G \circ \pi^A(1_A) = G(A)$.

$$
\begin{array}{ccc}
(\mathbf{d}P)^* & \xrightarrow{\ \mathbf{d}^*_P\ } & P^* \\
\Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}P}} & & \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{P}} \\
(\alpha^{n-1})^* & \xrightarrow[\ \mathbf{d}^*_{\alpha^n}\ ]{} & (\alpha^n)^*
\end{array}
$$

which, by Theorem 14.3, is preserved by F. By induction hypothesis the morphisms

$$(\eta_F)_{(\mathbf{d}P)^*} \qquad (\eta_F)_{(\alpha^{n-1})^*} \qquad (\eta_F)_{(\alpha^n)^*}$$

are isos, so we have that the morphism

$$(\eta_F)_{P^*} = (\eta_F)_{(\mathbf{d}P)^*} \times (\eta_F)_{(\alpha^n)^*}$$

is an iso, as well. $\Box$

Corollary 15.2 *We have a commuting triangle of adjoint equivalences*

$$
\begin{array}{ccc}
\omega Cat & & \\
\widehat{(-)}\Big\downarrow\Big\uparrow\widetilde{(-)} & \searrow^{\widehat{(-)}} \ \nwarrow_{\widetilde{(-)}} & \\
sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) & \underset{Ran_{\mathbf{k}}}{\overset{\mathbf{k}^*}{\rightleftarrows}} & sPb(\mathcal{S}^{op}, Set)
\end{array}
$$

In particular, the categories ωCat and $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ are equivalent.

Proof. It is enough to show that in the above diagram $\mathbf{k}^* \circ \widehat{(-)} = \widehat{(-)}$. But this is clear. $\Box$

16 A monadic adjunction

In this section we show that the inclusion functor $\mathbf{e} : \mathbf{pPoly} \longrightarrow \omega Cat$ has a right adjoint which is monadic.

First we will give an outline of the proof. Consider the following diagram of categories and functors

$$
\begin{array}{ccc}
\mathbf{pPoly} & \xrightarrow{\ \ \mathbf{e}\ \ } & \omega Cat \\
\widehat{(-)}\Big\downarrow\Big\uparrow\widetilde{(-)} & & \widehat{(-)}\Big\downarrow\Big\uparrow\widetilde{(-)} \\
sPb((\mathbf{pOpeCard})^{op}, Set) & \underset{\mathbf{j}^*}{\overset{Lan_{\mathbf{j}}}{\rightleftarrows}} & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)
\end{array}
$$

where $\mathbf{e}$ is just an inclusion of positive-to-one polygraphs into ω-categories and $\mathbf{j} = (-)^* : \mathbf{pOpeCard} \longrightarrow \mathbf{pOpeCard}_\omega$ is the essentially surjective inclusion functor. We have already shown (Proposition 13.4, Corollary 15.2) that the vertical functors constitute two adjoint equivalences. The proof that $\mathbf{e}$ has a right adjoint takes a few steps. We begin by presenting $Lan_{\mathbf{j}}$ as a familially representable functor (or local right adjoint). Then we check that $Lan_{\mathbf{j}}$ is well defined, i.e., we will check that the functor $Lan_{\mathbf{j}}(F)$, the left Kan extension of special pullbacks preserving functor F, preserves special pullbacks. Next we shall check that the above square commutes, i.e., $\widehat{(-)} \circ \mathbf{e} = Lan_{\mathbf{j}} \circ \widehat{(-)}$. This will reduce the problem of monadicity of ωCat over $\widehat{\mathbf{pOpe}}$ to verification whether $\mathbf{j}^*$, the left adjoint to $Lan_{\mathbf{j}}$, is monadic. The last statement is verified directly checking assumptions of Beck's monadicity theorem.

We describe the left Kan extension along $\mathbf{j}$ in a convenient way, c.f. [CWM], as a familially representable functor.

Proposition 16.1 *The functor of left Kan extension*

$$Lan_{\mathbf{j}} : \widehat{\mathbf{pOpeCard}} \longrightarrow \widehat{\mathbf{pOpeCard}}_\omega$$

along the functor

$$\mathbf{j} : \mathbf{pOpeCard} \to \mathbf{pOpeCard}_\omega$$

is defined, for $F \in \widehat{\mathbf{pOpeCard}}$, as follows. For a positive opetopic cardinal S, we have

$$Lan_{\mathbf{j}}(F)(S^*) = \coprod_{a:S^* \to T^* \text{ inner}} F(T) \xleftarrow{\kappa_a^{S^*}} F(T)$$

where the coproduct is taken over all up to iso inner maps in $\mathbf{pOpeCard}_\omega$ with the domain S^*, with the coprojections as shown.

If $h : S_1^* \longrightarrow S_2^*$ is an ω-functor and $a_2 : S_2^* \to T_2^*$ is inner, by Lemma 10.3, we can form a diagram

$$\begin{array}{ccc} S_1^* & \xrightarrow{h} & S_2^* \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ T_1^* & \xrightarrow[(h')^*]{} & T_2^* \end{array}$$

with a_1 inner and h' a map of positive opetopic cardinals, i.e., the map $(h')^*$ is an outer map. $Lan_{\mathbf{j}}(h)$ is so defined that, for any h, h', a_1, a_2 as above, the diagram

$$\begin{array}{ccc} Lan_{\mathbf{j}}(F)(S_2^*) = \coprod_{a_2:S_2^* \to T_2^* \text{ inner}} F(T_2) & \xleftarrow{\kappa_{a_2}^{S_2^*}} & F(T_2) \\ {\scriptstyle Lan_{\mathbf{j}}(F)(h)}\downarrow & & \downarrow{\scriptstyle F(h')} \\ Lan_{\mathbf{j}}(F)(S_1^*) = \coprod_{a_1:S_1^* \to T_1^* \text{ inner}} F(T_1) & \xleftarrow[\kappa_{a_1}^{S_1^*}]{} & F(T_1) \end{array}$$

commutes.

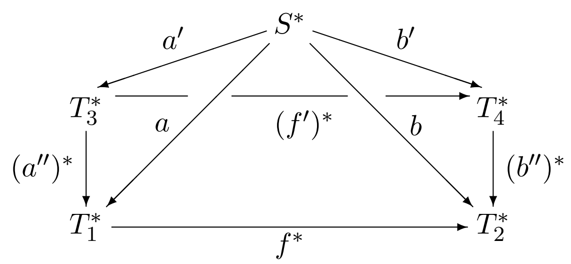
Proof. Fix F in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ for the whole proof. Let S be a positive opetopic cardinal. Then $Lan_{\mathbf{j}}(F)(S)$ is the colimit of the following functor

$$\mathbf{j}^{op} \downarrow S \xrightarrow{\pi^S} (\mathbf{pOpeCard})^{op} \xrightarrow{F} Set,$$

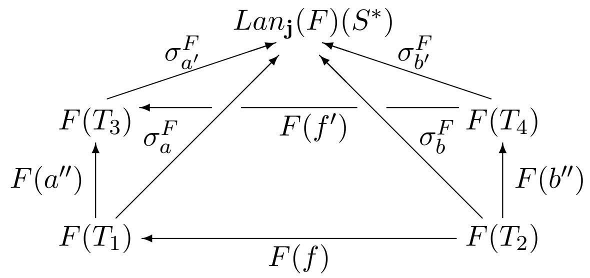
i.e., $Lan_{\mathbf{j}}(F)(S) = (F \circ \pi^S, \sigma^F)$. A map $f : a \longrightarrow b$ in $\mathbf{j}^{op} \downarrow S$, is a commuting triangle

$$\begin{array}{ccc} & S^* & \\ {\scriptstyle a}\swarrow & & \searrow{\scriptstyle b} \\ T_1^* & \xrightarrow[f^*]{} & T_2^* \end{array}$$

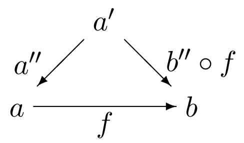
in $\mathbf{pOpeCard}_\omega$, and hence by Lemma 10.3 we can take the inner-outer factorizations, c.f. 10, of both $a = (a'')^* \circ a'$ and $b = (b'')^* \circ b'$, with a' and b' inner maps. Then, again by Lemma10.3, there is a morphism $f' : a' \longrightarrow b'$ in $\mathbf{pOpeCard}$ which must be an iso. In this way we get a commuting diagram



in $\mathbf{pOpeCard}_\omega$, which corresponds to the following part of the colimiting cocone:



Thus if there is a morphism $f : a \to b$ between two objects in $\mathbf{j}^{op} \downarrow S$, we have a commuting diagram

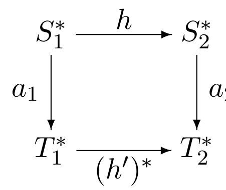


in $\mathbf{j}^{op} \downarrow S$ with a' being the inner part of both a and b. There are no other comparison maps between these objects. But this says that in fact

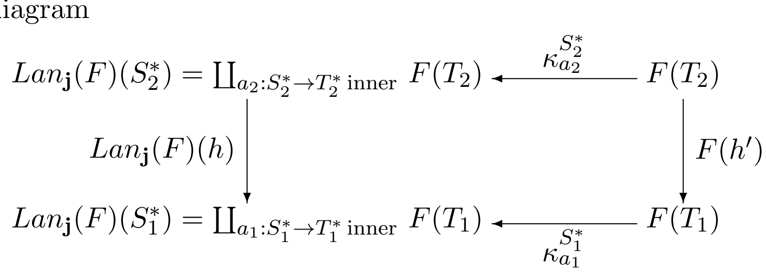
$$Lan_{\mathbf{j}}(F)(S^*) = \textstyle\coprod_{a:S^*\to T^* \text{ inner}} F(T) \xleftarrow{\ \kappa_a^{S^*}\ } F(T),$$

where the coproduct is taken over all (up to iso) inner maps with the domain S^*, with the coprojections as shown.

To define $Lan_{\mathbf{j}}(F)$ on morphisms, fix an ω-functor $h : S_1^* \longrightarrow S_2^*$ in **pOpeCard** and an inner map $a_2 : S_2^* \to T_2^*$. By Lemma 10.3, we can form a diagram



with a_1 inner and $(h')^*$ outer. $Lan_{\mathbf{j}}(h)$ is so defined that, for any h', a_1, a_2 as above, the diagram



commutes. This shows that the functor $Lan_{\mathbf{j}}$ is a familially representable functor. □

Lemma 16.2 *The functor of the left Kan extension along* $\mathbf{j}$ *restricts to*

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_\omega)^{op}, Set),$$

i.e., whenever $F : (\mathbf{pOpeCard})^{op} \longrightarrow Set$ *preserves special pullbacks, so does* $Lan_{\mathbf{j}}(F) : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$*. Moreover,* $Lan_{\mathbf{j}}$ *is the left adjoint to*

$$\mathbf{j}^* : sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set).$$

Proof. Note that once the first part of the statement will be proved, the part following 'moreover' will follow immediately.

Fix F in $sPb((\mathbf{pOpeCard})^{op}, Set)$ for the whole proof. We shall use the description of $Lan_{\mathbf{j}}(F)$ from Proposition 16.1 to show that $Lan_{\mathbf{j}}(F)$ preserves special pullbacks. So assume that S_1 and S_2 are positive opetopic cardinals such that

$$\mathbf{c}^{(k)}(S_1) = \mathbf{d}^{(k)}(S_2),$$

i.e., we have a pushout

$$\begin{array}{ccc} S_1 & \xrightarrow{\kappa_1} & S_1 \oplus_k S_2 \\ {\scriptstyle \mathbf{c}^{(k)}_{S_1}} \uparrow & & \uparrow {\scriptstyle \kappa_2} \\ \mathbf{c}^{(k)}(S_1) & \xrightarrow[\mathbf{d}^{(k)}_{S_2}]{} & S_2 \end{array}$$

in **pOpeCard**. We need to show that the square

$$\begin{array}{ccc} Lan_{\mathbf{j}}(F)(S_1) & \xleftarrow{Lan_{\mathbf{j}}(F)(\kappa_1)} & Lan_{\mathbf{j}}(F)(S_1 \oplus_k S_2) \\ {\scriptstyle Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})} \downarrow & & \downarrow {\scriptstyle Lan_{\mathbf{j}}(F)(\kappa_2)} \\ Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}(S_1)) & \xleftarrow[Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})]{} & Lan_{\mathbf{j}}(F)(S_2) \end{array}$$

is a pullback in Set, i.e., that the square

$$\begin{array}{ccc} \coprod_{a:S_1^* \to T^* \text{ inner}} F(T) & \xleftarrow{Lan_{\mathbf{j}}(F)(\kappa_1)} & \coprod_{a:(S_1;_k S_2)^* \to T^* \text{ inner}} F(T) \\ {\scriptstyle Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})} \downarrow & & \downarrow {\scriptstyle Lan_{\mathbf{j}}(F)(\kappa_2)} \\ \coprod_{a:(\mathbf{c}^{(k)}(S_1))^* \to T^* \text{ inner}} F(T) & \xleftarrow[Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})]{} & \coprod_{a:S_2^* \to T^* \text{ inner}} F(T) \end{array}$$

is a pullback in Set. So suppose we have

$$x_1 \in F(T_1) \xrightarrow{\kappa^{s^*}_{a_1}} \coprod_{a:S_1^* \to T^* \text{ inner}} F(T)$$

$$x_2 \in F(T_2) \xrightarrow{\kappa^{s^*}_{a_2}} \coprod_{a:S_2^* \to T^* \text{ inner}} F(T)$$

such that

$$Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})(x_1) = Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})(x_2),$$

i.e., we have a commuting diagram in **pOpeCard**

$$\begin{array}{ccccc} S_1^* & \xleftarrow{(\mathbf{c}^{(k)}_{S_1})^*} & (\mathbf{c}^{(k)}(S_1))^* = (\mathbf{d}^{(k)}(S_2))^* & \xrightarrow{(\mathbf{d}^{(k)}_{S_2})^*} & S_2^* \\ {\scriptstyle a_1} \downarrow & & \downarrow {\scriptstyle a_0} & & \downarrow {\scriptstyle a_2} \\ T_1^* & \xleftarrow[f_1^*]{} & T^* & \xrightarrow[f_2^*]{} & T_2^* \end{array}$$

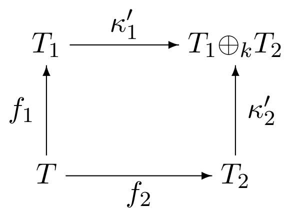
such that

$$F(f_1)(x_1) = F(f_2)(x_2).$$

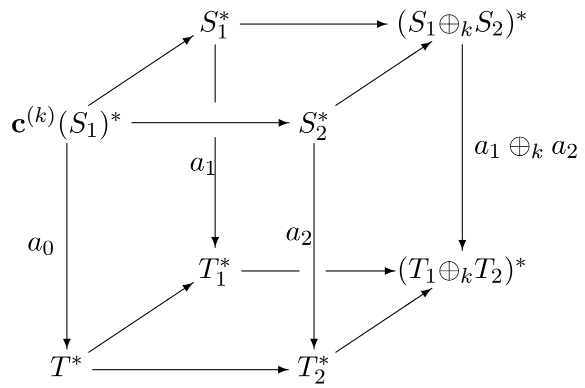
By Proposition 10.2,

$$\mathbf{c}^{(k)}a_1 = a_0 = \mathbf{d}^{(k)}a_2 \qquad f_1^* = (\mathbf{c}_{T_1}^{(k)})^* \qquad f_2^* = (\mathbf{d}_{T_2}^{(k)})^*,$$

and the square



is a special pushout. We have a commuting diagram

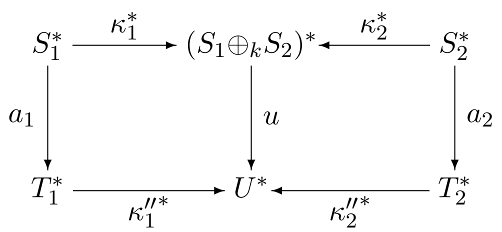


where the bottom square is the above square, and the top square is the one we formed earlier. All the horizontal morphisms are outer. Since a_1 and a_2 are inner, $a_1(S_1) = T_1$ and $a_2(S_2) = T_2$, we have

$$(a_1 \oplus_k a_2)(S_1 \oplus_k S_2) = a_1(S_1) \oplus_k a_2(S_2) = T_1 \oplus_k T_2,$$

i.e., $a_1 \oplus a_2 : (S_1 \oplus_k S_2)^* \longrightarrow (T_1 \oplus_k T_2)^*$ is inner, as well. So in fact all vertical morphisms in the above diagram are inner.

Suppose we have another inner map u and outer maps κ_1'', κ_2'' so that the squares



commute. A diagram chasing shows that

$$\kappa_1''^* \circ f_1^* \circ a_1 = \kappa_2''^* \circ f_2^* \circ a_1.$$

As inner-outer factorization is essentially unique, it follows that

$$\kappa_1''^* \circ f_1^* = \kappa_2''^* \circ f_2^*.$$

By the universal property of the pushout $(T_1 \oplus_k T_2)^*$, we have an ω-functor

$$v : (T_1 \oplus_k T_2)^* \longrightarrow U^*$$

such that

$$\kappa''_1 = u \circ \kappa'_1 \qquad \kappa''_2 = u \circ \kappa'_2.$$

Then again, by a diagram chasing, we get

$$u \circ \kappa_i = v \circ (a_1 \oplus_k a_2) \circ \kappa_i,$$

for $i = 1, 2$. Hence, by universal property of the pushout $(S_1 \oplus_k S_2)^*$, we have that $u = v \circ (a_1 \oplus_k a_2)$. But both u and $(a_1 \oplus_k a_2)$ are inner so by uniqueness of factorization, see Lemma 10.3, v must be an iso, as well. This means that we need to find an

$$x \in F(T_1 \oplus_k T_2) \xrightarrow{\kappa^{(S_1 \oplus_k S_2)^*}_{(a_1 \oplus_k a_2)}} \textstyle\coprod_{a:(S_1 \oplus_k S_2)^* \to T^* \text{ inner}} F(T)$$

such that

$$Lan_{\mathbf{j}}(F)(\kappa_1)(x) = x_1, \quad Lan_{\mathbf{j}}(F)(\kappa_2)(x) = x_2.$$

But F sends special pushouts in **pOpeCard** to pullbacks in Set so the square

$$\begin{array}{ccc} F(T_1) & \xleftarrow{F(\kappa'_1)} & F(T_1 \oplus_k T_2) \\ {\scriptstyle F(f_1)} \downarrow & & \downarrow {\scriptstyle F(\kappa'_2)} \\ F(T) & \xleftarrow[F(f_2)]{} & F(T_2) \end{array}$$

is a pullback in Set. Thus indeed there is a unique $x \in F(T_1 \oplus_k T_2)$ such that $F(\kappa'_i)(x) = x_i$ for $i = 1, 2$. This shows that $Lan_{\mathbf{j}}(F)$ preserves special pullbacks. $\square$

Lemma 16.3 *The following square*

$$\begin{array}{ccc} \mathbf{pPoly} & \xrightarrow{\quad \mathbf{e} \quad} & \omega Cat \\ {\scriptstyle \widehat{(-)}} \downarrow & & \downarrow {\scriptstyle \widehat{(-)}} \\ sPb((\mathbf{pOpeCard})^{op}, Set) & \xrightarrow[Lan_{\mathbf{j}}]{} & sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \end{array}$$

commutes, up to an isomorphism.

Proof. We shall define two natural transformations φ and ψ which are mutually inverse, i.e., for a positive-to-one polygraph Q we define

$$Lan_{\mathbf{j}}\left(\mathbf{pPoly}((-)^*, Q)\right) \underset{\psi_Q}{\overset{\varphi_Q}{\rightleftarrows}} \omega Cat((-)^*, Q).$$

Let $a : S^* \longrightarrow T^*$ be an inner map and $g : T^* \to Q$ be a polygraph map, i.e., g is in the following coproduct/

$$g \in \mathbf{pPoly}(T^*, Q) \xrightarrow{\kappa^{S^*}_a} \textstyle\coprod_{S^* \to R^* \text{ inner}} \mathbf{pPoly}(R^*, Q).$$

Then we put

$$\varphi_Q(g) = g \circ a.$$

On the other hand, for an ω-functor $f : S^* \to Q \in \omega Cat(S^*, Q)$, by Proposition 12.2.4, we have a factorization

$$\begin{array}{ccc} S^* & \xrightarrow{\quad f \quad} & Q \\ & {\scriptstyle f^{in}} \searrow \quad \nearrow {\scriptstyle \tau_{f(S)}} & \\ & T^*_{f(S)} & \end{array}$$

Then we put

$$\psi_Q(f) = \tau_{f(S)} \in \mathbf{pPoly}(T^*_{f(S)}, Q) \xrightarrow{\kappa^{S^*}_{fin}} \coprod_{S^* \to R^* \text{ inner}} \mathbf{pPoly}(R^*, Q).$$

The fact that these transformations are mutually inverse follows from the fact that the above factorization is essentially unique.

The verifications that these transformations are natural is left for the reader. □

Theorem 16.4 *The functor*

$$\mathbf{j}^* : sPb((\mathbf{pOpeCard}_\omega)^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set)$$

is monadic.

Proof. We are going to verify Beck's conditions for monadicity. As $\mathbf{j}$ is essentially surjective, $\mathbf{j}^*$ is conservative. By Lemma 16.2, the adjunction $Lan_{\mathbf{j}} \dashv j^*$ restricts to the above categories. So $\mathbf{j}^*$ has a left adjoint. It remains to show that $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ has coequalizers of $\mathbf{j}^*$-contractible coequalizer pairs and that $\mathbf{j}^*$ preserves them. To this aim, let us assume that we have a parallel pair

$$A \underset{G}{\overset{F}{\rightrightarrows}} B$$

of morphisms in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ such that

$$A((-)^*) \underset{G_{(-)^*}}{\overset{F_{(-)^*}}{\rightrightarrows}} B((-)^*) \xrightarrow{\quad q \quad} Q, \qquad t : B((-)^*) \to A((-)^*), \quad s : Q \to B((-)^*)$$

is a split coequalizer in $sPb((\mathbf{pOpeCard})^{op}, Set)$, i.e., the following equations

$$q \circ s = 1_Q \qquad q \circ G_{(-)^*} = q \circ F_{(-)^*} \qquad F_{(-)^*} \circ t = 1_{B((-)^*)} \qquad G_{(-)^*} \circ t = s \circ q$$

hold. We are going to construct a special pullbacks preserving functor

$$C : (\mathbf{pOpeCard}_\omega)^{op} \longrightarrow Set$$

and a natural transformation

$$H : B \longrightarrow C$$

so that the diagram in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$

$$A \underset{G}{\overset{F}{\rightrightarrows}} B \xrightarrow{\quad H \quad} C$$

is a coequalizer, and $H_{(-)^*} = q$.

The functor C on a morphism $f : T_1^* \longrightarrow T_2^*$ is defined as in the diagram

$$\begin{array}{ccc} C(T_1^*) & \xrightarrow{\quad C(f) \quad} & C(T_2^*) \\ \| & & \| \\ Q(T_1) & & Q(T_2) \\ {\scriptstyle s_{T_1}} \downarrow & & \uparrow {\scriptstyle q_{T_2}} \\ B(T_1^*) & \xrightarrow[\quad B(f) \quad]{} & B(T_2^*) \end{array}$$

i.e., $C(T_i) = Q(T_i)$, for $i = 1,2$ and $C(f) = q_{T_2} \circ B(f) \circ s_{T_1}$.

The natural transformation H is given by

$$H_{T^*} = q_T,$$

for $T \in \mathbf{pOpeCard}$. It remains to verify that

1. C is a functor;
2. H is a natural transformation;
3. $C((-)^*) = Q$;
4. $H_{(-)^*} = q$;
5. C preserves the special pullbacks;
6. H is a coequalizer.

Ad 1. Let

$$T_1^* \xleftarrow{\ f\ } T_2^* \xleftarrow{\ g\ } T_3^*$$

be a pair of morphisms in $\mathbf{pOpeCard}_\omega$. We calculate

$$C(g) \circ C(f) = q_{T_3} \circ B(g) \circ s_{T_2} \circ q_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= q_{T_3} \circ B(g) \circ G_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ G_{T_3^*} \circ A(g) \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ F_{T_3^*} \circ A(g) \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(g) \circ F_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(g) \circ 1_{B(T_2)^*} \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(g) \circ B(f) \circ s_{T_1} =$$
$$= q_{T_3} \circ B(f \circ g) \circ s_{T_1} = C(f \circ g),$$

i.e., C preserves compositions. If T is a positive opetopic cardinal, we also have

$$C(1_{T^*}) = q_T \circ B(1_{T^*}) \circ s_T = q_T \circ s_T = 1_{Q(T)} = 1_{C(T^*)},$$

i.e., C preserves identities, as well.

Ad 2. Let $f : T_2^* \longrightarrow T_1^*$ be a morphism in $\mathbf{pOpeCard}_\omega$. We have

$$H_{T_2^*} \circ B(f) = q_{T_2} \circ B(f) =$$

$$= q_{T_2} \circ B(f) \circ F(T_1^*) \circ t_{T_1} =$$
$$= q_{T_2} \circ F(T_2^*) \circ A(f) \circ t_{T_1} =$$
$$= q_{T_2} \circ G(T_2^*) \circ A(f) \circ t_{T_1} =$$
$$= q_{T_2} \circ B(f) \circ G(T_1^*) \circ t_{T_1} =$$
$$= q_{T_2} \circ B(f) \circ s_{T_1} \circ q_{T_1} =$$
$$= C(f) \circ q_{T_1} = C(f) \circ H_{T_1^*},$$

i.e., H is a natural transformation.

Ad 3. Let $f : T_2 \longrightarrow T_1$ be a morphism in **pOpeCard**. Thus q is natural with respect to f. So we have

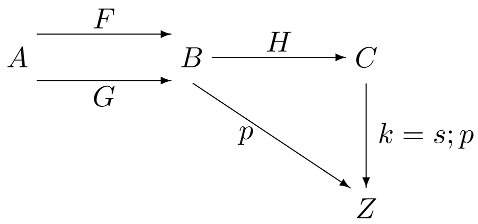
$$C(f^*) = q_{T_2} \circ B(f^*) \circ s_{T_1} = Q(f) \circ q_{T_1} \circ s_{T_1} = Q(f) \circ 1_{T_1} = Q(f),$$

i.e., $C_{(-)^*} = Q$.

Ad 4. $H_{(-)^*} = q$ holds by definition.

Ad 5. Since special pullbacks involve only the outer morphisms (i.e., those that come from **pOpeCard**), and Q preserves special pullbacks, so does C.

Ad 6. Finally, we shall show that H is a coequalizer. Let $p : B \longrightarrow Z$ be a natural transformation in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$ such that $pF = pG$. We put $k = s;p : C \longrightarrow Z$, so that we have a diagram



We need to verify that k is a natural transformation in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$, such that $p = H;k$. Then the uniqueness of k will follow from the fact that q is a split epi. Let $f : T_2^* \longrightarrow T_1^*$ be a morphism in $\mathbf{pOpeCard}_\omega$. Then

$$k_{T_2^*} \circ C(f) = k_{T_2^*} \circ q_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= p_{T_2^*} \circ s_{T_2} \circ q_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= p_{T_2^*} \circ G_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= p_{T_2^*} \circ F_{T_2^*} \circ t_{T_2} \circ B(f) \circ s_{T_1} =$$

$$= p_{T_2^*} \circ B(f) \circ s_{T_1} =$$

$$= D(f) \circ p_{T_1^*} \circ s_{T_1} = D(f) \circ k_{T_1^*},$$

i.e., k is a natural transformation and hence H is indeed a coequalizer of F and G in $sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$, as required. □

Theorem 16.5 *The nerve functor*

$$\widehat{(-)} : \omega Cat \longrightarrow sPb((\mathbf{pOpeCard})^{op}, Set)$$

sending the ω-category C to the presheaf

$$\omega Cat((-)^*, C) : (\mathbf{pOpeCard})^{op} \longrightarrow Set$$

is monadic.

Proof. This is obtained by combining the previous theorem with Corollaries 13.5 and 15.2. □

Proposition 16.6 *The functor*

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

preserves connected limits.

Proof. This is a consequence of Lemma 16.1, where $Lan_{\mathbf{j}} : \widehat{\mathbf{pOpeCard}} \longrightarrow \widehat{\mathbf{pOpeCard}}_\omega$ is described as a familially representable functor. In particular, it preserves connected limits. The above functor is a restriction of a familially representable functor to the category of functors preserving special pullbacks. Since limits commute with limits, the functor

$$Lan_{\mathbf{j}} : sPb((\mathbf{pOpeCard})^{op}, Set) \longrightarrow sPb((\mathbf{pOpeCard}_\omega)^{op}, Set)$$

preserves the connected limits, as well. $\Box$

Theorem 16.7 *The embedding functor*

$$\mathbf{e} : \mathbf{pPoly} \longrightarrow \omega Cat$$

preserves connected limits. $\Box$

Proof. This follows immediately from Propositions 13.4, 16.6, Lemma 16.3 and Corollary 15.2. $\Box$

17 More on monadic adjunctions and distributive laws

We have shown that ωCat is monadic over **pPoly** with the free functor being the embedding $\mathbf{pPoly} \to \omega Cat$. We also know that the category of positive-to-one polygraphs is equivalent to the category of presheaves on **pOpe** and to the subcategory of special pullback preserving functors $sPb((\mathbf{pOpeCard})^{op}, Set)$ of the presheaf category $\widehat{\mathbf{pOpeCard}}$. Because of the last equivalence we shall freely use the notation $X(Q)$ when X is a presheaf on **pPoly** and Q ranges over all positive opetopic cardinals. In this section we shall describe explicitly the whole strongly cartesian monad $(T_\omega, \eta_\omega, \mu_\omega)$ on $\widehat{\mathbf{pOpe}}$ whose category of algebras is equivalent to ωCat. We also show that this monad decomposes into two other strongly cartesian monads of 'pure composition' (T_c, η_c, μ_c) and of 'adding identities' $(T_\iota, \eta_\iota, \mu_\iota)$, in analogy with the decomposition of the strongly cartesian free monoid monad T_{mon} into free semigroup monad and pointed set monad, c.f. [TTT, p. 258]. In particular, the nerve theorem, c.f. [W, BMW], applies.

The monad T_ω

We write $P \overset{q}{\to\!\!\!\!\circ}\, Q$ to indicate that the map q is inner, i.e., it is an ω-functor between ω-categories P^* and Q^* so that $q(P) = Q$. Let $u : X \to Y$ be a map of presheaves on **pOpe**, and S be a positive opetope. Then $T_\omega(X)(S)$ is given by the coproduct†

$$T_\omega(X)(S) = \coprod_{S \overset{q}{\to\!\!\!\!\circ} Q} X(Q) = \{\langle x, q\rangle | S \overset{q}{\to\!\!\!\!\circ} Q \in \mathbf{pOpeCard},\ \ x : Q \to X \in X(Q)\},$$

with coprojections

$$\kappa^1_q : X(Q) \longrightarrow \coprod_{S \overset{q}{\to\!\!\!\!\circ} Q} X(Q) = T_\omega(X)(S)$$

$$X(Q) \ni x \mapsto \langle x, q\rangle.$$

† For example, Q can be of the form $S_1 \oplus_n S_2$, with S_1, S_2 opetopes of the same dimension n as S, with q mapping the generator S to the free composite of the generators S_1 and S_2, so that an element of $X(Q)$ is a pair of an element in $X(S_1)$ and an element in $X(S_2)$. Then the (Q, q)-component of a T_ω-algebra α will provide an actual composition of these elements.

Moreover we set

$$T_\omega(X)_f : T_\omega(X)(S) \longrightarrow T_\omega(X)(S')$$
$$\langle x, q\rangle \mapsto \langle x \circ f', q'\rangle,$$

where (q', f') is the inner-outer factorization of $q \circ f$

$$\begin{array}{ccc} S' & \xrightarrow{f} & S \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ Q' & \xrightarrow[f']{} & Q \end{array}$$

and

$$T_\omega(u)_S : T_\omega(X)(S) \longrightarrow T_\omega(Y)(S)$$
$$\langle x, q\rangle \mapsto \langle u \circ x, q\rangle.$$

The iteration $T^2_\omega(X)(S)$ is given by the coproduct

$$T^2_\omega(X)(S) = \coprod_{S \xrightarrow{q'} R} \ \coprod_{R \xrightarrow{q} Q} X(Q) =$$

$$= \{\langle x, q, q'\rangle | S \xrightarrow{q'} R, R \xrightarrow{q} Q \in \mathbf{pOpeCard}, \ \ x \in X(Q)\},$$

with coprojections

$$\kappa^2_{q,q'} : X(R) \longrightarrow \coprod_{S \xrightarrow{q'} Q} \ \coprod_{Q \xrightarrow{q} R} X(R) = T^2_\omega(X)(S)$$

$$X(R) \ni x \mapsto \langle x, q, q'\rangle.$$

The unit is given by

$$((\eta_\omega)_X)_S = \kappa^1_{id_S} : X(S) \longrightarrow T_\omega(X)(S) = \coprod_{S \xrightarrow{q} Q} X(Q)$$

$$X(S) \ni x \mapsto \langle x, id_S\rangle$$

and the multiplication is the unique map commuting with the following coprojections

$$\begin{array}{ccc} T^2_\omega(X)(S) = \coprod_{S \xrightarrow{q'} Q} \coprod_{Q \xrightarrow{q} R} X(R) & \xrightarrow{((\mu_\omega)_X)_S} & \coprod_{S \xrightarrow{k} R} X(R) = T_\omega(X)(S) \\ & {\scriptstyle \kappa^2_{q,q'}} \nwarrow \quad \nearrow {\scriptstyle \kappa^1_{q \circ q'}} & \\ & X(R) & \end{array}$$

i.e.,

$$\langle x, q, q'\rangle \mapsto \langle x, q \circ q'\rangle.$$

The factorization of a morphism

$$f : P \longrightarrow T_\omega(1)$$

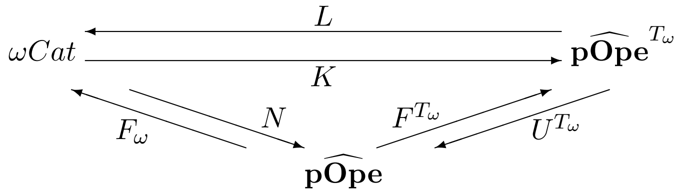
through an inner map q_f is

$$P \xrightarrow[q_f]{} T_\omega(Q_f) \longrightarrow T_\omega(1),$$

where $q_f = f_P(id_P)$.

The comparison functor K

Now we will describe the comparison functors between the category of ω-categories ωCat and the category of T_ω-algebras $\widehat{\mathbf{pOpe}}^{T_\omega}$, i.e., we shall define the functor K in the diagram



and its left adjoint L.[†]

If $H : \mathcal{C} \to \mathcal{C}'$ is an ω-functor, and $f : S \to S'$ a morphism in **pOpe**, then

$$K(\mathcal{C})(S) = \omega Cat(S^*, \mathcal{C}),$$

and

$$K(\mathcal{C})(f) : K(\mathcal{C})(S') \to K(\mathcal{C})(S)$$
$$h : S'^* \to \mathcal{C} \mapsto h \circ f.$$

Moreover, the T_ω-algebra map

$$\xi_{\mathcal{C}} : T_\omega(K(\mathcal{C})) \longrightarrow K(\mathcal{C}),$$

for $S \in \mathbf{pOpe}$ is given by

$$(\xi_{\mathcal{C}})_S : T_\omega(K(\mathcal{C}))(S) \longrightarrow K(\mathcal{C})(S)$$
$$\langle S \overset{q}{\twoheadrightarrow} S', S'^* \overset{h}{\to} \mathcal{C}\rangle \mapsto h \circ q : S^* \to \mathcal{C}.$$

In particular, for $P, S \in \mathbf{pOpe}$,

$$K(P^*)(S) = \coprod_{S \overset{h}{\twoheadrightarrow} S'} \mathbf{pOpe}(S', P) \cong \{\langle h, k\rangle | h : S \twoheadrightarrow S',\ k : S' \to P \in \mathbf{pOpe}\} \cong T_\omega(P)(S).$$

For an ω-functor $f : P^* \to Q^*$ in $\mathbf{pOpe}_\omega$, the map

$$K(f) : T_\omega(P)(S) \longrightarrow T_\omega(Q)(S)$$

is given by

$$\langle q : S \twoheadrightarrow S', x : S' \to P\rangle \mapsto \langle q' : S \twoheadrightarrow S'', x' : S'' \to Q\rangle,$$

where (q', x') is the inner-outer factorization of the map in $\mathbf{pOpe}_\omega$

$$S^* \xrightarrow{\ q\ } S'^* \xrightarrow{\ x^*\ } P^* \xrightarrow{\ f\ } Q^*.$$

† Here, in reference to Section 1, we can set $F_\omega = \mathbf{e} \circ \widetilde{(-)} \circ Ran_{\mathbf{i}}$, i.e., F_ω is $\mathbf{e}$ up to the characterization of **pPoly** as $\widehat{\mathbf{pOpe}}$. One can chase a candidate for N similarly. One then could embark on proving the commutation of the triangles, which could provide an alternative direct proof of the monadicity result of Section 16.

The comparison functor L

Below we describe explicitly the left adjoint, the essential inverse to the functor K.

For a T_ω-algebra $(X, \xi : T_\omega(X) \to X)$, the ω-category $L((X, \xi)$ is defined as follows. The set of n-cells is

$$L(X,\xi)_n = X(\alpha^n),$$

and the map

$$m_{n,k,n} : X(\alpha^n) \times_{X(\alpha^k)} X(\alpha^n) \cong X(\alpha^{n,k,n}) \longrightarrow X(\alpha^n),$$

composing n-cells compatible over dimension k is the composition of the maps

$$X(\alpha^{n,k,n}) \xrightarrow{\kappa_{m_{n,k,n}}} T_\omega(X)(\alpha^n) \xrightarrow{\quad \xi_{\alpha^n} \quad} T_\omega(\alpha^n).$$

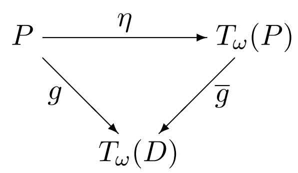
Moreover, as there is a unique map from α^n to any opetopic cardinal Q of dimension less or equal n $\alpha^n \twoheadrightarrow Q$, we have

$$L(T_\omega(P))_n = T_\omega(P)(\alpha^n) = \coprod_{din(S') \leq n} \mathbf{pOpe}(S', P).$$

Generic maps and the generic closure.

Recall the notion of a T_ω-generic map from [W, BMW].

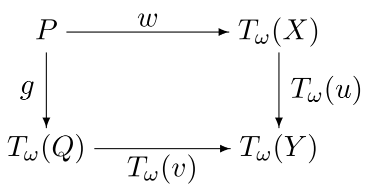
Let $g : P \to T_\omega(D)$ be a map in $\widehat{\mathbf{pOpe}}$. Then we have a unique extension $\overline{g}$ as in the diagram



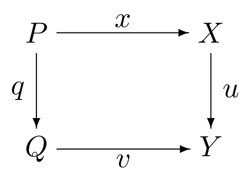
The morphism $g : P \to T_\omega(D)$ is T_ω-generic iff z is an isomorphism, where

$$\overline{g}_P(id_P) = \langle P \overset{q}{\twoheadrightarrow} P', P' \overset{z}{\to} D\rangle.$$

To see this, consider maps $w \to T_\omega(X)$, $v : Q \to Y$ and $u : X \to Y$ in $\widehat{\mathbf{pOpe}}$, so that the diagram



commutes. Then if $w(id_P) = \langle q', x'\rangle$ with $x = x' \circ q' : P \to X$, and $g(id_P) = \langle q, id_Q\rangle$, then the above commutation is equivalent to the commutation of the square



Since (q, v) is an inner-outer factorization and u is outer, there is an outer map $d : Q \to X$ making the square commute. Hence $T_\omega(d)$ is the lift in the previous square showing that g is indeed generic. Moreover, if z as above is not an isomorphism then we do not have a lifting d in general, and hence g is not a generic morphism.

Thus, since any positive opetopic cardinal is a codomain of a T_ω-generic map whose domain is a positive opetope, in fact α^n, for some $n \in \omega$, see page 43, the T_ω-generic closure of **pOpe** consists of all opetopic cardinals, i.e., **pOpeCard** is the T_ω-closure of **pOpe**.

T_ω as a local right adjoint

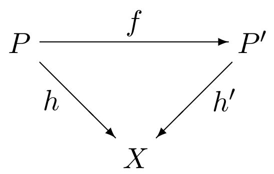
The functor T_ω is a local right adjoint as it is familialy representable. Below we describe explicitly the left adjoint $L_{\omega,1}$ to the functor

$$T_{\omega,1} : \widehat{\mathbf{pOpe}} \longrightarrow \widehat{\mathbf{pOpe}}/T_\omega(1).$$

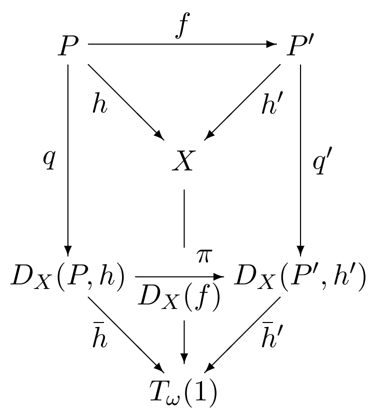
Let $\pi : X \to T_\omega(1)$ be an object of $\widehat{\mathbf{pOpe}}/T_\omega(1)$. We have a functor

$$D_X : \int_{\mathbf{pOpe}} X \longrightarrow \widehat{\mathbf{pOpe}}$$

such that the image of a morphism



in $\int_{\mathbf{pOpe}} X$ is the map $D_X(f) : D_X(P, h) \to D_X(P', h')$ defined from the diagram



where $(q, \bar{h})$ is an inner-outer factorization of $\pi \circ h$, $(q', \bar{h}')$ is an inner-outer factorization of $\pi \circ h'$, and $D_X(f)$ is the unique map making the whole diagram commute. Then $L_{\omega,1}(X, \pi)$ is the colimit of the functor D_X.

The distributive law.

If we replace in the above formulas the inner maps by inner epis (resp. inner monos), we still get strongly cartesian monads $(T_{\omega,\iota}, \eta_{\omega,\iota}, \mu_{\omega,\iota})$ $((T_{\omega,c}, \eta_{\omega,c}, \mu_{\omega,c}))$ on $\widehat{\mathbf{pOpe}}$.

These monads do compose to the monad $(T_\omega, \eta_\omega, \mu_\omega)$. This is because there is a (cartesian) distributive law

$$\lambda_\omega : T_{\omega,c} \circ T_{\omega,\iota} \longrightarrow T_{\omega,\iota} \circ T_{\omega,c}.$$

For X in $\widehat{\mathbf{pOpe}}$ and S in $\mathbf{pOpe}$, both $((T_{\omega,c} \circ T_{\omega,\iota})_X)_S$ and $((T_{\omega,\iota} \circ T_{\omega,c})_X)_S$ are given by double coproducts with coprojections as displayed

$$\sigma^{c,\iota}_{m,e} : X(R) \longrightarrow \coprod_{S \overset{m}{\rightarrowtail} Q\ mono} \ \coprod_{Q \overset{e}{\twoheadrightarrow} R\ epi} X(R) = ((T_{\omega,c} \circ T_{\omega,\iota})_X)_S$$

$$\sigma^{\iota,c}_{e',m'} : X(R) \longrightarrow \coprod_{S \overset{e'}{\twoheadrightarrow} Q'\ epi} \ \coprod_{Q' \overset{m'}{\rightarrowtail} R\ mono} X(R) = ((T_{\omega,\iota} \circ T_{\omega,c})_X)_S.$$

The component

$$((\lambda_\omega)_X)_S : ((T_{\omega,c} \circ T_{\omega,\iota})_X)_S \longrightarrow ((T_{\omega,\iota} \circ T_{\omega,c})_X)_S$$

of the distributive law λ_ω is the unique map making all the following triangles

$$\begin{array}{ccc} \coprod_{S \overset{m}{\rightarrowtail} Q\ mono} \coprod_{Q \overset{e}{\twoheadrightarrow} R\ epi} X(R) & \xrightarrow{((\lambda_\omega)_X)_S} & \coprod_{S \overset{e'}{\twoheadrightarrow} Q'\ epi} \coprod_{Q' \overset{m'}{\rightarrowtail} R\ mono} X(R) \\ & {}^{\sigma^{c,\iota}_{m,e}}\nwarrow \qquad \nearrow^{\sigma^{\iota,c}_{e',m'}} & \\ & X(R) & \end{array}$$

commute, where e', m' is the epi-mono factorization of the inner map $e \circ m$, i.e., we have the following square of inner epi's and mono's in $\mathbf{pOpe}_\omega$

$$\begin{array}{ccc} S & \overset{m}{\rightarrowtail} & Q \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle e} \\ Q' & \underset{m'}{\rightarrowtail} & R \end{array}$$

that commutes.

All the above constructions and considerations can be to truncated to the level n, for $n \in \omega$. Thus $n\mathbf{Cat}$ is monadic over $\widehat{pOpe_n}$ and the corresponding monad (T_n, η_n, μ_n), the n-truncation of the monad $(T_\omega, \eta_\omega, \mu_\omega)$ is strongly cartesian decomposing into two strongly cartesian monads $(T_{n,\iota}, \eta_{n,\iota}, \mu_{n,\iota})$ and $(T_{n,c}, \eta_{n,c}, \mu_{n,c})$, related by a distributive law

$$\lambda_n : T_{n,c} \circ T_{n,\iota} \longrightarrow T_{n,\iota} \circ T_{n,c},$$

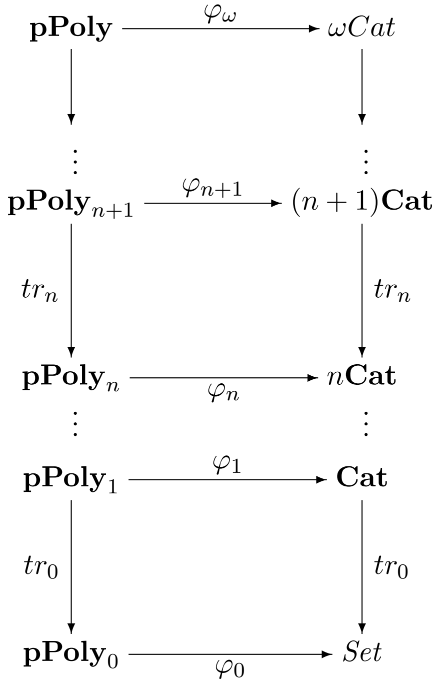
so that we have

$$T_n = T_{n,\iota} \circ T_{n,c}$$

as monads.

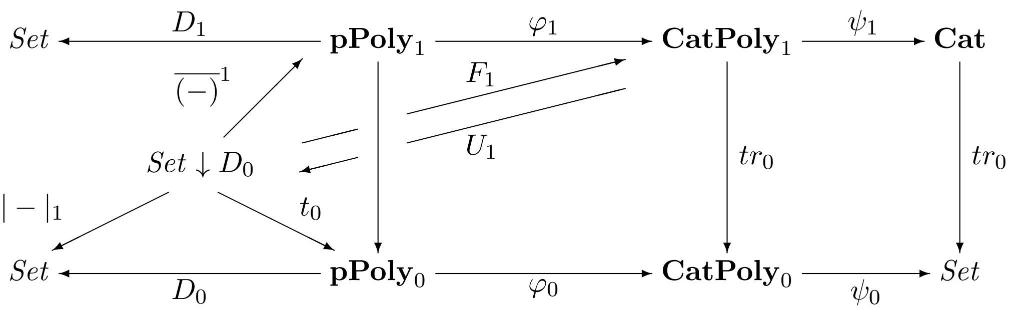
18 Appendix: a definition of positive-to-one polygraphs

The category of positive-to-one polygraphs $\mathbf{pPoly}$, a replete subcategory of the category of ω-categories ωCat, is a limit of a tower of categories of positive-to-one polygraphs of dimension n $\mathbf{pPoly}_n$, replete subcategories of the category of n-categories $n\mathbf{Cat}$. Thus we shall describe a diagram



Having defined the subcategory $\mathbf{pPoly}_n$ of $n\mathbf{Cat}$, we define the category $\mathbf{CatPoly}_{n+1}$ to be a non-full subcategory of the category of $(n+1)$-categories $(n+1)\mathbf{Cat}$ such that an $(n+1)$-category C is an object of $\mathbf{CatPoly}_{n+1}$ iff its truncation to $n\mathbf{Cat}$ is in $\mathbf{pPoly}_n$, and an $(n+1)$-functor $f : C \to D$ is $\mathbf{CatPoly}_{n+1}$ iff its truncation to $n\mathbf{Cat}$ is in $\mathbf{pPoly}_n$.

The first two stages of the above tower are built as follows.



The categories $\mathbf{pPoly}_0$ and $\mathbf{CatPoly}_0$ are the category of sets, the functors φ_0 and ψ_0 are the identity on Set, and the functor D_0 sends set X to its product $X \times X$. $\mathbf{CatPoly}_1$ is $\mathbf{Cat}$ and ψ_1 is an identity on $\mathbf{Cat}$. Having this data, we form a comma category $Set \downarrow D_0$ with projections

$$|-|_1 : Set \downarrow D_0 \to Set \quad \text{and} \quad t_0 : Set \downarrow D_0 \to \mathbf{pPoly}_0.$$

There is a forgetful functor from $U_1 : \mathbf{CatPoly}_1 = \mathbf{Cat} \to Set \downarrow D_0$ sending a category $(C, d, c, \circ, i)$ to (C, d, c). The functor U_1 has a left adjoint F_1 building a free category on a graph. Then $\mathbf{pPoly}_1$ is the replete subcategory of $\mathbf{Cat}$ such that

$$Set \downarrow D_0 \xrightarrow{\overline{(-)}^1} \mathbf{pPoly}_1 \xrightarrow{\varphi_1} \mathbf{Cat}$$

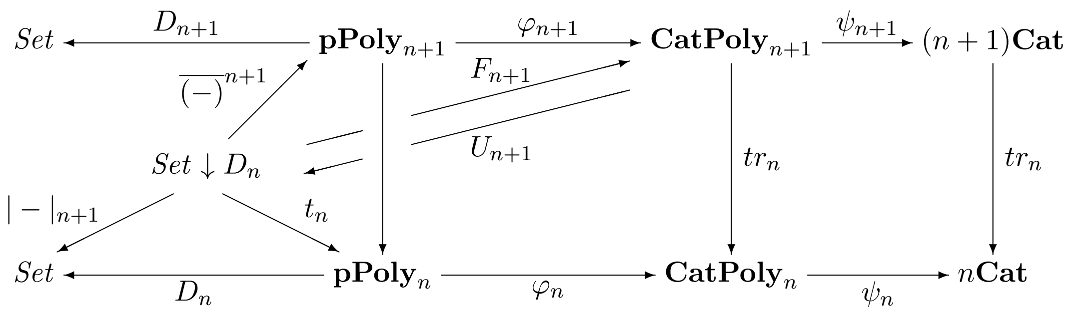
is a full and faithful/essentially surjective factorization. Thus $\mathbf{pPoly}_1$ is a category of free categories over graphs with functors that send generating morphisms to generating morphisms. Finally, we define the functor

$$D_1 : \mathbf{pPoly}_1 \longrightarrow Set$$

$$X \mapsto \{\langle x, y\rangle \in X_1 |\ x \| y,\ \ y \text{ is a generator and } x \text{ is a non-identity morphism}\},$$

i.e., sending a free category X to the set of parallel pairs of morphisms $\langle x, y\rangle$, i.e., $d(x) = d(y)$ and $c(x) = c(y)$, noted $x\|y$, such that y is a generating morphism and x is a non-identity morphism.

The inductive stage in the construction of the above tower is similar to the second one.



Having defined the subcategory $\mathbf{pPoly}_n$ of $n\mathbf{Cat}$ and the functor $D_n : \mathbf{pPoly}_n \to Set$, we built the other parts of the above diagram. We form a comma category $Set \downarrow D_n$ with projections

$$|-|_{n+1} : Set \downarrow D_n \to Set \quad \text{ and } \quad t_n : Set \downarrow D_n \to \mathbf{pPoly}_n.$$

There is a forgetful functor from $U_{n+1} : \mathbf{CatPoly}_{n+1} \to Set \downarrow U_n$ sending an $(n+1)$-category C in $\mathbf{CatPoly}_{n+1}$ to its n-truncation together with the set of $(n+1)$-cells and functions assigning their domains and codomains. The functor U_{n+1} has a left adjoint F_{n+1} – building free $(n+1)$)-categories.Then $\mathbf{pPoly}_{n+1}$ is the replete subcategory of $\mathbf{CatPoly}_{n+1}$ such that

$$Set \downarrow D_n \xrightarrow{\overline{(-)}^{n+1}} \mathbf{pPoly}_{n+1} \xrightarrow{\varphi_{n+1}} \mathbf{CatPoly}_{n+1}$$

is a full and faithful/essentially surjective factorization. Thus $\mathbf{pPoly}_{n+1}$ is a category of free $(n+1)$-categories whose generators/indeterminates are cells of positive-to-one shapes, and of $(n+1)$-functors that send indeterminates to indeterminates.[†] Finally, we define the functor

$$D_{n+1} : \mathbf{pPoly}_{n+1} \longrightarrow Set$$

$$X \mapsto \{\langle x, y\rangle \in X_{n+1} |\ x \| y,\ \ y \text{ is an indeterminate and } x \text{ is a non-identity cell}\},$$

i.e., sending a free category X to the set of parallel pairs of cells $\langle x, y\rangle$, i.e., $d(x) = d(y)$ and $c(x) = c(y)$, noted $x\|y$, such that y is an indeterminate and x is a non-identity map.[‡]

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[†] More precisely, F_{n+1} builds an $(n+1)$-category whose n-truncation is a positive-to-one polygraph of dimension n and whose $(n+1)$-morphisms are freely obtained from a set of generators. By construction, every positive-to-one polygraph of dimension $n+1$ comes with such a set of $(n+1)$-generators, which is left implicit here, but is explicit in the definition given in [Bu, Poly].

[‡] The definition of the functors D_n guarantees that the shapes of generators in all dimensions are positive-to-one. Note also that by construction the truncation $tr_n : (n+1)\mathbf{Cat} \to n\mathbf{Cat}$ restricts and corestricts to $tr_n = \mathbf{CatPoly}_{n+1} \to \mathbf{CatPoly}_n$ and $tr_n : \mathbf{pPoly}_{n+1} \to \mathbf{pPoly}_n$.

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